DIFFERENTIAL GEOMETRY

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Contents

Preface

These are lecture notes for the courses "Differentiable Manifolds I" and "Differentiable Manifolds II", that I am lecturing at UIUC. This course is usually taken by graduate students in Mathematics in their first or second year of studies. The background for this course is a basic knowledge of analysis, algebra and topology.

My main aim in writing up these lectures notes is to offer a written version of the lectures. This should give a chance to students to concentrate more on the class, without worrying about taking notes. It offers also a guide for what material was covered in class. These notes do not replace the recommended texts for this course, quite the contrary: I hope they will be a stimulus for the students to consult those works. In fact, some of these notes follow the material in theses texts.

These notes are organized into "Lectures". Each of these lectures should correspond approximately to 1 hour and 30 minutes of classroom time. However, some lectures do include more material than others, which correspond to different rhythms in class. The exercises at the end of each lecture are a very important part of the course, since one learns a good deal about mathematics by solving exercises. Moreover, sometimes the exercises contain results that were mentioned in class, but not proved, and which are used in later lectures. The students should also keep in mind that the exercises are not homogeneous: this is in line with the fact that in mathematics when one faces for the first time a problem, one usually does not know if it has an easy solution, a hard solution or if it is an open problem.

These notes are a modified version of similar lectures in portuguese that I have used at IST-Lisbon. For the portuguese version I have profited from comments from Ana Rita Pires, Georgios Kydonakis, Miguel Negrão, Miguel Olmos, Ricardo Inglês, Ricardo Joel, José Natário and Roger Picken. Since this is the first english version of these notes, they contain too many typos and mistakes. I will be grateful for any corrections and suggestions for improvement that are sent to me.

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Part 1. Basic Concepts

The notion of a *smooth manifold* of dimension d makes precise the concept of a space which locally looks like the usual euclidean space $\mathbb{R}^d$. Hence, it generalizes the usual notions of curve (locally looks like $\mathbb{R}^1$) and surface (locally looks like $\mathbb{R}^2$). This course consists of a precise study of this fundamental concept of Mathematics and some of the constructions associated with it. We will see that many constructions familiar in *infinitesimal analysis* (i.e., calculus) extend from euclidean space to smooth manifolds. On the other hand, the *global analysis* of manifolds requires new techniques and methods, and often elementary questions lead to open problems.

In this first series of lectures we will introduce the most basic concepts of Differential Geometry, starting with the precise notion of a smooth manifold. The main concepts and ideas to keep in mind from these first series of lectures are:

- Lecture 0: A *manifold as a subset of Euclidean space*, and the various *categories of manifolds*: topological, smooth and analytic manifolds.
- Lecture 1: The *abstract notion of smooth manifold* (our objects) and *smooth map* (our morphisms).
- Lecture 2: A technique of gluing called *Partions of unity*.
- Lecture 3: *Manifolds with boundary* and *smooth maps between manifolds with boundary*.
- Lecture 4: *Tangent vector*, *tangent space* (our infinitesimal objects) and the *differential of a smooth map* (our infinitesimal morphisms).
- Lecture 5: Important classes of smooth maps: *immersions*, *submersions* and *local difeomorphisms*. *Submanifolds* (our sub-objects).
- Lecture 6: *Embedded sub manifolds* and the *Whitney Embedding Theorem*, showing that any smooth manifold can be embedded in some Euclidean space $\mathbb{R}^n$.
- Lecture 7 *Foliations*, which are certain partitions of a manifold into submanifolds, a very useful generalization of the notion of manifold.
- Lecture 8: *Quotients of manifolds*, i.e., smooth manifolds obtained from other smooth manifolds by taking equivalence relations.

Recall that the *Euclidean space of dimension* d is:

$$\mathbb{R}^n := \left\{(x^1, \dots, x^n) : x^1, \dots, x^n \in \mathbb{R}\right\}$$

We will also denote by $x^i : \mathbb{R}^n \to \mathbb{R}$ the i-th coordinate function in $\mathbb{R}^n$. If $U \subset \mathbb{R}^n$ is an open set, a map $f : U \to \mathbb{R}^m$ is called a **smooth map** if all its partials derivatives of every order:

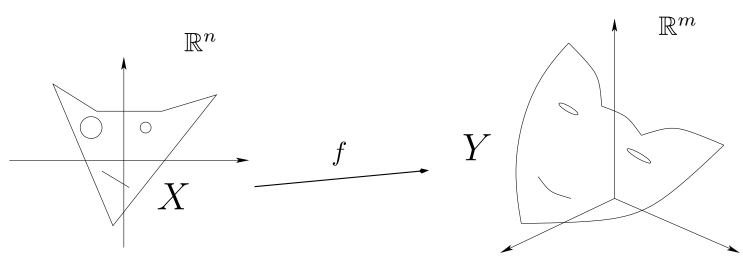
$$\frac{\partial^{i_1+\cdots+i_r} f^j}{\partial x^{i_1} \cdots \partial x^{i_r}}(x),$$

exist and are continuous functions in U. More generally, given any subset $X \subset \mathbb{R}^n$ and a map $f : X \to \mathbb{R}^m$, where X is not necessarily an open set, we say that f is a **smooth map** if every $x \in X$ has an open neighborhood $U \subset \mathbb{R}^n$ where there exists a smooth map $F : U \to \mathbb{R}^m$ such that $F|_X = f$.

A very basic property which we leave as an exercise is that:

Proposition 0.1. *Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ and $Z \subset \mathbb{R}^p$. If $f : X \to Y$ and $g : Y \to Z$ are smooth maps, then $g \circ f : X \to Z$ is also a smooth map.*

A bijection $f : X \to Y$, where $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$, with inverse map $f^{-1} : Y \to X$, such that both f and f^{-1} are smooth, is called a **diffeomorphism** and we say that X and Y are **diffeomorphic subsets**.

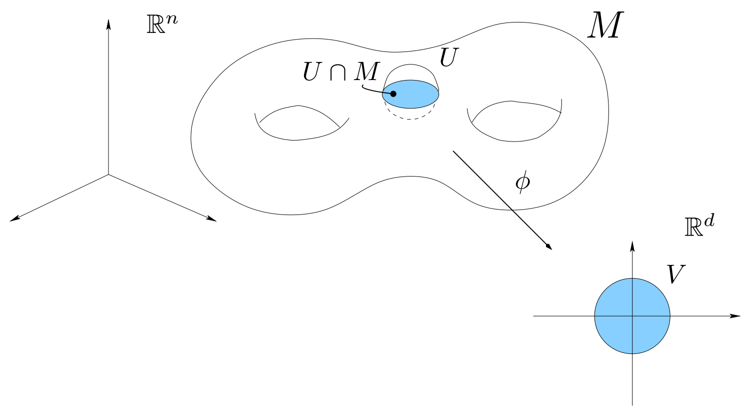


One would like to study properties of sets which are invariant under diffeomorphisms, characterize classes of sets invariant under diffeomorphisms, etc. However, in this definition, the sets X and Y are just too general, and it is hopeless to try to say anything interesting about classes of such diffeomorphic subsets. One must consider nicer subsets of Euclidean space: for example, it is desirable that the subset has at each point a tangent space and that the tangent spaces vary smoothly.

Recall that a subset $X \subset \mathbb{R}^n$ has an induced topology, called the *relative topology*, where the relative open sets are just the sets of the form $X \cap U$, where $U \subset \mathbb{R}^n$ is an open set.

Definition 0.2. *A subset $M \subset \mathbb{R}^n$ is called a **smooth manifold of dimension** d if each $p \in M$ has a neighborhood $M \cap U$ which is diffeomorphic to an open set $V \subset \mathbb{R}^d$.*

The diffeomorphism $\phi : M \cap U \to V$, in this definition, is called a **coordinate system**. The inverse map $\phi^{-1} : V \to M \cap U$, which by assumption is smooth, is called a **parameterization**.



We have the **category of smooth manifolds** where:

- the objects are smooth manifolds;
- the morphisms are smooth maps.

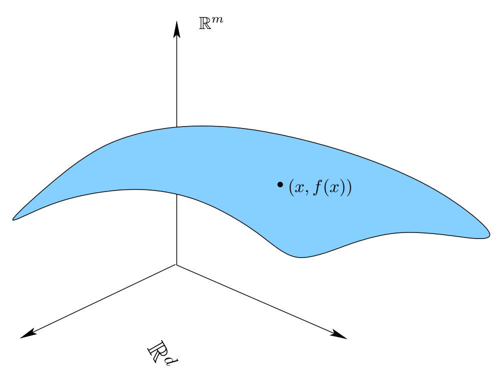
The reason they form a category is because the composition of smooth maps is a smooth map and the identity is also a smooth map.

EXAMPLES 0.3.

1. *Any open subset $U \subset \mathbb{R}^d$ is itself a smooth manifold of dimension d: the inclusion $i : U \hookrightarrow \mathbb{R}^d$ gives a global defined coordinate chart.*
2. *If $f : \mathbb{R}^d \to \mathbb{R}^m$ is any smooth map, its graph:*

$$\mathrm{Graph}(f) := \{(x, f(x)) : x \in \mathbb{R}^d\} \subset \mathbb{R}^{d+m}$$

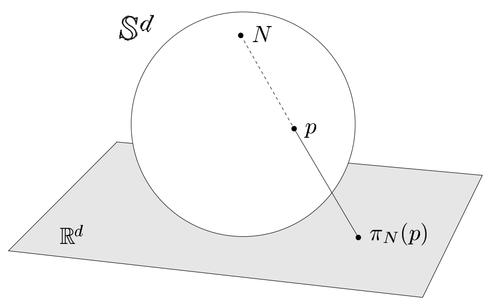
is a smooth manifold of dimension d: the map $x \mapsto (x, f(x))$ is a diffeomorphism $\mathbb{R}^d \to \mathrm{Graph}(M)$, so gives a global parametrization of $\mathrm{Graph}(f)$.



*3. The **unit** d**-sphere** is the subset of $\mathbb{R}^{d+1}$ formed by all vectors of length 1:*

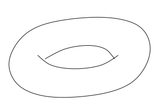
$$\mathbb{S}^d := \{x \in \mathbb{R}^{d+1} : ||x|| = 1\}.$$

This is a d-dimensional manifold which does not admit a global parametrization. However we can cover the sphere by two coordinate systems: if we let $N = (0, \ldots, 0, 1)$ and $S = (0, \ldots, 0, -1)$ denote the north and south poles, then stereographic projection relative to N and S give two coordinate systems $\pi_N : \mathbb{S}^d - \{N\} \to \mathbb{R}^d$ and $\pi_S : \mathbb{S}^d - \{S\} \to \mathbb{R}^d$.

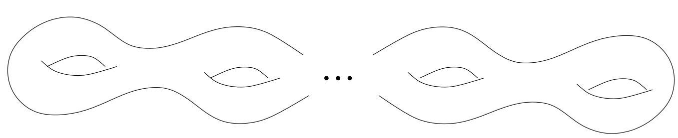


4. The only connected manifolds of dimension 1 are the line $\mathbb{R}$ and the circle $\mathbb{S}^1$. What this statement means is that any connected manifold of dimension 1 is diffeomorphic to $\mathbb{R}$ or to $\mathbb{S}^1$.

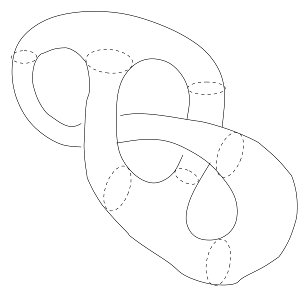
5. The manifolds of dimension 2 include the compact surfaces of genus g. For $g = 0$ this is the sphere $\mathbb{S}^2$. For $g = 1$ this is the torus:



*For $g > 1$, the **compact surface of genus** g is a g-holed torus:*



You should note, however, that a compact surface of genus g can be embedded in $\mathbb{R}^3$ in many forms. Here is one example (can you figure out what is the genus of this surface?):



You should note that in the definitions we have adopted so far in this lecture we have chosen the smooth category, where differentiable maps have all partial derivatives of all orders. We could have chose other classes, such as continuous maps, C^k-maps, or analytic maps([1]). This would lead us to the categories of **topological manifolds**, C^k **manifolds** or **analytic manifolds**. Note that in each such category we have an appropriate notion of equivalence: for example, two topological manifolds X and Y are equivalent if and only if there exists a homeomorphism between them, i.e., a continuous bijection $f : X \to Y$ such that the inverse is also continuous.

Examples 0.4.

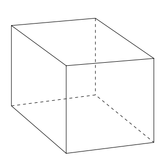
1. *Let $I = [-1, 1]$. The unit cube d-dimensional cube is the set:*

$$I^d = \{(x^1, \dots, x^d) \in \mathbb{R}^{d+1} : x^i \in I, \text{ for all } i = 1, \dots, n\}.$$

The boundary of the cube

$$\partial I^d = \{(x^1, \dots, x^d) \in I^d : x^i = -1 \textbf{ or } 1, \text{ for some } i = 1, \dots, n\}.$$

is a topological manifold of dimension $d - 1$, which is not a smooth manifold.



[1]We shall also use the term C^k**-map**, $k = 1, 2, ..., +\infty$, for a map whose partial derivatives of all orders up to k exist and are continuous, and we make the conventions that a C^0**-map** is simply a continuous map and a C^ω**-map** means an analytic map. A C^k-map which is invertible and whose inverse is also a C^k-map is called a C^k**-equivalence** or a C^k**-isomorphism**.

2. *If $f : \mathbb{R}^d \to \mathbb{R}^l$ is any map of class C^k, its graph:*

$$\text{Graph}(f) := \{(x, f(x)) : x \in \mathbb{R}^d\} \subset \mathbb{R}^{d+l}$$

is a C^k-manifold of dimension d. Similarly, if f is any analytic map then Graph(f) *is an analytic manifold.*

Most of the times we will be working with smooth manifolds. However, there are many situations where it is desirable to consider other categories of manifolds, so you should keep them in mind.

You may wonder if the dimension d that appears in the definition of a manifold is a well defined integer, in other words if a manifold $M \subset \mathbb{R}^n$ could be of dimension d and d', for distinct integers $d \neq d'$. The reason that this cannot happen is due to the following important result:

Theorem 0.5 (Invariance of Domain)**.** *Let $U \subset \mathbb{R}^n$ be an open set and let $\phi : U \to \mathbb{R}^n$ be a 1:1, continuous map. Then $\phi(U)$ is open.*

The reason for calling this result "invariance of domain" is that a domain is a connected open set of $\mathbb{R}^n$, so the result says that the property of being a domain remains invariant under a continuous, 1:1 map. The proof of this result requires some methods from algebraic topology and so we will not give it here. We leave it as an exercise to show that the invariance of domain implies that the dimension of a manifold is a well defined integer.

Homework.

1. Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ and $Z \subset \mathbb{R}^p$. If $f : X \to Y$ and $g : Y \to Z$ are smooth maps, show that $g \circ f : X \to Z$ is also a smooth map.

2. Let $f : \mathbb{R}^d \to \mathbb{R}^m$ be a map of class C^k, $k = 0, \ldots, \omega$. Show that $\phi : \mathbb{R}^d \to$ Graph(f), $x \mapsto (x, f(x))$, is a C^k-equivalence.

3. Show that the sphere $\mathbb{S}^d$ and the boundary of the cube ∂I^{d+1} are equivalent topological manifolds.

4. Consider the set SL$(2, \mathbb{R})$ formed by all 2×2 matrices with real entries and determinant 1:

$$\text{SL}(2, \mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 1 \right\} \subset \mathbb{R}^4.$$

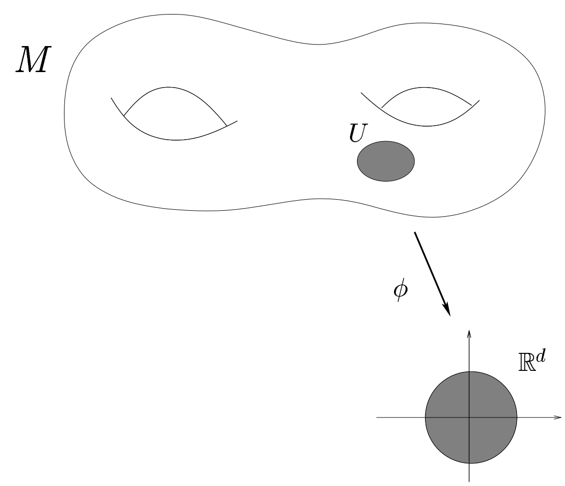
Show that SL$(2, \mathbb{R})$ is a 3-dimensional smooth manifold.

5. Use invariance of domain to show that the notion of dimension of a topological manifold is well defined.

In many situations manifolds do not arise naturally as subsets of Euclidean space. We will see several examples of this later. For that reason, the definition of manifold that we have seen in the previous lecture is often not the most useful one. We need a different definition of a manifold, where M is not assumed a priori to be a subset of some $\mathbb{R}^n$. For this more abstract definition of manifold we need the set M to have a notion of proximity, in other words, we need M to be furnished with a topology. At this point, it maybe useful to remind yourself of the basics of point set topology.

In this more general context, the definition of a topological manifold is very simple:

Definition 1.1. *A topological space M is called a* ***topological manifold of dimension*** *d if every $p \in M$ has a neighborhood $U \subset M$ homeomorphic to some open subset $V \subset \mathbb{R}^d$.*



Some times one also calls a topological manifold a **locally Euclidean space**. In this more general context we still use the same notation as before: we call $\phi : U \to \mathbb{R}^d$ a **system of coordintaes** or a **chart**, and the functions $\phi^i = x^i \circ \phi$ are called **coordinate functions**. We shall denote a system of coordinates by (U, ϕ). Often we write x^i instead of ϕ^i for the coordinate functions, in which case we may denote the system of coordinates by $(U, x^1, \ldots, x^d)$. We say that a system of coordinates is (U, ϕ) *centered at a point* $p \in M$ if $\phi(p) = 0$.

There is a **tacit assumption** about the underlying topology of a manifold, that we will also adopt here, and which is the following:

Manifolds are assumed to be Hausdorff and second countable

This assumption has significant implications, as we shall see shortly, which are very useful in the study of manifolds (e.g., existence of partitions of unity or of Riemannian metrics). On the other hand, it means that in any construction of a manifold we have to show that the underlying topology satisfies these assumptions. This is often easy since, for example, any metric space satisfies these assumptions.

It should be noted, however, that non-Hausdorff manifolds do appear sometimes, for example when one forms quotients of (Hausdorff) manifolds (see Lecture 8). Manifolds which are not second countable can also appear (e.g., in sheaf theory), although we will not meet them in the course of these lectures. We limit ourselves here to give one such example.

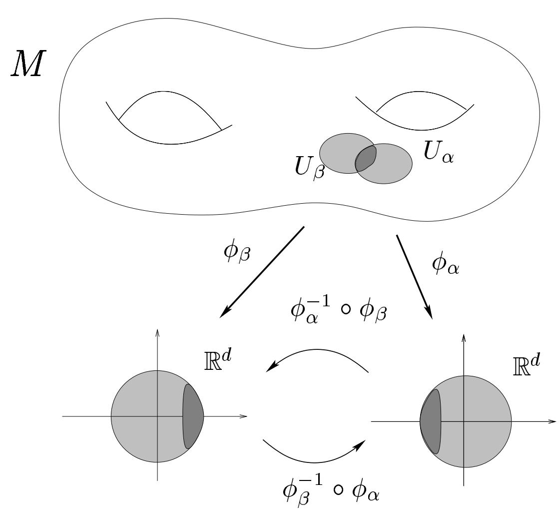
EXAMPLE 1.2.
Consider $M = \mathbb{R}^2$ with the topology generated by sets of the form $U \times \{y\}$, where $U \subset \mathbb{R}$ is open and $y \in \mathbb{R}$. It is easy to see that this topology does not have a countable basis. However, M is a topological one-dimensional manifold with coordinate charts $(U \times \{y\}, \phi_y)$ given by $\phi_y(x, y) = x$.

Of course we are interested in smooth manifolds. The definition is slightly more involved:

Definition 1.3. *A **smooth structure** on a topological d-manifold M is a collection of coordinate systems $\mathcal{C} = \{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ which satisfies the following properties:*

(i) The collection $\mathcal{C}$ covers M: $\bigcup_{\alpha \in A} U_\alpha = M$;
*(ii) For all $\alpha, \beta \in A$, the **transition function** $\phi_\alpha \circ \phi_\beta^{-1}$ is a smooth map;*
(iii) The collection $\mathcal{C}$ is maximal: if (U, ϕ) any coordinate system such that $\phi \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ \phi^{-1}$ are smooth maps for all $\alpha \in A$, then $(U, \phi) \in \mathcal{C}$.

*The pair $(M, \mathcal{C})$ is called a **smooth manifold** of dimension d.*



Given a topological manifold, a collection of coordinate systems which satisfies (i) and (ii) in the previous definition is called an **atlas**. Given an atlas $\mathcal{C}_0 = \{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ there exists a unique maximal atlas $\mathcal{C}$ which contains $\mathcal{C}_0$: it is enough to define $\mathcal{C}$ to be the collection of all **smooth coordinate systems** relative to $\mathcal{C}$, i.e., all coordinate systems (U, ϕ) such that $\phi \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ \phi^{-1}$ are both smooth, for all $(U_\alpha, \phi_\alpha) \in \mathcal{C}_0$. For this reason, one often defines a smooth structure by specifying some atlas, and it is then implicit that the smooth structure is the one associated with the corresponding maximal atlas.

It should be clear from this definition that one can define in a similar fashion **manifolds of class** C^k for any $k = 1, \dots, +\infty, \omega$, by requiring the transition functions to be of class C^k. In these lectures, we shall concentrate on the case $k = +\infty$.

Examples 1.4.

1. *The* ***standard differential structure on Euclidean space*** $\mathbb{R}^d$ *is the maximal atlas that contains the coordinate system* $(\mathbb{R}^d, i)$*, where* $i : \mathbb{R}^d \to \mathbb{R}^d$ *is the identity map. It is a non-trivial fact that the Euclidean space* $\mathbb{R}^4$ *has an infinite number of smooth structures, with the same underlying topology, but which are not equivalent to this one (in a sense to be made precise later). These are called* ***exotic smooth structures****. It is also known that* $\mathbb{R}^d$*, for* $d \neq 4$*, has no exotic smooth structures.*

2. *If* $M \subset \mathbb{R}^n$ *is a* d*-dimensional manifold in the sense of Definition 0.2, then* M *carries a natural smooth structure: the coordinate systems in Definition 0.2 form a maximal atlas (exercise) for the topology on* M *induced from the usual topology on* $\mathbb{R}^n$*. We shall see later in Lecture 6, that the Whitney Embedding Theorem shows that, conversely, any smooth manifold* M *arises in this way. Henceforth, we shall refer to a manifold* $M \subset \mathbb{R}^n$ *in the sense of Definition 0.2 as an* ***embedded manifold*** *in* $\mathbb{R}^n$*.*

3. *If* M *is a* d*-dimensional smooth manifold with smooth structure* $\mathcal{C}$ *and* $U \subset M$ *is an* ***open subset****, then* U *with the relative topology is also a smooth* d*-dimensional manifold with smooth structure given by:*

$$\mathcal{C}_U = \{(U_\alpha \cap U, \phi_\alpha|_{U_\alpha \cap U}) : (U, \phi_\alpha) \in \mathcal{C}\}.$$

4. *If* M *and* N *are smooth manifolds then the* ***Cartesian product*** $M \times N$*, with the product topology, is a smooth manifold: in* $M \times N$ *we consider the maximal atlas that contains all coordinate systems of the form* $(U_\alpha \times V_\beta, \phi_\alpha \times \psi_\beta)$*, where* (U_α, ϕ_α) *and* (V_β, ψ_β) *are smooth coordinate systems of* M *and* N*, respectively. It should be clear that* $\dim M \times N = \dim M + \dim N$*. More generally, if* $M_1, \dots, M_k$ *are smooth manifolds then* $M_1 \times \cdots \times M_k$ *is a smooth manifold of dimension* $\dim M_1 + \cdots + \dim M_k$*. For example, the* d***-torus*** $\mathbb{T}^d = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ *and the* ***cylinders*** $\mathbb{R}^n \times \mathbb{S}^m$ *are smooth manifolds of dimensions* d *and* $n + m$*, respectively.*

5. *The* ***projective*** d***-dimensional space*** *is the set*

$$\mathbb{P}^d = \left\{ L \subset \mathbb{R}^{d+1} : L \textit{ is a straight line through the origin} \right\}.$$

We can think of $\mathbb{P}^d$ as the quotient space $\mathbb{R}^{d+1} - \{0\} / \sim$ where $\sim$ is the equivalence relation:

$$(x^0, \ldots, x^d) \sim (y^0, \ldots, y^d) \text{ if and only if } (x^0, \ldots, x^d) = \lambda(y^0, \ldots, y^d),$$

for some $\lambda \in \mathbb{R} - 0$. If take on $\mathbb{P}^d$ the quotient topology, then it becomes a topological manifold of dimension d: if we denote by $[x^0 : \cdots : x^d]$ the equivalence class of $(x^0, \ldots, x^d) \in \mathbb{R}^{d+1}$, then for each $\alpha = 0, \ldots, n$ we have the coordinate system (U_α, ϕ_α) where:

$$U_\alpha = \left\{[x^0 : \cdots : x^d] : x^\alpha \neq 0\right\},$$

$$\phi_\alpha : U_\alpha \to \mathbb{R}^d, \quad [x^0 : \cdots : x^d] \mapsto (\frac{x^0}{x^\alpha}, \ldots, \frac{\widehat{x^\alpha}}{x^\alpha}, \ldots, \frac{x^d}{x^\alpha})$$

(the symbol $\widehat{a}$ means that we omit the term a). We leave it as an exercise to check that the transition functions between these coordinate functions are smooth, so they form an atlas on $\mathbb{P}^d$. Note that in this example $\mathbb{P}^d$ does not arise naturally as a subset of some Euclidean space.

We have established what are our objects. Now we turn to the morphisms.

Definition 1.5. *Let M and N be smooth manifolds.*

*(i) A function $f : M \to \mathbb{R}$ is called a **smooth function** if $f \circ \phi^{-1}$ is smooth for all smooth coordinate systems (U, ϕ) of M.*

*(ii) A map $\Psi : M \to N$ is called a **smooth map** if $\tau \circ \Psi \circ \phi^{-1}$ is smooth for all smooth coordinate systems (U, ϕ) of M and (V, τ) of N.*

*A smooth map $\Psi : M \to N$ which is invertible and whose inverse is smooth is called a **diffeomorphism**. In this case we say that M and N are **diffeomorphic manifolds**.*

Note that to check that a map $\Psi : M \to N$ is smooth, it is enough to verify that for each $p \in M$, there exist a smooth coordinate system (U, ϕ) of M and (V, τ) of N, with $p \in U$ and $\Psi(p) \in V$, such that $\tau \circ \Psi \circ \phi^{-1}$ is a smooth map. Also, a smooth function $f : M \to \mathbb{R}$ is just a smooth map where $\mathbb{R}$ has its standard smooth structure.

Clearly, the composition of two smooth maps, whenever defined, is a smooth map. The identity map is also a smooth map. So we have the **category of smooth manifolds**, whose objects are the smooth manifolds and whose morphisms are the smooth maps.

Just as we did for maps between subsets of Euclidean space, when $X \subset M$ and $Y \subset N$ are arbitrary subsets of some smooth manifolds, we will say that $\Psi : X \to Y$ is a **smooth map** if for each $p \in X$ there is an open neighborhood $U \subset M$ and a smooth map $F : U \to N$ such that $F|_{U \cap M} = \Psi|_{U \cap M}$.

The set of smooth maps from X to Y will be denoted $C^\infty(X; Y)$. When $Y = \mathbb{R}$, we use $C^\infty(X)$ instead of $C^\infty(X; \mathbb{R})$.

Examples 1.6.

1. *If $M \subset \mathbb{R}^n$ is an embedded manifold, any smooth function $F : U \to \mathbb{R}$ defined on an open $\mathbb{R}^{d+1} \supset U \supset M$ induces, by restriction, a smooth function $f : M \to \mathbb{R}$. Conversely, every smooth function $f : M \to \mathbb{R}$ is the restriction of some smooth function $F : U \to \mathbb{R}$ defined on some open set $\mathbb{R}^{d+1} \supset U \supset M$. To see this we will need the partitions of unity to be introduced in the next lecture.*

   You should also check that if $M \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^m$ are embedded manifolds then $\Psi : M \to N$ is a smooth map if and only if for every $p \in M$ there exists an open neighborhood $U \subset \mathbb{R}^n$ of p and a smooth map $F : U \to \mathbb{R}^m$ such that $\Psi|_{U \cap M} = F|_{U \cap M}$. This shows that the notion of smooth map in Definition 1.5 extends the notion we have introduced in the previous lecture.

2. *The map $\pi : \mathbb{S}^d \to \mathbb{P}^d$ defined by:*
$$\pi(x^0, \dots, x^d) = [x^0 : \cdots : x^d],$$
is a smooth map. Moreover, any smooth function $F : \mathbb{S}^d \to \mathbb{R}$ which is invariant under inversion (i.e., $F(-x) = F(x)$), induces a smooth function $f : \mathbb{P}^d \to \mathbb{R}$: the function f is the unique one that makes the following diagram commutative:
$$\begin{tikzcd} \mathbb{S}^d \arrow[r, "F"] \arrow[d, "\pi"'] & \mathbb{R} \\ \mathbb{P}^d \arrow[ur, dashed, "f"'] & \end{tikzcd}$$
Conversely, every smooth function in $C^\infty(\mathbb{P}^d)$ arises in this way.

If we are given two smooth structures $\mathcal{C}_1$ and $\mathcal{C}_2$ on the same manifold M we say that they are **equivalent smooth structures** if there is a diffeomorphism $\Psi : (M, \mathcal{C}_1) \to (M, \mathcal{C}_2)$.

Example 1.7.

On the line $\mathbb{R}$ the identity map $\mathbb{R} \to \mathbb{R}$, $x \mapsto x$, gives a chart which defines a smooth structure $\mathcal{C}_1$. We can also consider the chart $\mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$, and this defines a distinct smooth structure $\mathcal{C}_2$ on $\mathbb{R}$ (why?). However, these two smooth structures are equivalent since the map $x \mapsto x^3$ gives a diffeomorphism from $(M, \mathcal{C}_2)$ to $(M, \mathcal{C}_1)$.

It is known that every topological manifold of dimension less or equal than 3 has a unique smooth structure. For dimension greater than 3 the situation is much more complicated, and not much is known. However, as we have mentioned before, the smooth structures on $\mathbb{R}^d$, compatible with the usual topology, are all equivalent if $d \neq 4$, and there are uncountably many inequivalent **exotic smooth structures** on $\mathbb{R}^4$. On the other hand, for the sphere $\mathbb{S}^d$ there are no exotic smooth structures for $d \leq 6$ but Milnor found that $\mathbb{S}^7$ has 27 inequivalent smooth structures. Its known, e.g., that $\mathbb{S}^{31}$ has more than 16 million inequivalent smooth structures!

Homework.

1. Let M be a topological manifold. Show that M is locally compact, i.e., every point of M has a compact neighborhood.

2. A normal topological space with a countable basis is metrizable. Use this to show that every topological manifold M is metrizable.
Hint: A topological space is called normal if for every disjoint pair of closed sets A_1 and A_2, there exist disjoint open sets O_1 and O_2 such that $A_1 \subset O_1$ and $A_2 \subset O_2$.

3. Let M be a *connected* topological manifold. Show that M is path connected. If, additionally, M is a smooth manifold, show that for any $p, q \in M$ there exists a smooth path $c : [0,1] \to M$ with $c(0) = p$ and $c(1) = q$.
Hint: Given any smooth path $c : [0,1] \to \mathbb{R}^n$ there is a smooth function $\tau : \mathbb{R} \to \mathbb{R}$, with $\tau(t) = 0$ if $t \leq 0$, $\tau(t) = 1$ if $t \geq 1$, and $\tau'(t) > 0$ if $t \in]0,1[$, so that $c_\tau := c \circ \tau : [0,1] \to \mathbb{R}^n$ is a new smooth path with the same image as c and $c'_\tau(0) = c'_\tau(1) = 0$.

4. Let $\phi : \mathbb{R}^m \to \mathbb{R}^n$ be a diffeomorphism. Use the chain rule to deduce that one must have $m = n$. Use this result to conclude that if M and N are diffeomorphic smooth manifolds then $\dim M = \dim N$, without appealing to invariance of domain.

5. Compute the transition functions for the atlas of projective space $\mathbb{P}^d$ and show that they are smooth. Show also that:
(a) $\mathbb{P}^1$ is diffeomorphic to $\mathbb{S}^1$;
(b) $\mathbb{P}^d - \mathbb{P}^{d-1}$ is diffeomorphic to the open disc $D^n = \{x \in \mathbb{R}^d : ||x|| < 1\}$, where we identify $\mathbb{P}^{d-1}$ with the subset $\{[x^0 : \cdots : x^d] : x^d = 0\} \subset \mathbb{P}^d$.

6. Show that is $M \subset \mathbb{R}^n$ is a d-dimensional manifold in the sense of Definition 0.2, then M carries a natural smooth structure.
Note: One sometimes says that M is an **embedded** manifold in $\mathbb{R}^n$ or a d-**surface** in $\mathbb{R}^n$. When $d = 1$, one says that M is a **curve**, when $d = 2$ one says that M is a **surface**, and when $k = n-1$ one says that M is an **hypersurface**.

7. Let $M \subset \mathbb{R}^n$ be a subset with the following property: for each $p \in M$, there exists an open set $U \subset \mathbb{R}^n$ containing p and diffeomorphism $\Phi : U \to V$ onto an open set $V \subset \mathbb{R}^n$, such that:

$$\Phi(U \cap M) = \left\{ q \in V : q^{d+1} = \cdots = q^n = 0 \right\}.$$

Show that M is a smooth manifold of dimension d (in fact, M is an embedded manifold or a d-surface in $\mathbb{R}^n$; see the previous exercise).

8. Let M be a set and assume that one has a collection $\mathcal{C} = \{(U_\alpha, \phi_\alpha) : \alpha \in A\}$, where $U_\alpha \subset M$ and $\phi_\alpha : U_\alpha \to \mathbb{R}^d$, satisfying the following properties:
(a) For each $\alpha \in A$, $\phi_\alpha(U_\alpha) \subset \mathbb{R}^n$ is open and $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a bijection
(b) For each $\alpha, \beta \in A$, the sets $\phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ are open.

(c) For each $\alpha, \beta \in A$, with $U_\alpha \cap U_\beta \neq \emptyset$ the map $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is smooth.
(d) There is a countable set of U_α that cover M.
(e) For any $p, q \in M$, with $p \neq q$, either there exists a U_α such that $p, q \in U_\alpha$, or there exists U_α and U_β, with $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \emptyset$.

Show that there exists a unique smooth structure on M such that the collection $\mathcal{C}$ is an atlas.

9. Let $M = \mathbb{C} \cup \{\infty\}$. Let $U := M - \{\infty\} = \mathbb{C}$ and $\phi_U : U \to \mathbb{C}$ be the identity map and let $V := M - \{0\}$ and $\phi_V : V \to \mathbb{C}$ be the map $\phi_V(z) = 1/z$, with the convention that $\phi(\infty) = 0$. Use the previous exercise to show that M has a unique smooth structure with atlas $\mathcal{C} := \{(U, \phi_U), (V, \phi_V)\}$. Show that M is diffeomorphic to $\mathbb{S}^2$.
Hint: Be careful with item (e)!

10. Let M and N be smooth manifolds and let $\Psi : M \to N$ be a map. Show that the following statements are equivalent:
(i) $\Psi : M \to N$ is smooth.
(ii) For every $p \in$M there are coordinate systems (U, ϕ) of M and (V, τ) of N, with $p \in U$ and $\Phi(p) \in V$, such that $\tau \circ \Psi \circ \phi^{-1}$ is smooth.
(iii) There exist smooth atlases $\{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ and $\{(U_\beta, \psi_\beta) : \beta \in B\}$ of M and N, such that for each $\alpha \in A$ and $\beta \in B$, $\psi_\beta \circ \Psi \circ \phi_\alpha^{-1}$ is smooth.

11. Let M and N be smooth manifolds and let $\Phi : M \to N$ be a map. Show that:
(i) If Φ is smooth, then for every open set $U \subset M$ the restriction $\Phi|_U : U \to N$ is a smooth map.
(ii) if every $p \in M$ has an open neighborhood U such that the restriction $\Phi|_U : U \to N$ is a smooth map, then $\Phi : M \to N$ is smooth.

Lecture 2. Manifolds with Boundary

There are many spaces, such as the closed unit ball, a solid donought or the Möbius strip, which just fail to be a manifold because they have a "boundary". One can remedy this situation by trying to enlarge the notion of manifold so that it includes this possibility. The clue to be able to include boundary points is to understand what is the local model around points in the "boundary" and this turns out to be the **closed half-space** $\mathbb{H}^d$:

$$\mathbb{H}^d := \{(x^1, \ldots, x^d) \in \mathbb{R}^d : x^d \geq 0\}.$$

We will denote the **open half-space** by:

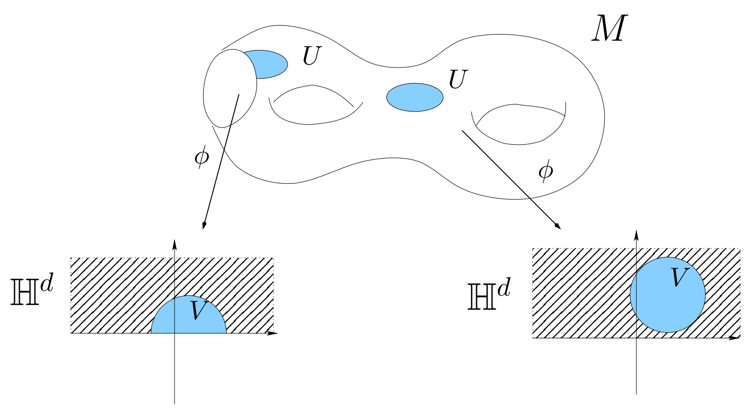
$$\operatorname{Int} \mathbb{H}^d =: \{(x^1, \ldots, x^d) \in \mathbb{R}^d : x^d > 0\}.$$

and the **boundary of the closed half-space** by:

$$\partial \mathbb{H}^d =: \{(x^1, \ldots, x^d) \in \mathbb{R}^d : x^d = 0\}.$$

When $n = 0$, we have $\mathbb{H}^0 = \mathbb{R}^0 = \{0\}$, so $\operatorname{Int} H^0 = \mathbb{R}^0$ and $\partial \mathbb{H}^0 = \emptyset$.

Definition 2.1. *A* ***topological manifold with boundary of dimension*** *d is a topological space M such that every $p \in M$ has a neighborhood U which is homeomorphic to some open set $V \subset \mathbb{H}^d$.*



Just as we do for manifolds without boundary, we shall assume that all manifolds with boundary are Hausdorff and have a countable basis of open sets.

We shall use the same notations as before, so we call a homeomorphism $\phi : U \to V$ as in the definition a system of coordinates or a coordinate chart. Note that there are two types of open sets in $\mathbb{H}^d$ according to whether they intersect $\partial\,\mathbb{H}^d$ or not. These give rise to two types of coordinate systems $\phi : U \to V$, according to whether V intersects $\partial\,\mathbb{H}^d$ or not. In the first case, when $V \cap \partial\,\mathbb{H}^d = \emptyset$, we just have a coordinate system of the same sort as for manifolds without boundary, and we call it an **interior chart**. In the second case, when $V \cap \partial\,\mathbb{H}^d \neq \emptyset$, we call it a **boundary chart**.

Using Invariance of Domain (Theorem 0.5), one shows that:

Lemma 2.2. *Let M be a topological manifold with boundary of dimension d. If for some chart (U, ϕ) we have $\phi(p) \in \partial\,\mathbb{H}^d$, then this is also true for every other chart.*

Proof. Exercise. □

This justifies the following definition:

Definition 2.3. *Let M be a topological manifold with boundary of dimension d. A point $p \in M$ is called a* ***boundary point*** *if there exists some chart (U, ϕ) with $p \in U$, such that $\phi(p) \in \partial\,\mathbb{H}^d$. Otherwise, p is called an* ***interior point****.*

The set of boundary points of M will be denoted by $\partial\, M$ and is called the **boundary of M** and the set of interior points of M will be denoted by $\operatorname{Int} M$ and is called the **interior of M**. If on both sets we consider the topology induced from M, we have:

Proposition 2.4. *Let M be a topological manifold with boundary of dimension $d > 0$. Then $\operatorname{Int} M$ and ∂M are topological manifolds without boundary of dimension d and $d-1$, respectively. If N is another manifold with boundary and $\Psi : M \to N$ is a homeomorphism then Ψ restricts to homeomorphisms $\Psi|_{\partial M} : \partial M \to \partial N$ and $\Psi|_{\operatorname{Int} M} : \operatorname{Int} M \to \operatorname{Int} N$.*

Proof. Let $p \in \operatorname{Int} M$ and let $\phi : U \to V$ be a chart with $p \in U$ and $V \subset \mathbb{H}$. Then if we set $V_0 := V - \partial \mathbb{H}$ and $U_0 := \phi^{-1}(V_0)$, we have that U_0 is an open neighborhood of M, V_0 is open in $\mathbb{R}^d$, and $\phi|_{U_0} : U_0 \to V_0$ is a homeomorphism. This shows that $\operatorname{Int} M$ is a topological manifold without boundary of dimension d.

On the other hand, let $p \in \partial M$ and let $\phi : U \to V$ be a chart with $p \in U$ and $\phi(p) \in \partial \mathbb{H}$. Then if we set $V_0 := V \cap \partial \mathbb{H}$ and $U_0 := \phi^{-1}(V_0)$, we have that $U_0 = U \cap \partial M$ is an open neighborhood of ∂M, V_0 is open in $\partial \mathbb{H} \simeq \mathbb{R}^{d-1}$, and $\phi|_{U_0} : U_0 \to V_0$ is a homeomorphism. This shows that ∂M is a topological manifold without boundary of dimension $d-1$. $\square$

It is important not to confuse the notions of interior and boundary point for manifolds with boundary with the usual notions of interior and boundary point of a subset of a topological space. If M happens to be a manifold with boundary embedded in some $\mathbb{R}^n$ then the two notions may or may not coincide, as shown by the following examples.

EXAMPLES 2.5.

1. *$M = \mathbb{H}^d$ is itself a topological manifold with boundary of dimension d, where $\operatorname{Int} M = \operatorname{Int} \mathbb{H}^d$ and $\partial M = \partial \mathbb{H}^d$, so our notations are consistent. If we think of $\mathbb{H}^d \subset \mathbb{R}^d$, then these notions coincide with the usual notions of boundary and interior of $\mathbb{H}^d$ as a topological subspace of $\mathbb{R}^d$.*
2. *The closed unit disk:*
$$D^k = \overline{B^d} := \{x \in \mathbb{R}^d : ||x|| \leq 1\},$$
is a topological manifold with boundary of dimension d with interior the open unit ball B^d and boundary the unit sphere $\mathbb{S}^{d-1}$. If we think of $D^d \subset \mathbb{R}^d$, then these notions coincide with the usual notions of boundary and interior of D^d as a topological subspace of $\mathbb{R}^d$.
3. *The cube I^d is a topological manifold with boundary of dimension d. I^d and D^d are homeomorphic topological manifolds with boundary.*
4. *The Möbius strip $M \subset \mathbb{R}^3$ is a topological manifold with boundary $\partial M = \mathbb{S}^1$. Note that, as a topological subspace of $\mathbb{R}^3$, all points of M are boundary points!*

Now that we have the notion of chart for a topological manifold with boundary, we can define a **smooth structure** on a topological d-manifold with boundary M by exactly the same procedure as we did for manifolds without boundary: it is a collection of charts $\mathcal{C} = \{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ which satisfies the following properties:

(i) The collection $\mathcal{C}$ is an open cover of M: $\bigcup_{\alpha\in A} U_\alpha = M$;
(ii) For all $\alpha, \beta \in A$, the **transition function** $\phi_\alpha \circ \phi_\beta^{-1}$ is a smooth map;
(iii) The collection $\mathcal{C}$ is maximal: if (U, ϕ) any coordinate system such that $\phi \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ \phi^{-1}$ are smooth maps for all $\alpha \in A$, then $(U, \phi) \in \mathcal{C}$.

The pair $(M, \mathcal{C})$ is called a **smooth d-manifold with boundary**.

Again, given an atlas $\mathcal{C}_0 = \{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ (i.e., a collection satisfying (i) and (ii)), there exists a unique maximal atlas $\mathcal{C}$ which contains $\mathcal{C}_0$: it is enough to define $\mathcal{C}$ to be the collection of all **smooth coordinate systems** relative to $\mathcal{C}$, i.e., all coordinate systems (U, ϕ) such that $\phi\circ\phi_\alpha^{-1}$ and $\phi_\alpha\circ\phi^{-1}$ are both smooth, for all $(U_\alpha, \phi_\alpha) \in \mathcal{C}_0$.

The notion of smooth map $\Psi : M \to N$ between two manifolds with boundary is also defined in exactly the same way as in the case of manifolds without boundary.

Proposition 2.6. *Let M be a smooth manifold with boundary of dimension $d > 0$. Then $\operatorname{Int} M$ and ∂M are smooth manifolds without boundary of dimension d and $d-1$, respectively. If N is another smooth manifold with boundary and $\Psi : M \to N$ is a diffeomorphism then Ψ restricts to diffeomorphisms $\Psi|_{\partial M} : \partial M \to \partial N$ and $\Psi|_{\operatorname{Int} M} : \operatorname{Int} M \to \operatorname{Int} N$.*

Proof. Exercise. □

Although often one can work with manifolds with boundary much the same way as one can work with manifolds without boundary, some care must be taken. For example, the Cartesian product of two half-spaces is not a manifold with boundary (it is rather a *manifold with corners*, a notion we will not discuss). So the cartesian product of manifolds with boundary may not be a manifold with boundary. However, we do have the following result:

Proposition 2.7. *If M is a smooth manifold without boundary and N is a smooth manifold with boundary, then $M \times N$ is a smooth manifold with boundary with $\partial(M \times N) = M \times \partial N$ and $\operatorname{Int}(M \times N) = M \times \operatorname{Int} N$.*

Proof. Exercise. □

Example 2.8.
If M is a manifold without boundary and $I = [0, 1]$ then $M \times I$ is a manifold with boundary for which:

$$\operatorname{Int}(M \times I) = M\times]0,1[, \quad \partial(M \times I) = M \times \{0\} \cup M \times \{1\}.$$

It is very cumbersome to write always "manifold without boundary", so we agree to refer to manifolds without boundary simply as "manifolds", and add the qualitative "with boundary", whenever that is the case. You should be aware that in the literature it is common to use **non-bounded manifold** for a manifold in our sense, and to call a **closed manifold** a compact non-bounded manifold and **open manifold** a non-bounded manifold with no compact component.

Homework.

1. Using Invariance of Domain, show that if for some chart (U, ϕ) of a topological manifold with boundary one has $\phi(p) \in \partial\,\mathbb{H}^d$, then this is also true for every other chart.

2. Let $M \subset \mathbb{R}^d$ have the induced topology. Show that if M is a closed subset and a d-dimensional manifold with boundary then the topological boundary of M coincides with $\partial\, M$. Give a counterexample to this statement when M is not a closed subset.

3. Give the details of the proof of Proposition 2.6.

4. Show that if M is a smooth manifold without boundary and N is a smooth manifold with boundary, then $M \times N$ is a smooth manifold with boundary with $\partial\,(M \times N) = M \times \partial\, N$ and $\operatorname{Int}(M \times N) = M \times \operatorname{Int} N$.

5. A solid torus is the 3-manifold with boundary $D^2 \times \mathbb{S}^1$. What is the boundary of the solid torus? How does this generalize to dimensions larger than 3?

Lecture 3. Partitions of Unity

In this lecture we will study an important gluing technique for smooth manifolds. If M is a smooth manifold and $f \in C^\infty(M)$, we define the **support of** f to be the closed set:

$$\operatorname{supp} f \equiv \overline{\{p \in M : f(p) \neq 0\}}.$$

Also, given a collection $\mathcal{C} = \{U_\alpha : \alpha \in A\}$ of subsets of M we say that

- $\mathcal{C}$ is **locally finite** if, for all $p \in M$, there exists a neighborhood $p \in O \subset M$ such that $O \cap U_\alpha \neq \emptyset$ for only a finite number of $\alpha \in A$.
- $\mathcal{C}$ is a **cover** of M if $\bigcup_{\alpha \in A} U_\alpha = M$.
- $\mathcal{C}_0 = \{U_\beta : \beta \in B$ is **subcover** if $\mathcal{C}_0 \subset \mathcal{C}$ and $\mathcal{C}_0$ still covers M.
- $\mathcal{C}' = \{V_i : i \in I\}$ is a **refinement of a cover** $\mathcal{C}$ if it is itself a cover and for each $i \in I$ there exists $\alpha = \alpha(i) \in A$ such that $V_i \subset U_\alpha$.

Definition 3.1. *A **partition of unity** in a smooth manifold M is a collection $\{\phi_i : i \in I\} \subset C^\infty(M)$ such that:*

(i) the collection of supports $\{\operatorname{supp} \phi_i : i \in I\}$ is locally finite;
(ii) $\phi_i(p) \geq 0$ and $\sum_{i \in I} \phi_i(p) = 1$ for every $p \in M$.

A partition of unity $\{\phi_i : i \in I\}$ is called subordinated *to a cover $\{U_\alpha : \alpha \in A\}$ of M if for each $i \in I$ there exists $\alpha \in A$ such that $\operatorname{supp} \phi_i \subset U_\alpha$.*

Notice that the sum in (ii) is actually finite: by (i), for each $p \in M$ there is only a finite number of functions ϕ_i with $\phi_i(p) \neq 0$.

The existence of partitions of unity is not obvious, but we will see in this lecture that there are many partitions of unity on a manifold.

Theorem 3.2 (Existence of Partitions of Unity)**.** *Let M be a smooth manifold and let $\{U_\alpha : \alpha \in A\}$ be an open cover of M. Then there exists a countable partition of unity $\{\phi_i : i = 1, 2, \dots\}$, subordinated to the cover $\{U_\alpha : \alpha \in A\}$ and with* $\operatorname{supp} \phi_i$ *compact for all i.*

If we do not care about compact supports, for any open cover we can get partitions of unity with the same set of indices:

Corollary 3.3. *Let M be a smooth manifold and let $\{U_\alpha : \alpha \in A\}$ be an open cover of M. Then there exists a partition of unity $\{\phi_\alpha : \alpha \in A\}$ such that* $\operatorname{supp} \phi_\alpha \subset U_\alpha$ *for each $\alpha \in A$.*

Proof. By Theorem 3.2 there exists a countable partition of unity

$$\{\psi_i : i = 1, 2, \dots\}$$

subordinated to the cover $\{U_\alpha : \alpha \in A\}$. For each i we can choose a $\alpha = \alpha(i)$ such that $\operatorname{supp} \psi_i \subset U_{\alpha(i)}$. Then the functions

$$\phi_\alpha = \begin{cases} \sum_{\alpha(i)=\alpha} \psi_i, & \text{if } \{i : \alpha(i) = \alpha\} \neq \emptyset, \\ \\ 0 & \text{otherwise,} \end{cases}$$

form a partition of unity with $\operatorname{supp} \phi_\alpha \subset U_\alpha$, for all $\alpha \in A$. $\square$

Example 3.4.
For the sphere $\mathbb{S}^d$, consider the cover with the two opens sets $U_N := \mathbb{S}^d - N$ and $U_S := \mathbb{S}^d - S$. Then the corollary says that there exists a partition of unit subordinated to this cover with the same indices, i.e., a pair of non-negative smooth functions $\phi_N, \phi_S \in C^\infty(\mathbb{S}^d)$ with $\operatorname{supp} \phi_N \subset U_N$ *and* $\operatorname{supp} \phi_S \subset U_S$, *such that $\phi_N(p) + \phi_S(p) = 1$, for all $p \in \mathbb{S}^d$.*

Corollary 3.5. *Let $A \subset O \subset M$, where O is an open subset and A is a closed subset of a smooth manifold M. There exists a smooth function $\phi \in C^\infty(M)$ such that:*

(i) $0 \leq \phi(p) \leq 1$ *for each $p \in M$;*
(ii) $\phi(p) = 1$ *if $p \in A$;*
(iii) $\operatorname{supp} \phi \subset O$.

Proof. The open sets $\{O, M - A\}$ give an open cover of M. Therefore, by the previous corollary, there is a partition of unity $\{\phi, \psi\}$ with $\sup \phi \subset O$ and $\sup \psi \subset M - A$. The function ϕ satisfies (i)-(iii). $\square$

Roughly speaking, partitions of unity are used to "glue" local properties (i.e., properties that hold on domains of local coordinates), giving rise to global properties of a manifold, as shown in the proof of the following result.

Corollary 3.6 (Extension Lemma for smooth maps)**.** *Let M be a smooth manifold, $A \subset M$ a closed subset and $\Psi : A \to \mathbb{R}^n$ a smooth map. For any open set $A \subset U \subset M$ there exists a smooth map $\widetilde{\Psi} : M \to \mathbb{R}^n$ such that $\widetilde{\Psi}|_A = \Psi$ and* $\operatorname{supp} \widetilde{\Psi} \subset U$.

Proof. For each $p \in A$ we can find an open neighborhood $U_p \subset M$, such that we can extend $\Psi|_{U_p \cap A}$ to a smooth function $\widetilde{\Psi}_p : U_p \to \mathbb{R}^n$. By replacing U_p by $U_p \cap U$ we can assume that $U_p \subset U$. The sets $\{U_p, M - A; p \in A\}$ form an open cover of M so we can find a partition of unit $\{\phi_p : p \in A\} \cup \{\phi_0\}$, subordinated to this cover with $\operatorname{supp} \phi_p \subset U_p$. Now define $\widetilde{\Psi} : M \to \mathbb{R}^n$ by setting

$$\widetilde{\Psi} := \sum_{p \in A} \phi_p \widetilde{\Psi}_p.$$

Clearly $\widetilde{\Psi}$ has the required properties. □

We now turn to the proof of Theorem 3.2. There are two main ingredients in the proof. The first one is that topological manifolds are **paracompact**, i.e., every open cover has an open locally finite refinement. This is in fact a consequence of our assumption that manifolds are Hausdorff and second countable, and we will use the following more precise versions:

(a) Every open cover of a topological manifold M has a countable subcover.
(b) Every open cover of a topological manifold M has a countable, locally finite refinement consisting of open sets with compact closures.

The proofs are left to the exercises. The second ingredient is the existence of "very flexible" smooth functions, some times called **bump functions**:

- The function $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} \exp(-\frac{1}{x^2}), & x \neq 0, \\ \\ 0, & x = 0. \end{cases}$$

  is a smooth function.
- If $\delta > 0$, the function $g : \mathbb{R} \to \mathbb{R}$ defined by:

$$g(x) = f(x) f(\delta - x),$$

  is smooth, $g(x) > 0$ if $x \in]0, \delta[$ and $g(x) = 0$ otherwise.
- The function $h : \mathbb{R} \to \mathbb{R}$ defined by:

$$h(x) := \frac{\int_0^x g(t)\, dt}{\int_0^\delta g(t)\, dt},$$

  is smooth, non-decreasing, $h(x) = 0$ if $x \leq 0$ and $h(x) = 1$ if $x \geq \delta$.

Using these functions you should now be able to show that:

(c) there exists a function $\phi \in C^\infty(\mathbb{R}^d)$ such that $\phi(x) = 1$, if $x \in \overline{B_1(0)}$, and $\phi(x) = 0$, if $x \in B_2(0)^c$.

Proof of Theorem 3.2. By (b) above, we can assume that the open cover $\{U_\alpha : \alpha \in A\}$ is countable, locally finite, and the sets $\overline{U}_\alpha$ are compacts. If $p \in U_\alpha$, we can choose a coordinate system (V_p, τ), centered in p, with

$V_p \subset U_\alpha$, and such that $\overline{B_2(0)} \subset \tau(V_p)$. Now if ϕ the function defined in (c) above, we set:

$$\psi_p := \begin{cases} \phi \circ \tau, & \text{em } V_p, \\ \\ 0, & \text{em } M - V_p. \end{cases}$$

The function $\psi_p \in C^\infty(M)$ is non-negative and takes the value 1 in an open set $W_p \subset V_p$ which contains p. Since $\{W_p : p \in M\}$ is an open cover of M, by (a) above, there exists a countable subcover $\{W_{p_1}, W_{p_2}, \dots\}$ of M. Then the open cover $\{V_{p_1}, V_{p_2}, \dots\}$ is locally finite and subordinated to the cover $\{U_\alpha : \alpha \in A\}$. Moreover, the closures $\overline{V}_{p_i}$ are compact.

The sum $\sum_i \psi_{p_i}$ may not be equal to 1. To fix this we observe that

$$\psi = \sum_{i=1}^{+\infty} \psi_{p_i},$$

is well defined, of class C^∞ and $\phi(p) > 0$ for every $p \in M$. If we define:

$$\phi_i = \frac{\psi_{p_i}}{\psi},$$

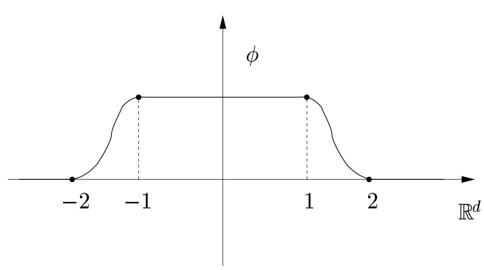
then the functions $\{\phi_1, \phi_2, \dots\}$ give a partition of unity, subordinated to the cover $\{U_\alpha : \alpha \in A\}$, with $\operatorname{supp} \phi_i$ compact for each $i = 1, 2, \dots$.

This completes the proof of Theorem 3.2. $\square$

Homework.

1. Show that $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \exp(-1/x^2)$ is a smooth function.

2. Show that there exists a function $\phi \in C^\infty(\mathbb{R}^d)$ such that $0 \le \phi(x) \le 1$, $\phi(x) = 1$ if $|x| \le 1$ and $\phi(x) = 0$ if $|x| > 2$.



3. Show that for a second countable topological space X, every open cover of X has a countable subcover.

Hint: If $\{U_\alpha : \alpha \in A\}$ is an open cover of X and $\mathcal{B} = \{V_j \in J\}$ is a countable basis of the topology of X, show that the collection $\mathcal{B}'$ formed by $V_j \in \mathcal{B}$ such that $V_j \subset U_\alpha$ for some α, is also a basis. Now, for each $V_j \in \mathcal{B}'$ choose some U_{α_j} containing V_j, and show that $\{U_{\alpha_j}\}$ is a countable subcover.

4. Show that a topological manifold is paracompact, in fact, show that every open cover of a topological manifold M has a countable, locally finite refinement consisting of open sets with compact closures.

Hint: Show first that M can be covered by open sets $O_1, O_2, \ldots,$ with compact closures and $\overline{O}_i \subset O_{i+1}$. Then given an arbitrary open cover $\{U_\alpha : \alpha \in A\}$ of M, choose for each $i \geq 3$ a finite subcover of the cover $\{U_\alpha \cap (O_{i+1} - \overline{O}_{i-2} : \alpha \in A\}$ of the compact set $\overline{O}_i - O_{i-1}$, and a finite subcover of the cover $\{U_\alpha \cap O_3 : \alpha \in A\}$ of the compact set $\overline{O}_2$. The collection of such open sets will do it.

5. Show that if $M \subset \mathbb{R}^n$ is an embedded manifold then a function $f : M \to \mathbb{R}$ is smooth if and only if there exists an open set $M \subset U \subset \mathbb{R}^n$ and a function $F : U \to \mathbb{R}$ such that $F|_M = f$.

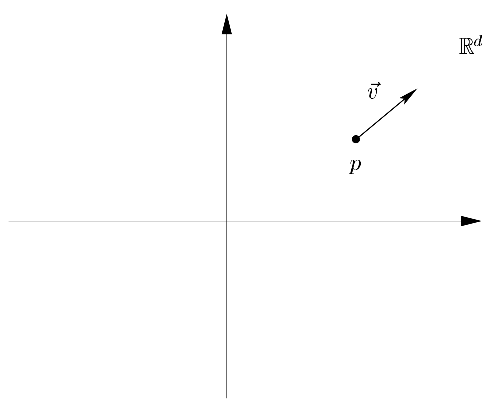
6. Show that the conclusion of the Extension Lemma for Smooth Maps may fail if $A \subset M$ is not assumed to be closed.

7. Show that Theorem 3.2 still holds for manifolds with boundary.

Lecture 4. Tangent Space and the Differential

The tangent space to $\mathbb{R}^d$ at $p \in \mathbb{R}^d$ is by definition the set:

$$T_p\mathbb{R}^d := \left\{(p, \vec{v}) : \vec{v} \in \mathbb{R}^d\right\}.$$



Note that this tangent space is a vector space over $\mathbb{R}$ where addition is defined by:

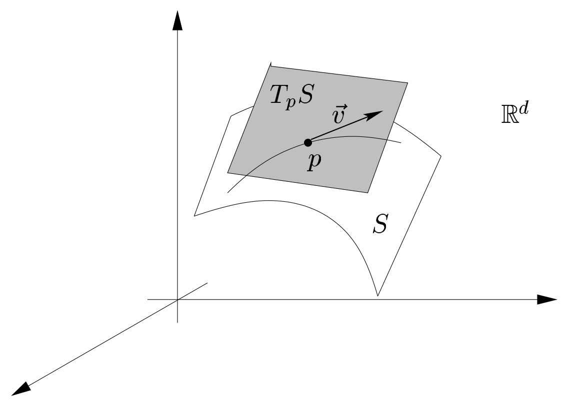
$$(p, \vec{v}_1) + (p, \vec{v}_2) \equiv (p, \vec{v}_1 + \vec{v}_2),$$

while multiplication is given by:

$$\lambda(p, \vec{v}) \equiv (p, \lambda\vec{v}).$$

Of course there is a natural isomorphism $T_p\mathbb{R}^d \simeq \mathbb{R}^d$, but in many situations it is better to think of $T_p\mathbb{R}^d$ as the set of vectors with origin at p.

This distinction is even more clear in the case of embedded manifolds, or d-surfaces, $S \subset \mathbb{R}^n$. In this case, we can define the tangent space to S at $p \in S$ to be the subspace $T_pS \subset T_p\mathbb{R}^n$ consisting of those tangent vectors $(p, \vec{v})$, for which there exists a smooth curve $c : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$, with $c(t) \in S$, $c(0) = p$ and $c'(0) = \vec{v}$.



A tangent vector $(p, \vec{v}) \in T_pS$ acts on smooth functions defined in a neighborhood of p: if $f : U \to \mathbb{R}$ is a smooth function defined on a open set U containing p then we can choose a smooth curve $c : (-\varepsilon, \varepsilon) \to U$, with $c(0) = p$ and $c'(0) = \vec{v}$, and set:

$$(p, \vec{v})(f) := \frac{d}{dt} f \circ c(0).$$

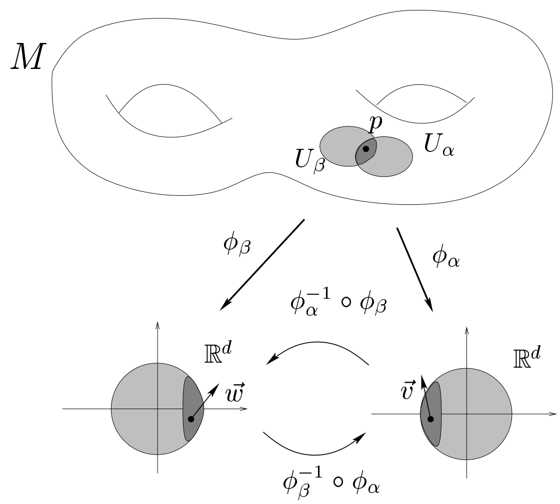
This operation does not depend on the choice of smooth curve c (exercise). In fact, this is just the usual notion of **directional derivative** of f at p in the direction $\vec{v}$.

We will now define the tangent space to an abstract manifold M at $p \in M$. There are several different approaches to define the tangent space at $p \in M$, which correspond to different points of view, all of them very useful. We shall give here three distinct descriptions and we leave it to the exercises to show that they are actually equivalent.

Description 1. Let M be a smooth d-dimensional manifold with an atlas $\mathcal{C} = \{(U_\alpha, \phi_\alpha) : \alpha \in A\}$. To each point $p \in M$ we would like to associate a copy of $\mathbb{R}^d$, so that each element $\vec{v} \in \mathbb{R}^d$ should represent a tangent vector. Of course if $p \in U_\alpha$, the system of coordinates ϕ_α gives an identification of an open neighborhood of p with $\mathbb{R}^d$. Distinct smooth charts will give different identifications, but they are all related by the transition functions.

This suggests one should consider triples $(p, \alpha, \vec{v}) \in M \times A \times \mathbb{R}^d$, with $p \in U_\alpha$, and that two such triples should be declared to be equivalent if

$$[p, \alpha, \vec{v}] = [q, \beta, \vec{w}] \qquad \text{iff} \qquad p = q \ \text{ and } \ (\phi_\alpha \circ \phi_\beta^{-1})'(\phi_\beta(p)) \cdot \vec{w} = \vec{v}.$$



Hence, we define a **tangent vector** to M at a point $p \in M$ to be an equivalence class $[p, \alpha, \vec{v}]$, and the **tangent space** at p to be the set of all such equivalence classes:

$$T_pM \equiv \left\{ [p, \alpha, \vec{v}] : \alpha \in A, \vec{v} \in \mathbb{R}^d \right\}.$$

We leave it as an exercise to check that the operations:

$$[p, \alpha, \vec{v}_1] + [p, \alpha, \vec{v}_2] := [p, \alpha, \vec{v}_1 + \vec{v}_2], \quad \lambda[p, \alpha, \vec{v}] := [p, \alpha, \lambda\vec{v}],$$

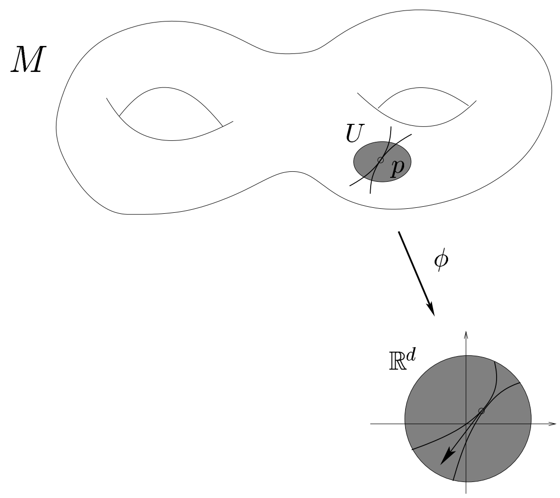
are well defined and give T_pM the structure of vector space over $\mathbb{R}$. Notice that we still have an isomorphism $T_pM \simeq \mathbb{R}^d$, but this isomorphism now depends on the choice of a chart.

Description 2. Again, fix $p \in M$. For this second description we will consider all smooth curves $c : (-\varepsilon, \varepsilon) \to M$, with $c(0) = p$. Two such smooth curves c_1 and c_2 will be declared equivalent if there exists some smooth chart (U, ϕ) with $p \in U$, such that

$$\frac{d}{dt}(\phi \circ c_1)(0) = \frac{d}{dt}(\phi \circ c_2)(0).$$

It should be clear that if this condition holds for some smooth chart around p, then it also holds for every other smooth chart around p belonging to the smooth structure.

We call a **tangent vector** at $p \in M$ an equivalence class of smooth curves $[c]$, and the set of all such classes is called the **tangent space** T_pM at the point p. Again, you should check that this tangent space has the structure of vector space over $\mathbb{R}$ and that T_pM is isomorphic to $\mathbb{R}^d$, through an isomorphism that depends on a choice of smooth chart.



Description 3. The two previous descriptions use smooth charts. Our third description has the advantage of not using charts, and it will be our final description of tangent vectors and tangent space.

Again we fix $p \in M$ and we look at the set of all smooth functions defined in some open neighborhood of p. Given two smooth functions $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$, where U and V are opens that contain p, we say that f and g define the same **germ at** p if there is an open set $W \subset U \cap V$ containing p and such that

$$f|_W = g|_W.$$

We denote by $\mathcal{G}_p$ the set of all germs at p. This set has the structure of an $\mathbb{R}$-algebra, where addition, product and multiplication by scalars are defined in the obvious way:

$$\begin{aligned} [f] + [g] &\equiv [f + g], \\ [f][g] &\equiv [fg], \\ \lambda[f] &\equiv [\lambda f]. \end{aligned}$$

Notice also that it makes sense to talk of the value of a germ $[f] \in \mathcal{G}_p$ at the point p, which is $f(p)$. On the other hand, the value of $[f] \in \mathcal{G}_p$ at any other point $q \neq p$ is not defined.

Definition 4.1. *A* ***tangent vector*** *at a point $p \in M$ is a linear derivation of $\mathcal{G}_p$, i.e., a map $\mathbf{v} : \mathcal{G}_p \to \mathbb{R}$ satisfying:*

(i) $\mathbf{v}([f] + \lambda[g]) = \mathbf{v}([f]) + \lambda\mathbf{v}([g])$*;*

(ii) $\mathbf{v}([f][g]) = \mathbf{v}([f])g(p) + f(p)\mathbf{v}([g])$*;*

The ***tangent space*** *at a point $p \in M$ is the set of all such tangent vectors and is denoted by T_pM.*

Since linear derivations can be added and multiplied by real numbers, it is clear that the tangent space T_pM has the structure of a real vector space.

EXAMPLE 4.2.
Let $(U,\phi) = (U,x^1,\ldots,x^d)$ be a coordinate system in M with $p \in U$. We define the tangent vectors $\frac{\partial}{\partial x^i}\big|_p \in T_pM$, $i = 1,\ldots,d$, to be the derivations

$$\frac{\partial}{\partial x^i}\bigg|_p ([f]) = \frac{\partial (f\circ\phi^{-1})}{\partial x^i}\bigg|_{\phi(p)} .$$

Notice that the tangent vector $\frac{\partial}{\partial x^i}\big|_p$ corresponds to the direction one obtains by freezing all coordinates but the i-th coordinate.

In order to check that T_pM is a vector space with dimension equal to $\dim M$, consider the set of all germs that vanish at p:

$$\mathcal{M}_p = \{[f] \in \mathcal{G}_p : f(p) = 0\}\,,$$

It is immediate to check that $\mathcal{M}_p \subset \mathcal{G}_p$ is a maximal ideal in $\mathcal{G}_p$. The k-th power of this ideal

$$\mathcal{M}_p^k = \underbrace{\mathcal{M}_p \cdots \mathcal{M}_p}_{k},$$

consists of germs that vanish to order k at p: if $[f] \in \mathcal{M}_p^k$ and (U,ϕ) is a coordinate system centered at p, then the smooth function $f\circ\phi^{-1}$ has vanishing partial derivatives at $\phi(p)$ up to order $k-1$. These powers form a tower of ideals

$$\mathcal{G}_p \supset \mathcal{M}_p \supset \mathcal{M}_p^2 \supset \cdots \supset \mathcal{M}_p^k \supset \ldots$$

Theorem 4.3. *The tangent space T_pM is naturally isomorphic to $(\mathcal{M}_p/\mathcal{M}_p^2)^*$ and has dimension* $\dim M$*.*

Proof. First we check that if $[c] \in \mathcal{G}_p$ is the germ of the constant function $f(x) = c$ then $\mathbf{v}([c]) = 0$, for any tangent vector $\mathbf{v} \in T_pM$. In fact, we have that

$$\mathbf{v}([c]) = c\mathbf{v}([1]),$$

and that

$$\mathbf{v}([1]) = \mathbf{v}([1][1]) = 1\mathbf{v}([1]) + 1\mathbf{v}([1]) = 2\mathbf{v}([1]),$$

hence $\mathbf{v}([1]) = 0$.

Now if $[f] \in \mathcal{G}_p$ and $c = f(p)$, we remark that

$$\mathbf{v}([f]) = \mathbf{v}([f] - [c]),$$

so the derivation $\mathbf{v}$ is completely determined by its effect on $\mathcal{M}_p$. On the other hand, any derivation vanishes on $\mathcal{M}_p^2$, because if $f(p) = g(p) = 0$, then

$$\mathbf{v}([f][g]) = \mathbf{v}([f])g(p) + f(p)\mathbf{v}([g]) = 0.$$

We conclude that every tangent vector $\mathbf{v} \in T_pM$ determines a unique linear transformation $\mathcal{M}_p \to \mathbb{R}$, which vanishes on $\mathcal{M}_p^2$. Conversely, if

$L \in (\mathcal{M}_p/\mathcal{M}_p^2)^*$ is a linear transformation, we can define a linear transformation $\mathbf{v} : \mathcal{G}_p \to \mathbb{R}$ by setting

$$\mathbf{v}([f]) \equiv L([f] - [f(p)]).$$

This is actually a derivation (exercise), so we conclude that $T_pM \simeq (\mathcal{M}_p/\mathcal{M}_p^2)^*$.

In order to verify the dimension of T_pM, we choose some system of coordinates $(U, x^1, \dots, x^d)$ centered at p, and we show that the tangent vector

$$\left.\frac{\partial}{\partial x^i}\right|_p \in T_pM, \qquad i = 1, \dots, d,$$

form a basis for T_pM. If $f : U \to \mathbb{R}$ is any smooth function, then $f \circ \phi^{-1} : \mathbb{R}^d \to \mathbb{R}$ is smooth in a neighborhood of the origin. This function can be expanded as:

$$f \circ \phi^{-1}(x) = f \circ \phi^{-1}(0) + \sum_{i=1}^{d} \frac{\partial (f \circ \phi^{-1})}{\partial x^i}(0) x^i + \sum_{i,j} g_{ij}(x) x^i x^j,$$

where the g_{ij} are some smooth functions in a neighborhood of the origin. It follows that we have the expansion:

$$f(q) = f(p) + \sum_{i=1}^{d} \left.\frac{\partial (f \circ \phi^{-1})}{\partial x^i}\right|_{\phi(p)} x^i(q) + \sum_{i,j} h_{ij}(q) x^i(q) x^j(q),$$

where $h_{ij} \in C^\infty(U)$, valid for any $q \in U$. We conclude that for any tangent vector $\mathbf{v} \in T_pM$:

$$\mathbf{v}([f]) = \sum_{i=1}^{d} \left.\frac{\partial (f \circ \phi^{-1})}{\partial x^i}\right|_{\phi(p)} \mathbf{v}([x^i]).$$

In other words, we have:

$$\mathbf{v} = \sum_{i=1}^{d} a^i \left.\frac{\partial}{\partial x^i}\right|_p,$$

where $a^i = \mathbf{v}([x^i])$. This shows that the vectors $(\partial/\partial x^i)|_p \in T_pM$, $i = 1, \dots, \dim M$ form a generating set. We leave it as an exercise to show that they are linearly independent. $\square$

From now on, given $\mathbf{v} \in T_pM$ and a smooth function f defined in some neighborhood of $p \in M$ we set:

$$\mathbf{v}(f) \equiv \mathbf{v}([f]).$$

Note that $\mathbf{v}(f) = \mathbf{v}(g)$ if f and g coincide in some neighborhood of p and that:

$$\mathbf{v}(f + \lambda g) = \mathbf{v}(f) + \lambda \mathbf{v}(g), \quad (\lambda \in \mathbb{R}),$$
$$\mathbf{v}(fg) = f(p)\mathbf{v}(g) + \mathbf{v}(f)g(p),$$

where $f + \lambda g$ and fg are defined in the intersection of the domains of f and g.

The proof of Theorem 4.3 shows that if $(U, \phi) = (U, x^1, \dots, x^d)$ is a coordinate system around p, then any tangent vector $\mathbf{v} \in T_pM$ can be written as:

$$\mathbf{v} = \sum_{i=1}^{d} a^i \left.\frac{\partial}{\partial x^i}\right|_p .$$

The numbers $a^i = \mathbf{v}(x^i)$ are called the **components of tangent vector v** in the coordinate system $(U, x^1, \dots, x^d)$. If we introduce the notation

$$\left.\frac{\partial f}{\partial x^i}\right|_p \equiv \left.\frac{\partial f \circ \phi^{-1}}{\partial x^i}\right|_{\phi(p)} ,$$

then we see that:

$$\mathbf{v}(f) = \sum_{i=1}^{d} a^i \left.\frac{\partial f}{\partial x^i}\right|_p .$$

On the other hand, given another coordinate system $(V, y^1, \dots, y^d)$ we find that

$$\left.\frac{\partial}{\partial y^j}\right|_p = \sum_{i=1}^{d} \left.\frac{\partial x^i}{\partial y^j}\right|_p \left.\frac{\partial}{\partial x^i}\right|_p .$$

Hence, in this new coordinate system we have

$$\mathbf{v} = \sum_{j=1}^{d} b^j \left.\frac{\partial}{\partial y^j}\right|_p , \quad \text{with } b^j = \mathbf{v}(y^j),$$

where the new components b^j are related to the old components a^i by the transformation formula:

$$a^i = \sum_{j=1}^{d} \left.\frac{\partial x^i}{\partial y^j}\right|_p b^j. \tag{4.1}$$

A smooth map between two smooth manifolds determines a linear transformation between the corresponding tangent spaces:

Definition 4.4. *Let $\Psi : M \to N$ be a smooth map. The **differential** of Ψ at $p \in M$ is the linear transformation $\mathrm{d}_p\Psi : T_pM \to T_{\Psi(p)}N$ defined by*

$$\mathrm{d}_p\Psi(\mathbf{v})(f) \equiv \mathbf{v}(f \circ \Psi),$$

where f is any smooth function defined in a neighborhood of $\Psi(p)$.

If $(U, \phi) = (U, x^1, \dots, x^d)$ is a coordinate system around p and $(V, \psi) = (V, y^1, \dots, y^e)$ is a coordinate system around $\Psi(p)$, we obtain

$$\mathrm{d}_p\Psi \cdot \left.\frac{\partial}{\partial x^i}\right|_p = \sum_{j=1}^{e} \left.\frac{\partial(\psi \circ \Psi \circ \phi^{-1})^j}{\partial x^i}\right|_{\phi(p)} \left.\frac{\partial}{\partial y^j}\right|_{\Psi(p)} .$$

The matrix formed by the partial derivatives $\frac{\partial(\psi\circ\Psi\circ\phi^{-1})^j}{\partial x^i}$ is often abbreviated to $\frac{\partial(y^j\circ\Psi)}{\partial x^i}$ and is called the **Jacobian matrix** of the smooth map Ψ relative to the specified system of coordinates.

The following result is an immediate consequence of the definitions and the usual chain rule for smooth maps between euclidean space:

Proposition 4.5 (Chain Rule)**.** *Let $\Psi : M \to N$ and $\Phi : N \to P$ be smooth maps. Then the composition $\Phi \circ \Psi$ is smooth and we have that:*

$$\mathrm{d}_p(\Phi\circ\Psi) = \mathrm{d}_{\Psi(p)}\Phi \circ \mathrm{d}_p\Psi.$$

Similarly, it is easy to prove the following proposition that generalizes a well known result:

Proposition 4.6. *If a smooth map $\Psi : M \to N$ has zero differential on a connected open set $U \subset M$, then Ψ is constant in U.*

A very important special case concerns real valued smooth functions $f : M \to \mathbb{R}$, which can be thought as smooth maps between M and the manifold $\mathbb{R}$, with its canonical smooth structure. In this case the differential at p is a linear transformation $\mathrm{d}_pf : T_pM \to T_{f(p)}\mathbb{R}$, and since we have a canonical identification $T_x\mathbb{R} \simeq \mathbb{R}$, the differential is an element in the dual vector space to T_pM.

Definition 4.7. *The **cotangent space** to M at a point p is the vector space T_p^*M dual to the tangent space T_pM:*

$$T_p^*M \equiv \{\omega : T_pM \to \mathbb{R},\ \textit{com}\ \omega\ \textit{linear}\}.$$

Of course we can define $\mathrm{d}_pf \in T_p^*M$ even if f is a smooth function *defined only* in a neighborhood of p. In particular, if choose a coordinate system $(U, x^1, \ldots, x^d)$ around p, we obtain elements

$$\left\{\mathrm{d}_px^1, \ldots, \mathrm{d}_px^d\right\} \subset T_p^*M.$$

It is then easy to check that

$$\mathrm{d}_px^i \cdot \left.\frac{\partial}{\partial x^j}\right|_p = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Hence:

Lemma 4.8. *For any coordinate system $(U, x^1, \ldots, x^d)$ of M around p, $\{\mathrm{d}_px^1, \ldots, \mathrm{d}_px^d\}$ is the basis of T_p^*M dual to the basis $\{\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^d}\right|_p\}$ of T_pM.*

Therefore, once we have fixed a coordinate system $(U, x^1, \ldots, x^d)$ around p, every element $\omega \in T_p^*M$ can be written in the basis $\left\{\mathrm{d}_px^1, \ldots, \mathrm{d}_px^d\right\}$:

$$\omega = \sum_{i=1}^d a_i \mathrm{d}_px^i, \quad \text{with } a_i = \omega(\partial/\partial x^i\big|_p).$$

If $(V, y^1, \ldots, y^d)$ is another coordinate system, we find

$$\omega = \sum_{j=1}^{d} b_j \mathrm{d}_p y^j, \quad \text{with } b_j = \omega(\partial/\partial y^j\big|_p),$$

and one checks easily that:

$$a_i = \sum_{j=1}^{d} \frac{\partial y^j}{\partial x^i}\bigg|_p b_j. \tag{4.2}$$

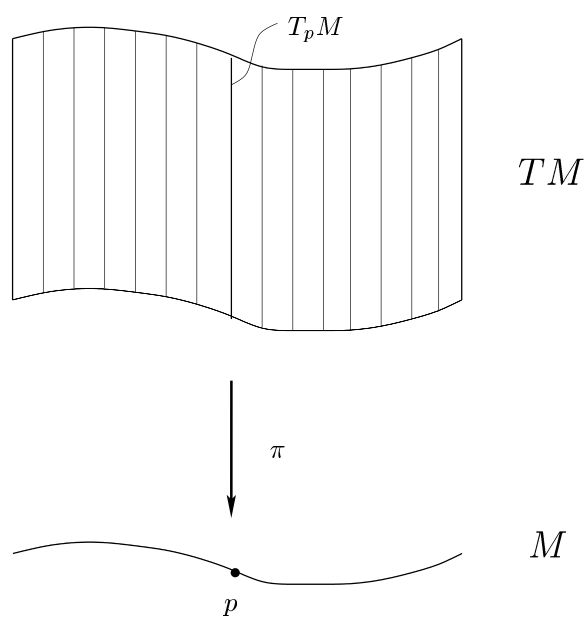
This transformation formula for the components of elements of T_p^*M should be compared with the corresponding transformation formula (4.1) for the components of elements de T_pM.

Let us turn now to the question of how the tangent spaces vary from point to point. We define the **tangent bundle** and the **cotangent bundle** to M as:

$$TM \equiv \bigcup_{p \in M} T_pM, \qquad T^*M \equiv \bigcup_{p \in M} T_p^*M.$$

Notice that we have natural projections $\pi : TM \to M$ and $\pi : T^*M \to M$, which associate to a tangent vector $\mathbf{v} \in T_pM$ and to a tangent covector $\omega \in T_p^*M$ the corresponding base point $\pi(\mathbf{v}) = p = \pi(\omega)$. The term "bundle" comes from the fact that we can picture TM (or T^*M) as a set of fibers (the spaces T_pM or T_p^*M), juxtaposed with each other, forming a manifold:

Proposition 4.9. *TM and T^*M have natural smooth structures of manifolds of dimension $2\dim M$, such that the projections in the base are smooth maps.*



Proof. We give the proof for TM. The proof for T^*M is similar and is left as an exercise.

Let $\{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ be an atlas for M. For each smooth chart $(U_\alpha, \phi_\alpha) = (U_\alpha, x^1, \dots, x^n)$, we define $\tilde{\phi}_\alpha : \pi^{-1}(U_\alpha) \to \mathbb{R}^{2d}$ by setting:

$$\tilde{\phi}_\alpha(\mathbf{v}) = (x^1(\pi(\mathbf{v})), \dots, x^d(\pi(\mathbf{v})), \mathrm{d}_{\pi(\mathbf{v})}x^1(\mathbf{v}), \dots, \mathrm{d}_{\pi(\mathbf{v})}x^d(\mathbf{v})).$$

One checks easily that the collection:

$$\left\{\tilde{\phi}_\alpha^{-1}(O) : O \subset \mathbb{R}^{2d} \text{ open}, \alpha \in A\right\}$$

is a basis for a topology of TM, which is Hausdorff and second countable. Now, we have that:

(a) TM is a topological manifold with local charts $(\pi^{-1}(U_\alpha), \tilde{\phi}_\alpha)$.
(b) For any pair of charts $(\pi^{-1}(U_\alpha), \tilde{\phi}_\alpha)$ and $(\pi^{-1}(U_\beta), \tilde{\phi}_\beta)$, the transition functions $\tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1}$ are smooth.

We conclude that the collection $\left\{(\pi^{-1}(U_\alpha), \tilde{\phi}_\alpha) : \alpha \in A\right\}$ is an atlas, and so defines on TM the structure of a smooth manifold of dimension $\dim TM = 2\dim M$. Finally, the map $\pi : TM \to M$ is smooth because for each α we have that $\phi_\alpha \circ \pi \circ \tilde{\phi}_\alpha^{-1} : \mathbb{R}^{2d} \to \mathbb{R}^d$ is just the projection in the first d components. $\square$

Let $\Psi : M \to N$ be a smooth map. We we will denote by $\mathrm{d}\Psi : TM \to TN$ induced map on the tangent bundle which is defined by:

$$\mathrm{d}\Psi(\mathbf{v}) \equiv \mathrm{d}_{\pi(\mathbf{v})}\Psi(\mathbf{v}).$$

We call this map the **differential** of Ψ. We leave it as an exercise to check that $\mathrm{d}\Psi : TM \to TN$ is a smooth map between the smooth manifolds TM and TN.

If $f : M \to \mathbb{R}$ is a smooth function and $(U, x^1, \dots, x^d)$ is a system of coordinates around p, then from the definition we see that $\mathrm{d}_p f \in T_p^*M$ satisfies:

$$\mathrm{d}_p f \cdot \left.\frac{\partial}{\partial x^i}\right|_p = \left.\frac{\partial f}{\partial x^i}\right|_p.$$

It follows that the expression for $\mathrm{d}f$ in local coordinates $(x^1, \dots, x^d)$ is:

$$\mathrm{d}f|_U = \sum_{i=1}^d \frac{\partial f}{\partial x^i}\mathrm{d}x^i.$$

Notice that in this formula all terms have been precisely defined, in contrast with some formulas one often finds, where heuristic manipulations with $\mathrm{d}f$ are done without much justifications!

We leave it to you to check that what we have done in this section extends to manifolds with boundary. One defines the **tangent space to a manifold with boundary** of dimension d at some point $p \in M$ exactly as in Definition 4.1. The tangent space at any point $p \in M$, even at points of the boundary, has dimension d. The tangent bundle TM is now a manifold with boundary of dimension $2\dim M$. Similarly, one defines the differential of a smooth map $\Psi : M \to N$ between manifolds with boundary and this gives a smooth map between their tangent bundles $\mathrm{d}\Phi : TM \to TN$.

For a manifold with boundary M of dimension $d > 0$, the boundary ∂M is a smooth manifold of dimension $d-1$. Hence, if $p \in \partial M$ we have two tangent spaces: T_pM, which has dimension d, and $T_p(\partial M)$, which has dimension $d-1$. We leave it as an exercise to check that the inclusion $i : \partial M \hookrightarrow M$ is a smooth and its differential $\mathrm{d}_p i : T_p(\partial M) \to T_pM$ is injective, at any point $p \in \partial M$. It follows that we can identify $T_p(\partial M)$ with its image in T_pM, so inside the tangent space to M at points of the boundary we have a well-defined subspace. It is common to denote this subspace also by $T_p(\partial M)$, a practice that we will also adopt here.

HOMEWORK.

1. Show that the 3 descriptions of tangent vectors given in this lecture are indeed equivalent.

2. In $\mathbb{R}^3$ consider the usual Cartesian coordinates (x, y, z). One defines **spherical coordinates** in $\mathbb{R}^3$ to be the smooth chart (U, ϕ), where $U = \mathbb{R}^3 - \{(x, 0, z) : x \geq 0\}$ and $\phi = (r, \theta, \varphi)$ is defined as usual by
   - $r(x, y, z) := \sqrt{x^2 + y^2 + z^2}$ is the distance to the origin;
   - $\theta(x, y, z)$ is the longitude, i.e., the angle in $]0, 2\pi[$ between the vector $(x, y, 0)$ and the x-axis;
   - $\varphi(x, y, z)$ is the co-latitude, i.e., the angle in $]0, \pi[$ between the vector (x, y, z) and the z-axis.

   Compute:
   (a) The components of the tangent vectors to $\mathbb{R}^3$ $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial \theta}$, $\frac{\partial}{\partial \varphi}$ in Cartesian coordinates;
   (b) The components of the tangent vectors to $\mathbb{R}^3$ $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ in spherical coordinates.

3. Let $M \subset \mathbb{R}^n$ be an embedded d-manifold. Show that if $\psi : V \to M \cap U$ is a parameterization of a neighborhood of $p \in M$, then the tangent space T_pM can be identified with the subspace $\psi'(q)(\mathbb{R}^d) \subset \mathbb{R}^n$.

4. Let $(U, x^1, \dots, x^d)$ be a local coordinate system in a manifold M. Show that the tangent vectors
$$\left.\frac{\partial}{\partial x^i}\right|_p \in T_pM, \qquad i = 1, \dots, d,$$
are linearly independent.

5. Show that T^*M has a smooth structure of manifold of dimension $2\dim M$, for which the projection $\pi : T^*M \to M$ is a smooth map.

6. Check that if M and N are smooth manifolds and $\Psi : M \to N$ is a smooth map, then $\mathrm{d}\Psi : TM \to TN$ is also smooth.

Lecture 5. Immersions, Submersions and Submanifolds

As we can expect from what we know from calculus in Euclidean space the properties of the differential of a smooth map between two smooth manifolds reflect the local behavior of the smooth map. In this lecture we will make this precise.

Definition 5.1. *Let $\Psi : M \to N$ be a smooth map:*

*(a) Ψ is called an **immersion** if $\mathrm{d}_p\Psi : T_pM \to T_{\Psi(p)}N$ is injective, for all $p \in M$;*

*(b) Ψ is called a **submersion** if $\mathrm{d}_p\Psi : T_pM \to T_{\Psi(p)}N$ is surjective, for all $p \in M$;*

*(a) Ψ is called an **étale**[2] if $\mathrm{d}_p\Psi : T_pM \to T_{\Psi(p)}N$ is an isomorphism, for all $p \in M$.*

Immersions, submersions and étales have local canonical forms. They are all consequences of the following general result:

Theorem 5.2 (Constant Rank Theorem)**.** *Let $\Psi : M \to N$ be a smooth map and $p \in M$. If $\mathrm{d}_q\Psi : T_qM \to T_{\Psi(q)}N$ has constant rank r, for all q in a neighborhood of p, then there are local coordinates $(U, \phi) = (U, x^1, \dots, x^m)$ for M centered at p and local coordinates $(V, \psi) = (V, y^1, \dots, y^n)$ for N centered at $\Psi(p)$, such that:*

$$\psi \circ \Psi \circ \phi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^r, 0, \dots, 0).$$

Proof. Let $(\tilde{U}, \tilde{\phi})$ and $(\tilde{V}, \tilde{\psi})$ be local coordinates centered at p and $\Psi(p)$, respectively. Then

$$\tilde{\psi} \circ \Psi \circ \tilde{\phi}^{-1} : \tilde{\phi}(\tilde{U} \cap \tilde{V}) \to \tilde{\psi}(\tilde{U} \cap \tilde{V})$$

is a smooth map from a neighborhood of zero in $\mathbb{R}^m$ to a neighborhood of zero in $\mathbb{R}^n$, whose differential has constant rank. Therefore, it is enough to consider the case where $\Psi : \mathbb{R}^m \to \mathbb{R}^n$ is a smooth map

$$(x^1, \dots, x^m) \mapsto (\Psi^1(x), \dots, \Psi^n(x)),$$

whose differential has constant rank in a neighborhood of the origin.

Let r be the rank of $\mathrm{d}\Psi$. Eventually after some reordering of the coordinates, we can assume that

$$\det \left[\frac{\partial \Psi^j}{\partial x^i}\right]_{i,j=1}^{r} (0) \neq 0.$$

It follows immediately from the Inverse Function Theorem, that the smooth map $\phi : \mathbb{R}^m \to \mathbb{R}^m$ defined by

$$(x^1, \dots, x^m) \to (\Psi^1(x), \dots, \Psi^r(x), x^{r+1}, \dots, x^m),$$

[2]We use this term provisionally. We shall see later in Corollary 5.5 that an étale map is the same thing as a local diffeomorphism.

is a diffeomorphism from a neighborhood of the origin. We conclude that:

$$\Psi \circ \phi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^r, \Psi^{r+1} \circ \phi^{-1}(x), \dots, \Psi^n \circ \phi^{-1}(x)).$$

Let q be any point in the domain of $\Psi \circ \phi^{-1}$. We can compute the Jacobian matrix of $\Psi \circ \phi^{-1}$ as:

$$\left[\begin{array}{c|c} I_r & 0 \\ \hline * & \frac{\partial(\Psi^j \circ \phi^{-1})}{\partial x^i}(q) \end{array}\right],$$

where I_r is the $r \times r$ identity matrix and where in the lower right corner $i, j > r$. Since this matrix has exactly rank r, we conclude that:

$$\frac{\partial(\Psi^j \circ \phi^{-1})}{\partial x^i}(q) = 0, \text{ if } i, j > r.$$

In other words, the components of $\Psi^j \circ \phi^{-1}$, for $j > r$, do not depend on the coordinates $x^{r+1}, \dots, x^m$:

$$\Psi^j \circ \phi^{-1}(x) = \Psi^j \circ \phi^{-1}(x^1, \dots, x^r), \text{ if } j > r.$$

Let us consider now the map $\psi : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\psi(y^1, \dots, y^n) = (y^1, \dots, y^r, y^{r+1} - \Psi^{r+1} \circ \phi^{-1}(y), \dots, y^n - \Psi^n \circ \phi^{-1}(y)).$$

We see that ψ is a diffeomorphism since its Jacobian matrix at the origin is given by:

$$\left[\begin{array}{c|c} I_r & 0 \\ \hline * & I_{e-r} \end{array}\right],$$

which is non-singular. But now we compute:

$$\psi \circ \Psi \circ \phi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^r, 0, \dots, 0).$$

□

An immediate corollary of this result is that an immersion of a m-manifold into a n-manifold, where necessarily $m \leq n$, locally looks like the *inclusion* $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$:

Corollary 5.3. *Let $\Psi : M \to N$ be an immersion. Then for each $p \in M$, there are local coordinates $(U, \phi) = (U, x^1, \dots, x^m)$ for M centered at p and local coordinates $(V, \psi) = (V, y^1, \dots, y^n)$ for N centered at $\Psi(p)$, such that: tais que:*

$$\psi \circ \Psi \circ \phi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^m, 0, \dots, 0).$$

Similarly, we conclude that a submersion of a m-manifold into a n-manifold, where necessarily $m \geq n$, locally looks like the *projection* $\mathbb{R}^m \twoheadrightarrow \mathbb{R}^n$:

Corollary 5.4. *Let $\Psi : M \to N$ be a submersion. Then for each $p \in M$, there are local coordinates $(U, \phi) = (U, x^1, \dots, x^m)$ for M centered at p and local coordinates $(V, \psi) = (V, y^1, \dots, y^n)$ for N centered at $\Psi(p)$, such that:*

$$\psi \circ \Psi \circ \phi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^n).$$

Since an étale is a smooth map which is simultaneously an immersion and a submersion, we conclude that an étale is just a local diffeomorphism:

Corollary 5.5. *Let $\Psi : M \to N$ be an étale. Then for each $p \in M$, there are local coordinates $(U, \phi) = (U, x^1, \ldots, x^d)$ for M centered at p and local coordinates $(V, \psi) = (V, y^1, \ldots, y^d)$ for N centered at $\Psi(p)$, such that:*

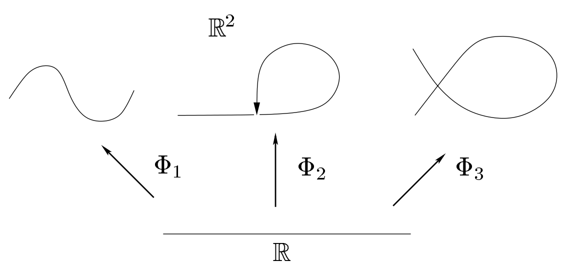
$$\psi \circ \Psi \circ \phi^{-1}(x^1, \ldots, x^d) = (x^1, \ldots, x^d).$$

Let us now turn to the study of subobjects in the category of smooth manifolds:

Definition 5.6. *A* ***submanifold*** *of a manifold M is a pair (N, Φ) where N is a manifold and $\Phi : N \to M$ is an injective immersion. When $\Phi : N \to \Phi(N)$ is a homeomorphism, where on $\Phi(N)$ one takes the relative topology, one calls the pair (N, Φ) an* ***embedded submanifold*** *and Φ an* ***embedding****.*

One sometimes uses the term **immersed submanifold** to emphasize that $\Phi : N \to M$ is only an immersion and reserves the term *submanifold* for embedded submanifolds.

Example 5.7.
The next picture illustrates various immersions of $N = \mathbb{R}$ in $M = \mathbb{R}^2$. Notice that $(\mathbb{R}, \Phi_1)$ is an embedded submanifold of $\mathbb{R}^2$, while $(\mathbb{R}, \Phi_2)$ is only an immersed submanifold of $\mathbb{R}^2$. On the other hand, Φ_3 is an immersion but it is not injective, so $(\mathbb{R}, \Phi_3)$ is not a submanifold of $\mathbb{R}^2$.

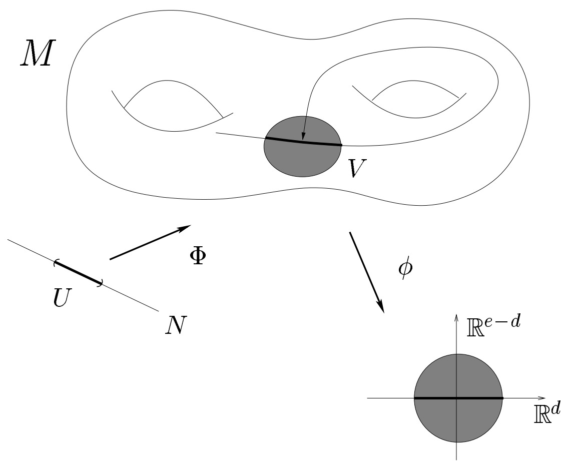


If (N, Φ) is a submanifold of M, then for each $p \in N$, the linear map $\mathrm{d}_p\Phi : T_pN \to T_{\Phi(p)}M$ is injective. Hence, we can always identify the tangent space T_pN with its image $\mathrm{d}_p\Phi(T_pN)$, which is a subspace of $T_{\Phi(p)}M$. From now on, we will use this identification, so that T_pN will always be interpreted as a subspace of $T_{\Phi(p)}M$.

The local canonical form (Corollary 5.3) implies immediately the following:

Proposition 5.8 (Local normal form for immersed submanifolds)**.** *Let (N, Φ) be a submanifold of dimension d of a manifold M. Then for all $p \in N$, there exists a neighborhood U of p and a coordinate system $(V, x^1, \ldots, x^m)$ for M centered at $\Phi(p)$ such that:*

$$\Phi(U) = \left\{ q \in V : x^{d+1}(q) = \cdots = x^m(q) = 0 \right\}.$$



Proof. By Corollary 5.3, for any $p \in N$ we can choose coordinates (U, ϕ) for N centered at p and coordinates $(V, \psi) = (V, x^1, \dots, x^m)$ for N centered at $\Phi(p)$, such that $\psi \circ \Phi \circ \phi^{-1} : \mathbb{R}^d \to \mathbb{R}^m$ is the inclusion. But then $\psi \circ \Phi(U)$ is exactly the set of points in $\psi(V) \subset \mathbb{R}^m$ with the last $m - d$ coordinates equal to 0.

□

You should notice (using the same notation as in the proposition) that, in general, $\Phi(N) \cap V \neq \Phi(U)$, so there could exist points in $\Phi(N) \cap V$ which *do not belong* to the slice $\left\{q \in V : x^{d+1}(q) = \cdots = x^m(q) = 0\right\}$.

However, whenever (N, Φ) is an embedded submanifold we find:

Corollary 5.9 (Local normal form for embedded submanifolds)**.** *Let* (N, Φ) *be an embedded submanifold of dimension* d *of a manifold* M*. For each* $p \in N$*, there exists a coordinate system* $(V, x^1, \dots, x^m)$ *of* M *centered at* $\Phi(p)$*, such that:*

$$\Phi(N) \cap V = \left\{q \in V : x^{d+1}(q) = \cdots = x^m(q) = 0\right\}.$$

Proof. Fix $p \in N$ and choose a a neighborhood U of p and a coordinate system $(V', x^1, \dots, x^e)$ centered at $\Phi(p)$, as in the proposition. Since (N, Φ) is assumed to be embedded, $\Phi(U)$ is an open subset of $\Phi(N)$ for the relative topology: there exists an open set $V'' \subset M$ such that $\Phi(U) = V' \cap \Phi(N)$. If we set $V = V' \cap V''$ the restrictions of the x^i to V, yield a coordinate system $(V, x^1, \dots, x^e)$ such that:

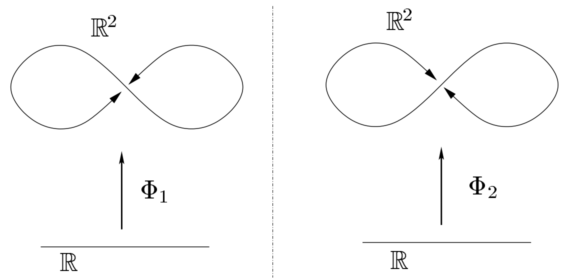
$$\Phi(N) \cap V = \left\{q \in V : x^{d+1}(q) = \cdots = x^e(q) = 0\right\}.$$

□

We would like to think of submanifolds of M simply as subsets of M. However, this in general is not possible, as shown by the following simple example.

Example 5.10.

There are two injective immersions $\Phi_i : \mathbb{R} \to \mathbb{R}^2$, $i = 1, 2$, whose images in $\mathbb{R}^2$ coincide with the infinite symbol: Hence, the infinite symbol by itself, does



not have a unique submanifold structure, and additional data must be specified.

The example of the infinity symbol shows one must be careful when we think of a submanifold of M as a subset of M. In order to see what can go wrong, we introduce the following equivalence relation:

Definition 5.11. *We say that (N_1, Φ_1) and (N_2, Φ_2) are **equivalent submanifolds** of M if there exists a diffeomorphism $\Psi : N_1 \to N_2$ such that the following diagram commutes:*

$$\begin{tikzcd} N_1 \arrow[r, "\Phi_1"] \arrow[dr, dashed, "\Psi"'] & M \\ & N_2 \arrow[u, "\Phi_2"'] \end{tikzcd}$$

If (N, Φ) is a submanifold of M we can consider the image $\Phi(N) \subset M$ with the unique smooth structure for which $\hat{\Phi} : N \to \Phi(N)$ is a diffeomorphism. Obviously, if we take this smooth structure on $\Phi(N)$, the inclusion $i : \Phi(N) \hookrightarrow M$ is an injective immersion and the following diagram commutes:

$$\begin{tikzcd} N \arrow[r, "\Phi"] \arrow[dr, "\hat{\Phi}"'] & M \\ & \Phi(N) \arrow[u, "i"'] \end{tikzcd}$$

Therefore, *every* submanifold (N, Φ) as a unique representative (A, i), where $A \subset M$ is a subset and $i : A \hookrightarrow M$ is the inclusion. We then say that $A \subset M$ is a submanifold.

Example 5.12.

If $A \subset M$ is an arbitrary subset, in general, there will be no smooth structure on A for which the inclusion $i : A \hookrightarrow M$ is an immersion. For example, the *subset $A = \{(x, |x|) : x \in \mathbb{R}\} \subset \mathbb{R}^2$ does not admit such a smooth structure (exercise).*

On the other hand, if A admits a smooth structure such that the inclusion $i : A \hookrightarrow M$ is an immersion, this smooth structure may not be unique: this is exactly what we saw Example 5.10.

Still, we have the following result:

Theorem 5.13. *Let $A \subset M$ be some subset of a smooth manifold and $i : A \hookrightarrow M$ the inclusion. Then:*

(i) For each choice of a topology in A there exists at most one smooth structure compatible with this topology and such that (A, i) is a submanifold of M.

(ii) If for the relative topology in A there exists a compatible smooth structure such that (A, i) is a submanifold of M, then this is the only topology in A for which there exists a compatible smooth structure such that (A, i) is a submanifold of M.

Example 5.14.

The sphere $\mathbb{S}^7 \subset \mathbb{R}^8$ is an embedded submanifold. We have mentioned before that the sphere $\mathbb{S}^7$ have smooth structures compatible with the usual topology but which are not equivalent to the standard smooth structure on the sphere. It follows that for these exotic smooth structures, $\mathbb{S}^7$ is not a submanifold of $\mathbb{R}^8$.

In order to prove Theorem 5.13, we observe that if (N, Φ) is a submanifold of M and $\Psi : P \to M$ is a smooth map such that $\Psi(P) \subset \Phi(N)$, the fact that Φ is 1:1 implies that Ψ factors through a map $\hat{\Psi} : P \to N$, i.e., we have a commutative diagram:

$$\begin{tikzcd} P \arrow[r, "\Psi"] \arrow[dr, dashed, "\hat{\Psi}"'] & M \\ & N \arrow[u, "\Phi"'] \end{tikzcd}$$

However, the problem is that, in general, the map $\hat{\Psi}$ *is not* smooth, as shown by the example of the infinite symbol.

Example 5.15.

Let $\Phi_i : \mathbb{R} \to \mathbb{R}^2$, $i = 1, 2$, be the two injective immersion whose images in $\mathbb{R}^2$ coincide with the infinite symbol, as in Example 5.10. Since $\Phi_1(\mathbb{R}) = \Phi_2(\mathbb{R})$, we have unique maps $\hat{\Phi}_{12} : \mathbb{R} \to \mathbb{R}$ e $\hat{\Phi}_{21} : \mathbb{R} \to \mathbb{R}$. such that $\hat{\Phi}_{12} \circ \Phi_2 = \Phi_1$ and $\hat{\Phi}_{21} \circ \Phi_1 = \Phi_2$. It is easy to check that $\hat{\Phi}_{12}$ and $\hat{\Phi}_{21}$ are not continuous, hence they are not smooth.

The next result shows that what may fail is precisely the continuity of the map $\hat{\Psi}$:

Proposition 5.16. *Let (N, Φ) be a submanifold of M, $\Psi : P \to M$ a smooth map such that $\Psi(P) \subset \Phi(N)$ and $\hat{\Psi} : P \to N$ the induced map.*

(i) If $\hat{\Psi}$ is continuous, then it is smooth.

(ii) If Φ is an embedding, then $\hat{\Psi}$ is continuous (hence smooth).

Proof. Assume first that $\hat{\Psi}$ is continuous. For each $p \in N$, choose $U \subset N$ and $(V, \phi) = (V, x^1, \ldots, x^m)$ as in Proposition 5.8, and consider the smooth map

$$\psi = \pi \circ \phi \circ \Phi : U \to \mathbb{R}^d,$$

where $\pi : \mathbb{R}^m \to \mathbb{R}^d$ is the projection $(x^1, \ldots, x^m) \mapsto (x^1, \ldots, x^d)$. The pair (U, ψ) is a smooth coordinate system for N centered at p. On the other hand, we see that

$$\psi \circ \hat{\Psi} = \pi \circ \phi \circ \Phi \circ \hat{\Psi} = \pi \circ \phi \circ \Psi,$$

is smooth in the open set $\hat{\Psi}^{-1}(U)$. Since the collection of all such open sets $\hat{\Psi}^{-1}(U)$ covers P, we conclude that $\hat{\Psi}$ is smooth, so (i) holds.

Now if Φ is an embedding, then every open set $U \subset N$ is of the form $\Phi^{-1}(V)$, where $V \subset M$ is open. Hence, $\hat{\Psi}^{-1}(U) = \hat{\Psi}^{-1}(\Phi^{-1}(V)) = \Psi^{-1}(V)$ is also open. We conclude that $\hat{\Psi}$ is continuous, so (ii) also holds. □

Proof of Theorem 5.13. (i) follows immediately from Proposition 5.16 (i).

On the other hand, to prove (ii), let (N, Φ) be a submanifold with $\Phi(N) = A$ and consider the following diagram:

$$\begin{tikzcd} N \arrow[r, "\Phi"] \arrow[dr, "\hat{\Phi}"'] & M \\ & A \arrow[u, "i"'] \end{tikzcd}$$

Since A is assume to have the relative topology, by Proposition 5.16 (ii), $\hat{\Phi}$ is smooth. Hence, $\hat{\Phi}$ is an invertible immersion so it is a diffeomorphism (exercise). We conclude that (N, Φ) is equivalent to (A, i), so (ii) holds.

□

The previous discussion justifies considering the following class of submanifolds, which is lies in between immersed submanifolds and embedded submanifolds:

Definition 5.17. *A* ***initial submanifold*** *of M is a submanifold (N, Φ) such that every $\Psi : P \to M$ with $\Psi(P) \subset \Phi(N)$ factors through a smooth map $\hat{\Psi} : P \to N$:*

$$\begin{tikzcd} P \arrow[r, "\Psi"] \arrow[dr, dashed, "\hat{\Psi}"'] & M \\ & N \arrow[u, "\Phi"'] \end{tikzcd}$$

Sometimes initial submanifolds are also called **regular immersed submanifolds** or **weakly embedded submanifolds**. The two different immersions of the infinity symbol that we saw above are not initial submanifols. On the other hand, Proposition 5.16 (ii) shows that embedded submanifolds are initial submanifolds. But you should be aware that there are many examples of initial submanifolds which are not embedded, such as in the following example.

EXAMPLE 5.18.
In the 2-torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ we have a family of submanifolds $(\mathbb{R}, \Phi_a)$, depending on the parameter $a \in \mathbb{R}$, defined by:

$$\Phi_a(t) = (e^{it}, e^{iat}).$$

If $a = m/n$ is rational, this is a closed curve, which turns m times in one torus direction and n times in the other torus direction, so this is an embedding.

If $a \notin \mathbb{Q}$ then the curve is dense in the 2-torus, so this is only an immersed submanifold. However, if $\hat{\Psi} : P \to \mathbb{R}$ is a map such that the composition $\Phi_a \circ \hat{\Psi}$ is smooth, then we see immediately that $\hat{\Psi} : P \to \mathbb{R}$ is continuous. By Proposition 5.16, we conclude that $\hat{\Psi}$ is smooth. Hence, (N, Φ_a) is a initial submanifold.

HOMEWORK.

1. Show that $\{(x, |x|) : x \in \mathbb{R}\}$ *is not* the image of an immersion $\Phi : \mathbb{R} \to \mathbb{R}^2$.

2. Show that there exists a diffeomorphism $\Psi : T\mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{R}^3$, which makes the following diagram commutative:

$$\begin{tikzcd} T\mathbb{S}^3 \arrow[rr, "\Psi"] \arrow[dr, "\pi"'] & & \mathbb{S}^3 \times \mathbb{R}^3 \arrow[dl, "\tau"] \\ & \mathbb{S}^3 & \end{tikzcd}$$

where $\tau : \mathbb{S}^3 \times \mathbb{R}^3 \to \mathbb{S}^3$ is the projection in the first factor and the restriction $\Psi : T_p\mathbb{S}^3 \to \mathbb{R}^3$ is linear for every $p \in \mathbb{S}^3$.
Hint: The 3-sphere is the set of quaternions of norm 1.

3. Let $\{y^1, \ldots, y^e\}$ be some set of smooth functions on a manifold M. Show that:
   (a) If $\{\mathrm{d}_p y^1, \ldots, \mathrm{d}_p y^e\} \subset T_p^*M$ is a linearly independent set, then the functions $\{y^1, \ldots, y^e\}$ is a part of a coordinate system around p.
   (b) If $\{\mathrm{d}_p y^1, \ldots, \mathrm{d}_p y^e\} \subset T_p^*M$ is a generating set, then a subset of $\{y^1, \ldots, y^e\}$ is a coordinate system around p.
   (c) If $\{\mathrm{d}_p y^1, \ldots, \mathrm{d}_p y^e\} \subset T_p^*M$ is a basis, then the functions $\{y^1, \ldots, y^e\}$ form a coordinate system around p.

4. Show that a submersion is an open map. What can you say about an immersion?

5. Let $\Phi : \mathbb{P}^2 \to \mathbb{R}^3$ be the map defined by

$$\Phi([x,y,z]) = \frac{1}{x^2+y^2+z^2}(yz, xz, xy).$$

Show that Φ is smooth and show that it only fails to be an immersion at 6 points. Make a sketch of the image of Φ.

6. Let M be a manifold, $A \subset M$, and $i : A \hookrightarrow M$ the inclusion. Show that (A, i) is a an embedded submanifold of M of dimension d, if and only if for each $p \in A$ there exists a coordinate system $(U, x^1, \dots, x^e)$ centered at p such that

$$A \cap U = \left\{p \in U : x^{d+1}(p) = \cdots = x^e(p) = 0\right\}.$$

7. Show that a subset $M \subset \mathbb{R}^n$ is a k-surface if and only it is an embedded submanifold (so this justifies us calling M and embedded manifold in $\mathbb{R}^n$).

8. One says that a subset S of a manifold M has **zero measure** if for every coordinate system (U, ϕ) of M, the set $\phi(S \cap U) \subset \mathbb{R}^d$ has zero measure. Show that if $\Phi : N \to M$ is an immersion then
   (a) Φ maps zero measure sets to zero measure sets;
   (b) If $\dim N < \dim M$ then $\Phi(N)$ has zero measure.

9. Show that for a submanifold (N, Φ) of a smooth manifold M the following are equivalent:
   (a) $\Phi(N) \subset M$ is a closed subset and (N, Φ) is embedded.
   (b) $\Phi : N \to M$ is a closed map (i.e, $\Phi(A)$ is closed whenever $A \subset N$ is a closed subset).
   (c) $\Phi : N \to M$ is a proper map (i.e., $\Phi^{-1}(K) \subset N$ is compact, whenever $K \subset M$ is compact).

   Use this to conclude that a submanifold (N, Φ) with N compact, is always an embedded submanifold.

10. Show that an invertible immersion $\Phi : N \to M$ is a diffeomorphism. Give a counterexample to this statement if N does not have a countable basis.

11. Let $\pi : \widetilde{M} \to M$ be a covering space of a smooth manifold M. Show that $\widetilde{M}$ has unique smooth structure for which the covering map π is a local diffeomorphism.

Lecture 6. Embeddings and Whitney's Theorem

Definition 6.1. *Let $\Psi : M \to N$ be a smooth map*

*(i) One calls $p \in M$ a **regular point** of Ψ if $\mathrm{d}_p\Psi : T_pM \to T_{\Psi(p)}N$ is surjective. Otherwise one calls p a **singular point** of Ψ;*

*(ii) One calls $q \in N$ a **regular value** of Ψ if every $p \in \Psi^{-1}(q)$ is a regular point. Otherwise one calls q a **singular value** of Ψ.*

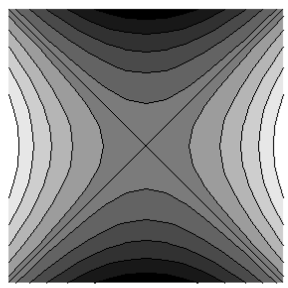
The following example gives some evidence for the use of the terms "regular" and "singular".

Example 6.2.
Let $\Psi : \mathbb{R}^2 \to \mathbb{R}$ be the map defined by

$$\Phi(x, y) = x^2 - y^2.$$

This map has Jacobian matrix $[2x\ 2y]$. Therefore, every $(x, y) \neq (0, 0)$ is a regular point of Ψ and $(0, 0)$ is a singular point of Ψ. On the other hand, 0 is a singular value of Ψ, while every other value is a regular value of Ψ.

If we consider a regular value $c \neq 0$, the level set $\Phi^{-1}(c)$ is a submanifold of $\mathbb{R}^2$ (an hyperboloid). On the other hand, for the singular value 0, we see that $\Phi^{-1}(0)$ is a union of two lines $x = \pm y$, which is not a manifold at the origin, where the two lines cross each other.



In fact, the level sets of regular values are always submanifolds:

Theorem 6.3. *Let $\Psi : M \to N$ be a smooth map and let $q \in N$ be a regular value of Ψ. Then $\Psi^{-1}(q) \subset M$ is an embedded submanifold of dimension $\dim M - \dim N$ and:*

$$T_m(\Psi^{-1}(q)) = \operatorname{Ker} \mathrm{d}_m\Psi.$$

Proof. If $q \in N$ is a regular value of Ψ there exists an open set $\Psi^{-1}(q) \subset O \subset M$ such that $\Psi|_O$ is a submersion. Therefore, for any $p \in \Phi^{-1}(q)$ we can choose coordinates $(U, x^1, \ldots, x^m)$ around p and coordinates $(V, y^1, \ldots, y^n)$ around q, such that Ψ is represented in these local coordinates by the projection

$$\mathbb{R}^m \to \mathbb{R}^n : \ (x^1, \ldots, x^m) \mapsto (x^1, \ldots, x^n).$$

Therefore, we see that

$$\Psi^{-1}(q) \cap U = \left\{ p \in U : x^1(p) = \cdots = x^n(p) = 0 \right\}.$$

It follows that $\Psi^{-1}(q)$ is an embedded submanifold of dimension $m - n = \dim M - \dim N$ (see Exercise 6 in the previous lecture). The statement about the tangent space to $\Psi^{-1}(q)$ is left as an exercise.

$\square$

Example 6.4.
Let $M = \mathbb{R}^{d+1}$ and let $\Psi : \mathbb{R}^{d+1} \to \mathbb{R}$ be the smooth map:

$$\Psi(x) = ||x||^2.$$

The Jacobian matrix of Ψ at x is given by:

$$\Psi'(x) = [2x^1, \ldots, 2x^{d+1}].$$

Since $\Psi'(x)$ has rank one if $||x|| > 0$, it follows that any $c = R^2 > 0$ is a regular value of Ψ. The theorem above then asserts that the spheres $\mathbb{S}^d_R = \Psi^{-1}(R)$ are embedded submanifolds of $\mathbb{R}^{d+1}$ of codimension 1. Note that for the differential structure on $\mathbb{S}^d$ that we have defined before, $\mathbb{S}^d$ is also an embedded submanifold of $\mathbb{R}^{d+1}$. Hence, that differential structure coincides with this one.

Not every embedded submanifold $S \subset M$ is of the form $\Psi^{-1}(q)$, for a regular value of some smooth map $\Psi : M \to N$. There are global obstructions that we will study later. Also, what happens at regular values can be very wild: using a partition of unity argument it is possible to show that for any closed subset $A \subset M$ of a smooth manifold, there exists a smooth function $f : M \to \mathbb{R}$ such that $f^{-1}(0) = A$.

If $N \subset M$ is a submanifold we call the **codimension** of N in M the integer $\dim M - \dim N$. Since a set with a single point is a manifold of dimension 0, the previous result can be restated as saying that if q is a regular value of Ψ, then $\Psi^{-1}(q)$ is an embedded submanifold with $\operatorname{codim} \Psi^{-1}(q) = \operatorname{codim} \{q\}$. In this form, the previous result can be generalized in the following very useful way:

Theorem 6.5. *Let $\Psi : M \to N$ be a smooth map and let $Q \subset N$ be an embedded submanifold. Assume that for all $p \in \Psi^{-1}(Q)$ one has:*

$$\operatorname{Im} \mathrm{d}_p\Psi + T_{\Psi(p)}Q = T_{\Psi(p)}N. \tag{6.1}$$

Then $\Psi^{-1}(Q) \subset M$ is an embedded submanifold with

$$\operatorname{codim} \Psi^{-1}(Q) = \operatorname{codim} Q$$

and:

$$T_m(\Psi^{-1}(Q)) = (\mathrm{d}_m\Psi)^{-1}(T_{\Psi(m)}Q).$$

Proof. Choose $p_0 \in \Psi^{-1}(Q)$ and set $q_0 = \Psi(p_0)$. Since $Q \subset N$ is assumed to be an embedded submanifold, we can choose a coordinate system $(V, \phi) = (V, y^1, \ldots, y^n)$ for N around q_0, such that

$$Q \cap V = \left\{ q \in V : y^{l+1}(q) = \cdots = y^n(q) = 0 \right\},$$

where $l = \dim Q$. Define a smooth map $\Phi : \Psi^{-1}(V) \to \mathbb{R}^{n-l}$ by

$$\Phi = (y^{l+1} \circ \Psi, \ldots, y^n \circ \Psi).$$

Then we see that $U = \Psi^{-1}(V)$ is an open subset of M which contains p_0 and such that $\Psi^{-1}(Q) \cap U = \Phi^{-1}(0)$. If we can show that 0 is a regular value of Φ, then by Theorem 6.3 it follows that for all $p_0 \in \Psi^{-1}(Q)$, there exists an open set $U \subset M$ such that $\Psi^{-1}(Q) \cap U$ is an embedded submanifold of M of codimension $n - l = \operatorname{codim} Q$. This implies that $\Psi^{-1}(Q)$ is an embedded submanifold of M, as claimed.

To check that 0 is a regular value of Φ note that $\Phi = \pi \circ \phi \circ \Psi$, where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-l}$ is the projection in the last $n - l$ components. Since π is a submersion, ϕ is a diffeomorphism and $\ker \mathrm{d}_q(\pi \circ \phi) = T_q Q$, for all $q \in Q \cap V$, it follows from (6.1), that $\mathrm{d}_p \Phi = \mathrm{d}_{\Psi(p)}(\pi \circ \phi) \cdot \mathrm{d}_p \Psi$ is surjective, for all $p \in \Psi^{-1}(Q) \cap U = \Phi^{-1}(0)$, i.e., 0 is a regular value of Φ.

The statement about the tangent space to $\Psi^{-1}(Q)$ is left as an exercise. □

The condition (6.1) appearing in the statement of the theorem is so important that one has a special name for it.

Definition 6.6. *Let $\Phi : M \to N$ be a smooth map. We say that Ψ is **transversal** to a submanifold $Q \subset N$, and we write $\Psi \pitchfork Q$, if:*

$$\operatorname{Im} \mathrm{d}_p \Psi + T_{\Psi(p)} Q = T_{\Psi(p)} N, \quad \forall p \in \Psi^{-1}(Q).$$

Notice that submersions $\Phi : M \to N$ are specially nice: they are transverse to every submanifold $Q \subset N$! So for a submersion the theorem shows that the inverse image of any submanifold is a submanifold.

A special case that justifies the use of the term "transversal" is when $M \subset N$ is a submanifold and $\Psi : M \hookrightarrow N$ is the inclusion. In this case, $\Psi^{-1}(Q) = M \cap Q$ and the transversality condition reduces to:

$$T_q M + T_q Q = T_q N, \quad \forall q \in M \cap Q.$$

Note that this condition is symmetric in M and Q. So in this case we simply say that M and Q **intersect transversely** and we write $M \pitchfork Q$. The previous result then gives that:

Corollary 6.7. *If $M, Q \subset N$ are embedded submanifolds such that $M \pitchfork Q$. Then $M \cap Q$ is an embedded submanifold of N with:*

$$\dim M \cap Q = \dim M + \dim Q - \dim N,$$

and

$$T_n(M \cap Q) = T_n M \cap T_n Q.$$

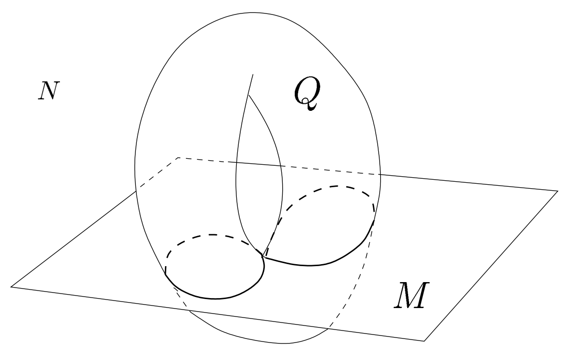
Although Theorem 6.5 and its corollary were stated for embedded submanifolds, you are asked in an exercise in this Lecture to check that these results still hold for immersed submanifolds.

Transversality plays an important role because of the following properties:

- *Transversality is a stable property:* If $\Phi : M \to N$ is transverse to Q then any map $\Psi : M \to N$ close enough to Φ is also transverse to Q.
- *Transversality is a generic property:* Any map $\Phi : M \to N$ can be approximated by a map $\widetilde{\Phi} : M \to N$ which is transverse to Q.

We shall not attempt to make precise these two statements, since we would need to introduce and study appropriate topologies on the space of smooth maps $C^\infty(M,N)$. In fact, transversality is one of the most important topics of study in *Differential Topology*.

On the other hand, when two manifolds do not intersect transversally, in general the intersection is not a manifold as illustrated by the following figure.



EXAMPLES 6.8.

1. *Let $M=\mathbb{S}^1\times\mathbb{R}$ be a cylinder. We can embed M in $\mathbb{R}^3$ as follows: we define a smooth map $\Phi: M\to\mathbb{R}^3$ by:*

$$\Phi(\theta,t)=(R\cos\theta, R\operatorname{sen}\theta, t),$$

wheer we identify $\mathbb{S}^1=[0,2\pi]/2\pi\mathbb{Z}$. This map is injective and its Jacobian matrix $\Phi'(\theta,t)$ has rank 2, hence Φ is an injective immersion.

The image of Φ is the subset of $\mathbb{R}^3$:

$$\left\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=R^2\right\}=\Psi^{-1}(c),$$

where $c=R^2$ and $\Psi:\mathbb{R}^3\to\mathbb{R}$ is the smooth map

$$\Psi(x,y,z)=x^2+y^2.$$

Since $\Psi'(x,y,z)=[2x,2y,0]\neq 0$ if $x^2+y^2=c\neq 0$, we conclude that any $c\neq 0$ is a regular value of Ψ. Hence, we have indeed an embedding of the cylinder $\mathbb{S}^1\times\mathbb{R}$ in $\mathbb{R}^3$.

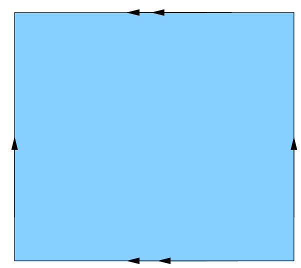
2. *The 2-**torus** $M=\mathbb{S}^1\times\mathbb{S}^1$ can also be embedded in $\mathbb{R}^3$ as follows/ We can think of the the two torus as $\mathbb{S}^1\times\mathbb{S}^1=[0,2\pi]/2\pi\mathbb{Z}\times[0,2\pi]/2\pi\mathbb{Z}$. Note that this amounts to think of the torus as a square of side 2π where we identify the sides of the square, as in the following figure:*

Now define $\Phi: M\to\mathbb{R}^3$ by:

$$\Phi(\theta,\phi)=((R+r\cos\phi)\cos\theta,(R+r\cos\phi)\operatorname{sen}\theta, r\operatorname{sen}\phi).$$

It is easy to check that if $R>r>0$, then Φ is an injective immersion whose image is the subset of $\mathbb{R}^3$:

$$\left\{(x,y,z)\in\mathbb{R}^3 : (x^2+y^2+z^2-R^2-r^2)^2+4R^2z^2=4R^2r^2\right\}=\Psi^{-1}(c),$$

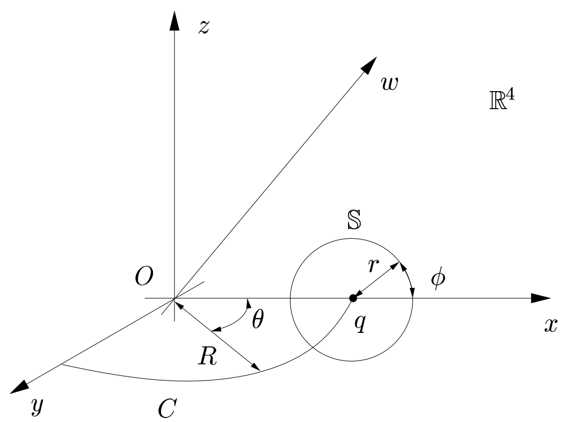


where $c = 4R^2r^2$ *and* $\Psi : \mathbb{R}^3 \to \mathbb{R}$ *is the smooth map*

$$\Psi(x, y, z) = (x^2 + y^2 + z^2 - R^2 - r^2)^2 + 4R^2z^2.$$

We leave it as an exercise to check that every $c \neq 0$ *is a regular value of* Ψ*, so this gives an embedding of* $\mathbb{S}^1 \times \mathbb{S}^1$ *in* $\mathbb{R}^3$*.*

3. *The* ***Klein bottle*** *is the subset* $K \subset \mathbb{R}^4$ *defined as follows: Let* Ox*,* Oy*,* Oz*, and* Ow*, be the coordinate axes in* $\mathbb{R}^4$ *and denote by* C *a circle of radius* R *in the plane* xOy*. Let* θ *be the angle coordinate on this circle (say, measured from the* Ox*-axis).*

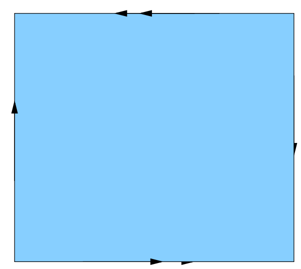


If $\mathbb{S}^1$ *is a circle of radius* r *in the plane* xOz*, with centre at* $q \in C$*, then* K *is the figure obtained by rotating this circle around the* Oz *axis so that when its center* $q \in C$ *as rotated an angle* θ*, the plane where* $\mathbb{S}^1$ *lies has rotated an angle* $\theta/2$ *around the* Oq*-axis in the 3-space* $OqOzOw$*. Let* ϕ *be the angle coordinate in the circle* $\mathbb{S}^1$ *(say, measured from the* Oq*-axis).*

Note that the points of K *with* $\theta \neq 0$ *and* $\phi \neq 0$ *can be parameterized by:* $\Phi_1 :]0, 2\pi[\times]0, 2\pi[\to \mathbb{R}^4$*:*

$$\Phi_1(\theta, \phi) = ((R + r\cos\phi)\cos\theta, (R + r\cos\phi)\operatorname{sen}\theta, r\operatorname{sen}\phi\cos\theta/2, r\operatorname{sen}\phi\operatorname{sen}\theta/2).$$

We can change the origin of θ *and* ϕ*, obtaining new parameterizations, which all together cover* K*. We leave it as an exercise to show that 3 parameterizations* Φ_1*,* Φ_2 *and* Φ_3 *are enough to cover* K*. Since for these parameterizations* *the transitions $\Phi_i \circ \Phi_j^{-1}$ are C^∞, we see that K is a 2-surface in $\mathbb{R}^4$. Also, we remark that these parameterizations amount to think of K as a square of side 2π where we identify the sides of the square, as in the following figure: Just*



like for the 2-torus, one checks that K is given by:

$$K = \Psi^{-1}(c, 0),$$

where $c = 4R^2r^2$ and $\Psi : \mathbb{R}^4 \to \mathbb{R}^2$ is the smooth map

$$\Psi(x, y, z) = ((x^2 + y^2 + z^2 + w^2 - R^2 - r^2)^2 + 4R^2(z^2 + w^2), y(z^2 - w^2) - 2xzw).$$

For $c \neq 0$, one checks that $(c, 0)$ is a regular value of Ψ, so we conclude that K is an embedded submanifold of $\mathbb{R}^4$.

Actually, any manifold can always be embedded in a Euclidean space of large enough dimension.

Theorem 6.9 (Whitney)**.** *Let M be a compact manifold. There exists an injective embedding $\Psi : M \to \mathbb{R}^m$, for some integer m.*

Proof. Since M is compact, we can find a finite collection of coordinate systems $\{(U_i, \phi_i) : i = 1, \dots, N\}$ such that:
(a) $\overline{B_1(0)} \subset \phi_i(U_i) \subset B_2(0)$;
(b) $\bigcup_{i=1}^N \phi_i^{-1}(B_1(0)) = M$.
Let $\lambda_i : M \to \mathbb{R}$, $i = 1, \dots, N$, be smooth functions such that

$$\lambda_i(p) = \begin{cases} 1 \text{ if } p \in \phi_i^{-1}(B_1(0)), \\ \\ 0 \text{ if } p \notin U_i. \end{cases}$$

Also, let $\psi_i : M \to \mathbb{R}^d$, $i = 1, \dots, N$, be smooth maps defined by:

$$\psi_i(p) = \begin{cases} \lambda_i \phi_i(p) \text{ if } p \in U_i, \\ \\ 0 \text{ if } p \notin U_i. \end{cases}$$

We claim that the smooth map $\Phi : M \to \mathbb{R}^{Nd+N}$ defined by:

$$\Phi(p) = (\psi_1(p), \lambda_1(p), \dots, \psi_N(p), \lambda_N(p))$$

is the desired embedding. In fact, we have that

(i) Φ is an immersion: if $p \in M$ then $p \in \phi_i^{-1}(B_1(0))$, for some i. Hence, we have that $\psi_i = \phi_i$ in a neighborhood p. We conclude that $\mathrm{d}_p\psi_i = \mathrm{d}_p\phi_i$ is injective. This shows that $\mathrm{d}_p\Phi$ is injective.

(ii) Φ is injective: Let $p, q \in M$, $p \neq q$, and choose i such that $p \in \lambda_i^{-1}(1)$. If $q \notin \lambda_i^{-1}(1)$, then $\lambda_i(p) \neq \lambda_i(q)$ so that $\Phi(p) \neq \Phi(q)$. On the other hand, if $q \in \lambda_i^{-1}(1)$, then $\psi_i(p) = \phi_i(p) \neq \phi_i(q) = \psi_i(q)$, since ϕ_i is injective. In any case, $\Phi(p) \neq \Phi(q)$, so Φ is injective.

Since M is compact, we conclude that Φ is an embedding.

$\square$

The previous result also holds for non-compact manifolds (see the exercises in this Lecture) and is valid also for manifolds with boundary.

This result is a weaker version of a famous theorem of Whitney: he showed that any smooth manifold (compact or not) of dimension d can be *embedded* in $\mathbb{R}^{2d}$. Note that there are smooth manifolds of dimension d which cannot be embedded in $\mathbb{R}^{2d-1}$ (e.g., the Klein bottle). On the other hand, for $d > 1$, Whitney also showed that any manifold of dimension d can be *immersed* in $\mathbb{R}^{2d-1}$. However, these results are not the best possible: Ralph Cohen in 1985 showed that a compact manifold of dimension d can be *immersed* in $\mathbb{R}^{2d-a(d)}$ where $a(d)$ is the number of 1's in the binary expression of d, and this is the best possible!! (e.g., every compact 5-manifold can immersed in $\mathbb{R}^8$, but there are compact 5-manifolds which cannot be immersed in $\mathbb{R}^7$). On the other hand, the best optimal *embedding* dimension is only known for a few dimensions.

Homework.

1. Consider the following sets of $n \times n$ matrices:
   - $O(n) = \left\{A : AA^T = I\right\}$ (orthogonal matrices);
   - $S(n) = \left\{A : A = A^T\right\}$ (symmetric matrices).

   Show that $O(n)$ and $S(n)$ are embedded submanifolds of the space $\mathbb{R}^{n^2}$ of all $n \times n$ matrices and check that they intersect transversely at I. Use this to conclude that there is a neighborhood of I where the only $n \times n$-matrix which is both orthogonal and symmetric is I itself.

2. Let $\Phi : \mathbb{P}^2 \to \mathbb{R}^4$ be the smooth map defined by
$$\Phi([x, y, z]) = \frac{1}{x^2 + y^2 + z^2}(x^2 - z^2, yz, xz, xy).$$
   Show that $(\mathbb{P}^2, \Phi)$ is an embedded submanifold in $\mathbb{R}^4$.

3. Furnish the details of the example of the Klein bottle K and show that K is a 2-surface in $\mathbb{R}^4$.

4. Let $\Psi : M \to N$ be a smooth map and let $q \in N$ be a regular value of Ψ. Show that
$$T_p\Psi^{-1}(q) = \{\mathbf{v} \in T_pM : \mathrm{d}_p\Psi \cdot \mathbf{v} = 0\}.$$

5. Let $\Psi : M \to N$ be a smooth map which is transversal to a submanifold $Q \subset N$ (not necessarily embedded). Show that $\Psi^{-1}(Q)$ is a submanifold of M (not necessarily embedded) and that

$$T_p\Psi^{-1}(Q) = \left\{ \mathbf{v} \in T_pM : \mathrm{d}_p\Psi \cdot \mathbf{v} \in T_{\Psi(p)}Q \right\}.$$

6. Let M and N be smooth manifolds and let $S \subset M \times N$ be a submanifold. Denote by $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ the projections on each factor. Show that the following are equivalent:
(a) S is the graph of a smooth map $\Phi : M \to N$;
(b) $\pi_M|_S$ is a diffeomorphism from S onto M;
(c) For each $p \in M$, the submanifolds S and $\{p\} \times N = \pi_M^{-1}(p)$ intersect transversely and the intersection consists of a single point.
Moreover, if any of these hold then S is an embedded submanifold.

7. Extend Theorem 6.5 to the case where $\Psi : M \to N$ is a smooth map between manifolds with boundary such that $\Psi(\partial M) = \partial N$. Show that the conclusion of the theorem may fail if this last condition is omitted.

The next sequence of exercises give a sketch of the proof of the weak Whitney's Embedding Theorem for non-compact manifolds.

8. Proof the following week version of **Sard's Theorem**: If $\Psi : M \to N$ is a smooth map between manifolds of the same dimension, then the set of singular values of Ψ has measure zero.

Note: The general Sard's Theorem states that for any smooth map Ψ the set of singular values has measure zero.

9. Using Sard's Theorem, show that if $\Phi : M \to N$ is a smooth map between smooth manifolds and $\dim M < \dim N$ then $\Phi(M)$ has measure zero.

10. Let $M \subset \mathbb{R}^n$ be a smooth submanifold of dimension d. Given $v \in \mathbb{R}^n - \mathbb{R}^{n-1}$ denote by $\pi_v : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the linear projection with kernel $\mathbb{R}v$. Show that if $n > 2d + 1$ there is a dense set of vectors $v \in \mathbb{R}^n - \mathbb{R}^{n-1}$ for which $\pi_v|_M$ is an injective immersion of M in $\mathbb{R}^{n-1}$. Conclude that any compact manifold with boundary of dimension d can be embedded in $\mathbb{R}^{2d+1}$.

Hint: Check that the proof given in the text of Whitney's embedding theorem is valid for compact manifolds with boundary. Then apply Sard's theorem in a clever way.

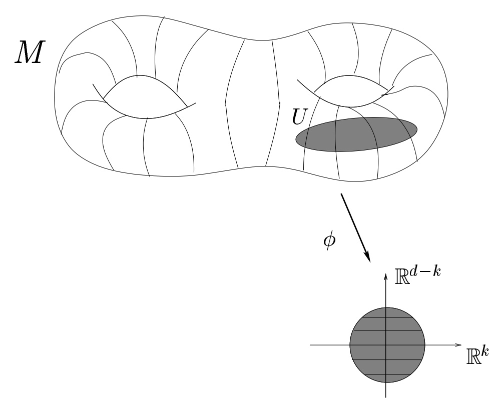
11. Using a smooth exhaustion function, show that any smooth manifold M of dimension d can be embedded in $\mathbb{R}^{2d+1}$.

Hint: If $f : M \to \mathbb{R}$ is a smooth exhaustion function, then by Sard's Theorem, in each interval $[i, i+1[$, the function f has a regular value a_i. It follows that the sets $E_0 = f^{-1}(]-\infty, a_2]$, $E_i = f^{-1}([a_{i-1}, a_{i+1}]$ $(i = 1, 2, \dots)$, are all compact submanifolds of M of dimension d to which the previous result can be applied. Now use a partition of unity to build an embedding of M in $\mathbb{R}^{2d+1}$.

A foliation is a nice decomposition of a manifold into submanifolds:

Definition 7.1. *Let M be a manifold of dimension d. A **foliation** of dimension k of M is a decomposition $\{L_\alpha : \alpha \in A\}$ of M into disjoint path-connected subsets satisfying the following property: for any $p \in M$ there exists a smooth chart $\phi = (x^1, \dots, x^k, y^1, \dots, y^{d-k}) : U \to \mathbb{R}^d = \mathbb{R}^k \times \mathbb{R}^{d-k}$, such the the connected components of $L_\alpha \cap U$ are the sets of*

$$\{p \in U : y^1(p) = const., \dots, y^{d-k}(p) = const.\}.$$



We will denote a foliation by $\mathcal{F} = \{L_\alpha : \alpha \in A\}$. The connected sets L_α are called **leaves** of $\mathcal{F}$ and a chart (U, ϕ) as in the definition is called an **foliated coordinate chart**. The connected components of $U \cap L_\alpha$ are called **plaques**.

A **path of plaques** is a collection of plaques $P_1, \dots, P_l$ such that $P_i \cap P_{i+1} \neq \emptyset$, for all $i = 1, \dots, l-1$. The integer l is called the **length of the path of plaques**. Two points $p, q \in M$ belong to the same leaf if and only if there exists a path of plaques $P_1, \dots, P_l$, with $p \in P_1$ and $q \in P_l$.

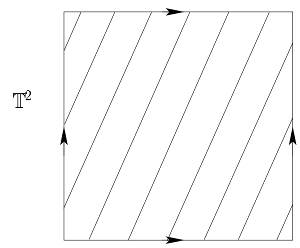
Each leaf of a k-dimensional foliation of M is a submanifold of M of dimension k. In general, these are only immersed submanifolds: a leaf can intersect a foliated coordinate chart an infinite number of times and accumulate overt itself. Before we check that leaves are submanifolds, let us look at some examples.

EXAMPLES 7.2.

1. *Let $\Phi : M \to N$ be a submersion. By the local normal form for submersions, the connected components of the fibers $\Phi^{-1}(q)$, where $q \in N$, form a foliation of M of codimension equal to the dimension of N. In this case, all leaves are actually embedded submanifolds.*
2. *In $\mathbb{R}^2$, take the foliation by straight lines with a fixed slope $a \in \mathbb{R}$. This is just a special case of the previous example, where $\Phi : \mathbb{R}^2 \to \mathbb{R}$, is given by:*

$$\Phi(x, y) = y - ax.$$

Now let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the torus. Then we have an induced foliation on $\mathbb{T}^2$, and there are two possibilities. If $a \in \mathbb{Q}$, the leaves are closed curves, hence they are embedded submanifolds. However, if $a \notin \mathbb{Q}$, then the leaves are dense in the torus, so they are only immersed submanifolds.



3. Let $\Phi : \mathbb{R}^3 \to \mathbb{R}$ be the smooth map defined by

$$\Phi(x, y, z) = f(x^2 + y^2)e^z,$$

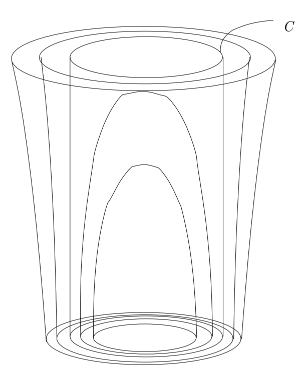
where $f \in C^\infty(\mathbb{R})$ is a smooth function with $f(0) = -1$, $f(1) = 0$ and $f'(t) > 0$. It is easy to check that Φ is a submersion and so determines a foliation $\mathcal{F}$ of $\mathbb{R}^3$ whose leaves are the pre-images $\{\Phi^{-1}(c)\}_{c\in\mathbb{R}}$. When $c = 0$ we obtain as leaf the cylinder $C = \{(x, y, z) : x^2 + y^2 = 1\}$. This cylinder splits the leaves into two classes:

- *The leaves with $c < 0$ lying in the interior of the cylinder C, which are all diffeomorphic to $\mathbb{R}^2$;*
- *The leaves with $c > 0$ lying in the exterior of the cylinder C which are all diffeomorphic to C;*

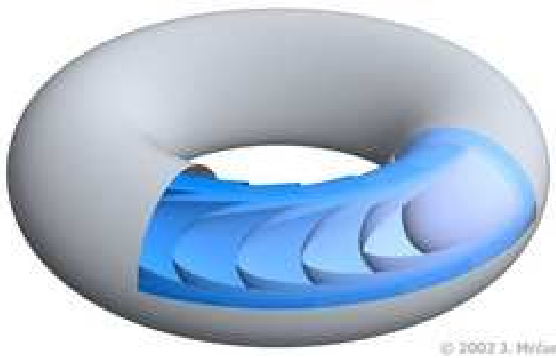
An explicit parameterization of the leaves with $c \neq 0$ is given by:

$$(x, y) \mapsto (x, y, \log(c/f(x^2 + y^2))).$$

For the first type of leaves, $c < 0$ and $x^2 + y^2 < 1$, while for the second type of leaves $c > 0$ and $x^2 + y^2 > 1$.



4. The foliation in the previous example is invariant under translations in the Oz-axis direction. If we identify $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$, we obtain a foliation in the quotient $\mathbb{R}^2 \times \mathbb{S}^1 = \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$. If we restrict this foliation to $\operatorname{Int} D^2 \times \mathbb{S}^1$, where $D^2 = \{(x, y) : x^2 + y^2 \leq 1\}$, we obtain a foliation of the solid 2-torus. This example suggests that foliations of manifolds with boundary are



also interesting. We will not pursue this topic, but you should be aware of the existence of foliations on manifolds with boundary.

5. The 3-sphere $\mathbb{S}^3$ can be obtained by "gluing" two solid 2-torus along its boundary:

$$\mathbb{S}^3 = T_1 \cup_\Phi T_2,$$

where $\Phi : \partial T_1 \to \partial T_2$ is a diffeomorphism that takes the meridians of ∂T_1 in the circles of latitude of ∂T_2, and vice-versa. Explicitly, if $\mathbb{S}^3 = \{(x, y, z, w) : x^2 + y^2 + z^2 + w^2 = 1\}$, then we can take:

$$T_1 = \{(x, y, z, w) \in \mathbb{S}^3 : x^2 + y^2 \leq 1/2\},$$
$$T_2 = \{(x, y, z, w) \in \mathbb{S}^3 : x^2 + y^2 \geq 1/2\}.$$

Each of these solid 2-torus admits a 2-dimensional foliation as in the previous example. One then obtains a famous 2-dimensional foliation of the sphere $\mathbb{S}^3$, called the ***Reeb foliation*** *of $\mathbb{S}^3$.*

Proposition 7.3. *Let $\mathcal{F}$ be a k-dimensional foliation of a smooth manifold M. Every leaf $L \in \mathcal{F}$ is a initial submanifold of dimension k.*

Proof. Let L be a leaf of $\mathcal{F}$. The topology of L is the topology generated by the plaques of L, i.e., the connected components of $L \cap U$, where U the domain of a foliated chart. For each plaque P, associated with a foliated chart $(U, \phi) = (U, x^1, \ldots, x^k, y^1, \ldots, y^{d-k})$, we consider the map $\psi : P \to \mathbb{R}^k$ obtain by choosing the first k-components:

$$\psi(p) = (x^1(p), \ldots, x^k(p)).$$

The pairs (P, ϕ) give charts for L, which shows that L is a Hausdorff topological manifold. The transition functions for these charts are clearly as C^∞, so we can consider the maximal atlas that contains all the charts (U,ψ). To check that L is a manifold, we only need to check that the topology admits a countable basis. For that we use the following lemma:

Lemma 7.4. *Let L be a leaf of $\mathcal{F}$ and $\{U_n : n \in \mathbb{Z}\}$ a covering of M by domains of foliated charts. The number of plaques of L in this covering, i.e., the number of connected components of $L \cap U_n$, $n \in \mathbb{Z}$, is countable.*

Fix a plaque P_0 of L in the covering $\{U_n : n \in \mathbb{Z}\}$. If a plaque P' belongs to L then there exists a **path of plaques** $P_1, \dots, P_l$ in the covering, with $P_i \cap P_{i+1} \neq \emptyset$ which connects P' to P_0. Therefore it is enough to check that the collection of such paths is countable.

For each path of plaques $P_1, \dots, P_l$ let us call l the length of the path. Using induction on n, we show that the collection of paths of length less or equal to n is countable:

- The collection of paths of length 1 has only one element hence is countable.
- Assume that the collection of paths of length $n-1$ is countable. Let $P_1, \dots, P_{n-1}$ be a path of length $n-1$, corresponding to domains of foliated charts $U_1, \dots, U_{n-1}$. In order to obtain a path of plaques of length n, we choose a domain of a foliated chart $U_n \neq U_{n-1}$ and we consider the plaques P', which are connected components of $L \cap U_n$, such that the intersection with P_{n-1} is non-empty. Now observe that:
$$(L \cap U_n) \cap P_{n-1} = U_n \cap P_{n-1},$$
intersections form an open cover of the plaque P_{n-1}. This cover has a countable subcover, so the collection of all such P' is countable. It follows that the collection of paths of length less or equal that n is countable.

We leave it as an exercise to check that the leaves are actually initial submanifolds. □

Corollary 7.5. *Each leaf of a foliation intersects the domain of a foliated chart at most a countable number of times.*

There are few constructions which allows one to obtain new foliations out of other foliations. The details of these constructions are left for the exercises.

Product of foliations. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be foliations of M_1 and M_2, respectively. Then the **product foliation** $\mathcal{F}_1 \times \mathcal{F}_2$ is a foliation of $M_1 \times M_2$ defined as follows: if $\mathcal{F}_1 = \{L^{(1)}_\alpha\}_{\alpha\in A}$ and $\mathcal{F}_2 = \{L^{(2)}_\beta\}_{\beta\in B}$, then
$$\mathcal{F}_1 \times \mathcal{F}_2 = \{L^{(1)}_\alpha \times L^{(2)}_\beta\}_{(\alpha,\beta)\in A\times B}.$$
It should be clear that $\dim(\mathcal{F}_1 \times \mathcal{F}_2) = \dim \mathcal{F}_1 + \dim \mathcal{F}_2$ and, hence, that $\operatorname{codim}(\mathcal{F}_1 \times \mathcal{F}_2) = \operatorname{codim} \mathcal{F}_1 + \operatorname{codim} \mathcal{F}_2$

Pull-back of a foliation. Let $\Phi : M \to N$ be a smooth map between smooth manifolds. If $\mathcal{F}$ is a foliation of N we will say that Φ **is transversal to** $\mathcal{F}$ and write $\Phi \pitchfork \mathcal{F}$ if Φ is transversal to every leaf L of $\mathcal{F}$:

$$\mathrm{d}_p\Phi(T_pM) + T_{\Phi(p)}L = T_{\Phi(p)}N, \quad \forall p \in L.$$

Whenever $\Phi \pitchfork \mathcal{F}$ one defines the **pull-back foliation** $\Phi^*(\mathcal{F})$ to be the foliation of M whose leaves are the connected components of $\Phi^{-1}(L)$, where $L \in \mathcal{F}$. It should be clear that $\operatorname{codim} \Phi^*(\mathcal{F}) = \operatorname{codim} \mathcal{F}$.

Suspension of a difeomorphism. The manifold $\mathbb{R} \times M$ has a foliation $\mathcal{F}$ of of dimension 1: the leaves are the sets $\mathbb{R} \times \{p\}$, where $p \in M$ (or if your prefer, the fibers of the projection $\pi : \mathbb{R} \times M \to M$). A difeomorphism $\Phi : M \to M$ induces an action of $\mathbb{Z}$ on $\mathbb{R} \times M$ by setting

$$n \cdot (t, p) = (t + n, \Phi^n(p)).$$

This action takes leaves of $\mathcal{F}$ into leaves of $\mathcal{F}$. The quotient $N = (\mathbb{R} \times M)/\mathbb{Z}$ is a manifold and has an induced foliation $\tilde{\mathcal{F}}$ whose leaves are the equivalence classes $[L]$ in N, where $L \in \mathcal{F}$. This foliation is called the **suspension of the diffemorphism** Φ.

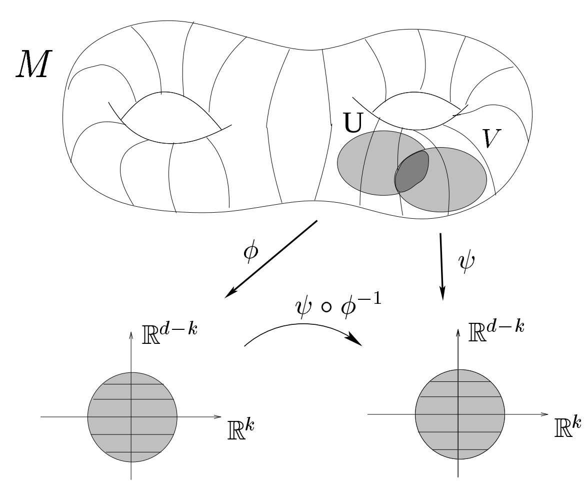
It is convenient to have alternative characterizations of foliations.

Foliations via smooth $\mathcal{G}^k_d$-structures. Let $\mathcal{F} = \{L_\alpha : \alpha \in A\}$ be a k-dimensional foliation of M. If (U, ϕ) and (V, ψ) are foliated charts then the change of coordinates $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is of the form:

$$\mathbb{R}^k \times \mathbb{R}^{d-k} \ni (x, y) \mapsto (h_1(x, y), h_2(y)) \in \mathbb{R}^k \times \mathbb{R}^{d-k}.$$

In other words, we have that the transition functions satisfy:

$$\frac{\partial(\psi \circ \phi^{-1})^j}{\partial x^i} = 0, \quad (i = 1, \ldots, k, \ j = k+1, \ldots, d). \tag{7.1}$$



Conversely, denote by $\mathcal{G}_d^k$ the local diffeomorphisms $\mathbb{R}^d \to \mathbb{R}^d$ that satisfy this condition. We can refine the notion of smooth structure by requiring that in Definition 1.3 the transition functions belong to $\mathcal{G}_d^k$, and we then speak of a **smooth $\mathcal{G}_d^k$-structure**. An ordinary smooth structure on M is just a $\mathcal{G}_d^d$-structure: the leaves are the connected components of M.

We have the following alternative description of a foliation:

Proposition 7.6. *Let M be a smooth d-dimensional manifold. Given a foliation $\mathcal{F} = \{L_\alpha : \alpha \in A\}$ of M of dimension k the collection of all foliated charts $\mathcal{C} = \{(U, \phi)\}$ defines a smooth $\mathcal{G}_d^k$-structure. Conversely, for every smooth $\mathcal{G}_d^k$-structure $\mathcal{C}$ on a topological space M, there is smooth structure that makes M into a d-dimensional manifold and there exists a foliation $\mathcal{F}$ of M of dimension k, for which the foliated charts are the elements of $\mathcal{C}$.*

Proof. We have shown above that every k-dimensional foliation of a d-dimensional manifold determines a smooth $\mathcal{G}_d^k$-structure. We will show that, conversely, given a smooth $\mathcal{G}_d^k$-structure $\mathcal{C} = \{(U, \phi)\}$ we can associate to it a smooth structure on M of dimension d and a k-dimensional foliation $\mathcal{F}$ of M.

It should be clear that a smooth $\mathcal{G}_d^k$-structure $\mathcal{C} = \{(U, \phi)\}$ determines a smooth structure on M of dimension d, since it is in particular an atlas. In order to build $\mathcal{F}$, we consider the sets $\phi^{-1}(\mathbb{R}^k \times \{c\})$, where $c \in \mathbb{R}^{d-k}$, which we call *plaques*. Since M is covered by all such plaques, we can define an equivalence relation in M by:

- $p \sim q$ if there exists a path of plaques $P_1, \ldots, P_l$ with $p \in P_1$ and $q \in P_l$.

Let $\mathcal{F}$ be the set of equivalence classes of $\sim$. We will show that $\mathcal{F}$ is a foliation of M.

Let $p_0 \in M$ and consider a plaque P_0 which contains p_0. Then

$$P_0 = \phi^{-1}(\mathbb{R}^k \times \{c_0\}),$$

for some smooth chart $(U, \phi) \in \mathcal{C}$ with $\phi(p_0) = (a_0, c_0) \in \mathbb{R}^k \times \mathbb{R}^{d-k}$. We claim that (U, ϕ) is a foliated chart: let $L \in \mathcal{F}$ be an equivalence class that intersects U. If $p \in U \cap L$, then $\phi(p) = (a, c) \in \mathbb{R}^k \times \mathbb{R}^{d-k}$, so we see that that the plaque

$$P = \phi^{-1}(\mathbb{R}^k \times \{c\}),$$

is contained in L. Since P is connected, it is clear that P is contained in the connected component of $L \cap U$ that contains p. We claim that this connected component is actually P, from which it will follow that (U, ϕ) is a foliated chart.

Let $q \in L \cap U$ be some point in the connected component of $L \cap U$ containing p. We claim that $q \in P$. By the definition of $\sim$, there exists a path of plaques $P_1, \ldots, P_l$, with $p \in P_1$ and $q \in P_l$, and such that $P_i \subset U$. Each plaque P_i is associated to a smooth chart $(U_i, \phi_i) \in \mathcal{C}$ such that

$$P_i = \phi_i^{-1}(\mathbb{R}^k \times \{c_i\}).$$

We can assume also that $U_1 = U$, $\phi_1 = \phi$, $P_1 = P$ and $c_1 = c$. Since $\phi_2 \circ \phi^{-1} \in \mathcal{G}_d^k$, we have that:

$$\phi_2^{-1}(\mathbb{R}^k \times \{c_2\}) \subset \phi_2^{-1} \circ \phi_2 \circ \phi^{-1} \circ (\mathbb{R}^k \times \{\bar{c}_2\}) = \phi^{-1}(\mathbb{R}^k \times \{\bar{c}_2\}),$$

for some $\bar{c}_2 \in \mathbb{R}^{d-k}$. Since $P_2 \cap P_1 \neq \emptyset$ and the plaques $\phi^{-1} \circ (\mathbb{R}^k \times \{c\})$ are disjoint, we conclude that $\bar{c}_2 = c_1$ and $P_2 \subset P_1 = P$. By induction, $P_i \subset P$ so $q \in P$, as claimed. $\square$

Foliations via Haefliger cocycles. We saw before that the connected components of the fibers of a submersion is an example of a foliation. Actually, every foliation is locally of this form: if $\mathcal{F} = \{L_\alpha\}_{\alpha \in A}$ is a foliation of M of dimension k, for any foliated chart:

$$\phi = (x^1, \ldots, x^k, y^1, \ldots, y^{d-k}) : U \to \mathbb{R}^d,$$

the projection in the last $(d-k)$-components gives a submersion:

$$\psi = (y^1, \ldots, y^{d-k}) : U \to \mathbb{R}^{d-k},$$

whose fibers are the connected components of $L_\alpha \cap U$. Given another foliated chart:

$$\bar{\phi} = (\bar{x}^1, \ldots, \bar{x}^k, \bar{y}^1, \ldots, \bar{y}^{d-k}) : \bar{U} \to \mathbb{R}^d,$$

with $U \cap \bar{U} \neq \emptyset$, for the corresponding submersion

$$\bar{\psi} = (\bar{y}^1, \ldots, \bar{y}^{d-k}) : \bar{U} \to \mathbb{R}^{d-k},$$

we have a change of coordinates of the form

$$\bar{\phi} \circ \phi^{-1}(x, y) = (h_1(x, y), h_2(y)),$$

where h_2 has Jacobian matrix

$$\left[\frac{\partial h_2^j}{\partial y^i}\right]_{i,j=1}^{d-k}$$

with rank $d-k$. We conclude that the submersions ψ and $\bar{\psi}$ differ by a local diffeomorphism: for every $p \in U \cap \bar{U}$ there exists an open neighborhood $p \in U_p \subset U \cap \bar{U}$ and a local diffeomorphism $\Psi : \mathbb{R}^{d-k} \to \mathbb{R}^{d-k}$, such that:

$$\bar{\psi}|_{U_p} = \Psi \circ \psi|_{U_p}.$$

This suggests another way of defining foliations:

Proposition 7.7. *Let M be a d-dimensional manifold. Every k-dimensional foliation $\mathcal{F}$ of M determines a maximal collection $\{\psi_i\}_{i \in I}$ of submersions $\psi_i : U_i \to \mathbb{R}^{d-k}$, where $\{U_i\}_{i \in I}$ is an open cover of M, and which satisfies the following property: for every $i, j \in I$ and $p \in U_i \cap U_j$, there exists a local diffeomorphism ψ_{ji}^p de $\mathbb{R}^{d-k}$, such that:*

$$\psi_j = \psi_{ji}^p \circ \psi_i,$$

in an open neighborhood U_p of p. Conversely, every such collection determines a foliation of M.

We have already seen how to a foliation we can associate a collection of submersions. We leave it as an exercise to prove the converse.

Given a collection of submersions $\{\psi_i\}_{i\in I}$, as in the proposition, we consider for each pair $i,j\in I$, the map

$$\psi_{ij}:U_i\cap U_j\to \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^{d-k}),\ p\longmapsto \psi_{ij}^p.$$

This map satisfies:

$$(\psi_{ji})^{-1}=\psi_{ji}\text{ em }U_i\cap U_j, \tag{7.2}$$

and the *cocycle condition*:

$$\psi_{ij}\circ\psi_{jk}\circ\psi_{ki}=1\text{ in }U_i\cap U_j\cap U_k. \tag{7.3}$$

We will see later, in Part IV of these notes, when we study the theory of fiber bundles, that these cocycles, called **Haefliger cocycles**, play a very important role.

Foliations appear naturally in many problems in differential geometry, and we shall see many other examples of foliations during the course of these lectures.

Homework.

1. Show that the leaves of a foliation are regularly immersed submanifolds.

2. Let $\mathcal{F}$ be the Reeb foliation of S^3 and let $\Phi:S^3\to N$ be a continuous map whose restriction to each leaf $\mathcal{F}$ is constant. Show that Φ is constant.

3. Proof Proposition 7.7.

4. Let $\mathcal{F}_1=\{L^{(1)}_\alpha\}_{\alpha\in A}$ and $\mathcal{F}_2=\{L^{(2)}_\beta\}_{\beta\in B}$ be foliations. Using your favorite definition of a foliation, show that the product $\mathcal{F}_1\times\mathcal{F}_2$ is a foliation:
$$\mathcal{F}_1\times\mathcal{F}_2:=\{L^{(1)}_\alpha\times L^{(2)}_\beta\}_{(\alpha,\beta)\in A\times B}.$$

5. Let $\Phi:M\to N$ be a smooth map and let $\mathcal{F}=\{L_\alpha\}_{\alpha\in A}$ be a foliation of N. Using your favorite definition of a foliation, show that the pull-back $\Phi^*(\mathcal{F})$ is a foliation:
$$\Phi^*(\mathcal{F}):=\{\text{connected components of }\Phi^{-1}(L_\alpha)\}_{\alpha\in A}.$$

6. Let $\mathcal{F}_1$ e $\mathcal{F}_2$ be two foliations of a smooth manifold M such that $\mathcal{F}_1\pitchfork\mathcal{F}_2$, i.e., such that
$$T_pM=T_pL^{(1)}+T_pL^{(2)},\qquad \forall p\in M,$$
where $L^{(1)}$ and $L^{(2)}$ are the leaves of $\mathcal{F}_1$ and $\mathcal{F}_2$ through p. Show that there exists a foliation $\mathcal{F}_1\cap\mathcal{F}_2$ of M whose leaves are the connected components of $L^{(1)}_\alpha\cap L^{(2)}_\beta$, and which satisfies $\operatorname{codim}\mathcal{F}=\operatorname{codim}\mathcal{F}_1+\operatorname{codim}\mathcal{F}_2$.

7. Given a foliation $\mathcal{F}$ of M, one denotes by $M/\mathcal{F}$ the space of leaves of $\mathcal{F}$ with the quotient topology. Try to describe for each of the examples given in the text their space of leaves.

Lecture 8. Quotients

We have seen before several constructions that produce new manifolds out of old manifolds, such as the product of manifolds or the pullback of sub manifolds under transversal maps. We will now study another important, but more delicate, construction: forming quotients of manifolds.

Let X be a topological space. If $\sim$ is an equivalence relation on X, we will denote by $X/\sim$ the set of equivalence classes of $\sim$ and by $\pi : X \to X/\sim$ the *quotient map* which associates to each $x \in X$ its equivalence class $\pi(x) = [x]$. In $X/\sim$ we consider the *quotient topology*: a subset $V \subset X/\sim$ is open if and only if $\pi^{-1}(V)$ is open. This is the largest topology in $X/\sim$ for which the quotient map $\pi : M \to M/\sim$ is continuous. We have the following basic result about the quotient topology which we leave as an exercise:

Lemma 8.1. *Let X be a Hausdorff topological space and let $\sim$ be an equivalence relation on X such that $\pi : X \to X/\sim$ is an open map. Then $X/\sim$ is Hausdorff if and only if the graph of $\sim$:*

$$R = \{(x, y) \in X \times X : x \sim y\},$$

is a closed subset of $X \times X$.

Let M be a smooth manifold and let $\sim$ be an equivalence relation on M. We would like to known when there exists a smooth structure on $M/\sim$, compatible with the quotient topology, such that $\pi : M \to M/\sim$ becomes a submersion. Before we can state a result that gives a complete answer to this question, we need one definition.

Recall that a continuous map $\Phi : X \to Y$, between two Hausdorff topological spaces is called a **proper map** if $\Phi^{-1}(K) \subset X$ is compact whenever $K \subset Y$ is compact. A proper map is always a closed map.

Definition 8.2. *A **proper submanifold** of M is a submanifold (N, Φ) such that $\Phi : N \to M$ is a proper map.*

By an exercise in Lecture 5, any proper submanifold is an embedded submanifold. Also, if $\Phi : N \to M$ is proper, then its image $\Phi(N)$ is a closed subset of M. Conversely, every embedded closed submanifold of M is a proper submanifold.

Theorem 8.3. *Let M be a smooth manifold and let $\sim$ be an equivalence relation on M. The following statements are equivalent:*

(i) There exists a smooth structure on $M/\sim$, compatible with the quotient topology, such that $\pi : M \to M/\sim$ is a submersion.
(ii) The graph R of $\sim$ is a proper submanifold of $M \times M$ and the restriction of the projection $p_1 : M \times M \to M$ to R is a submersion.

$$\begin{tikzcd} R \arrow[r, hook] & M \times M \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ M & & M \end{tikzcd}$$

Proof. We must show both implications:

(i) $\Rightarrow$ (ii). The graph of the quotient map, as for every smooth map, is a closed embedded submanifold:

$$\mathcal{G}(\pi) = \{(p, \pi(p)) : p \in M\} \subset M \times M/\sim,$$

Since $I \times \pi : M \times M \to M \times M/\sim$ is a submersion and

$$R = (Id \times \pi)^{-1}(\mathcal{G}(\pi)),$$

we conclude that $R \subset M \times M$ is an embedded closed submanifold, i.e., is a proper submanifold.

On the other hand, the map $(I \times \pi)|_R : R \to \mathcal{G}(\pi)$ is a submersion while $\mathcal{G}(\pi) \to M$, $(p, \pi(p)) \mapsto p$ is a diffeomorphism, hence their composition $p_1|R$ is a submersion.

(ii) $\Rightarrow$ (i). We split the proof into several lemmas. The first of these lemmas states that we can "straighten out" $\sim$:

Lemma 8.4. *For every $p \in M$, there exists a local chart $(U, (x^1, \dots, x^d))$ centered at p, such that*

$$\forall q, q' \in U,\ q \sim q' \text{ if and only if } x^{k+1}(q) = x^{k+1}(q'), \dots, x^d(q) = x^d(q'),$$

where k is an integer independent of p and $d = \dim M$.

To prove this lemma, let $\Delta \subset M \times M$ be the diagonal. Note that $\Delta \subset R \subset M \times M$, and since Δ and R are both embedded submanifolds of $M \times M$, we have that Δ is an embedded submanifold of R. Therefore, for each $p \in M$, there exists a neighborhood O of (p, p) in $M \times M$ and a submersion $\Phi : O \to \mathbb{R}^{d-k}$, where $d - k = \operatorname{codim} R$, such that:

$$(q, q') \in O \cap R \text{ if and only if } \Phi(q, q') = 0.$$

We have that $k \geq 0$, since $\Delta \subset R$ and $\operatorname{codim} \Delta = d$.

Next we observe that the differential of the map $q \mapsto \Phi(q, p)$ has maximal rank at $q = p$: in fact, after identifying $T_{(p,p)}(M \times M) = T_pM \times T_pM$, we see that $\mathrm{d}_{(p,p)}\Phi$ is zero precisely in the subspace formed by pairs $(\mathbf{v}, \mathbf{v}) \in T_pM \times T_pM$, and this subspace is complementary to the subspace formed by elements of the form $(\mathbf{v}, 0) \in T_pM \times T_pM$. We conclude that there exists a neighborhood V' of p such that $V' \times V' \subset O$, and the map $q \mapsto \Phi(q, p)$ is a submersion in V'. By the local canonical form for submersions, there exist a chart $(V, \phi) = (V, (u^1, \dots, u^k, v^1, \dots, v^{d-k}))$ centered at p, with $V \subset V'$, such that

$$\Phi \circ (\phi^{-1} \times \phi^{-1})(u^1, \dots, u^k, v^1, \dots, v^{d-k}, 0, \dots, 0) = (v^1, \dots, v^{d-k}).$$

In the domain of this chart, the points $q \in V$ such that $q \sim p$ are precisely the points satisfying $v^1(q) = 0, \dots, v^{d-k}(q) = 0$.

Now set $\widehat{\Phi} = \Phi \circ (\phi^{-1} \times \phi^{-1})$. The smooth map

$$\mathbb{R}^d \times \mathbb{R}^{d-k} \to \mathbb{R}^{d-k},\ (u, v, w) \mapsto \widehat{\Phi}((u, v), (0, w)),$$

satisfies

$$\widehat{\Phi}((u,v),(0,0)) = v.$$

so the matrix of partial derivatives $\partial\widehat{\Phi}^i/\partial v^j$, $(i,j = 1,\ldots,d-k)$ is non-degenerate. We can apply the Implicit Function Theorem to conclude that there exists a local defined smooth function $\mathbb{R}^k \times \mathbb{R}^{d-k} \to \mathbb{R}^{d-k}$, $(u,w) \mapsto v(u,w)$, such that:

$$\widehat{\Phi}((u,v),(0,w)) = 0 \text{ if and only if } v = v(u,w).$$

Since $v(0,w) = w$ is a solution, uniqueness implies that:

$$\phi(0,w) \sim \phi(0,w') \text{ if and only if } w = w'.$$

This shows that the map $(u,w) \mapsto (u, v(u,w))$ is a local diffeomorphism. Hence, there exists an open set U where

$$(x^1,\ldots,x^d) = (u^1,\ldots,u^k,w^1,\ldots,w^{d-k})$$

are local coordinates and in these coordinates:

$$\forall q,q' \in U,\ q \sim q' \text{ if and only if } x^{k+1}(q) = x^{k+1}(q'),\ldots,x^d(q) = x^d(q'),$$

so the lemma follows.

Since the functions $x^{k+1},\ldots,x^d$ given by this lemma induce well-defined functions $\bar{x}^{k+1},\ldots,\bar{x}^d$ on the quotient $M/\sim$, we consider the pairs of the form $(\pi(U),\bar{x}^{k+1},\ldots,\bar{x}^d)$:

Lemma 8.5. *The collection $\mathcal{C} = \{(\pi(U),\bar{x}^{k+1},\ldots,\bar{x}^d)\}$ gives $M/\sim$, with the quotient topology, the structure of a topological manifold of dimension $d-k$.*

First note that $\pi : M \to M/\sim$ is an open map: in fact, for any $V \subset M$, we have that

$$\pi^{-1}(\pi(V)) = p_1|_R((p_2|_R)^{-1}(V)).$$

By assumption, $p_1|_R$ is a submersion hence is an open map. Therefore, if $V \subset M$ is open then $\pi^{-1}(\pi(V))$ is also open, so $\pi(V) \subset M/\sim$ is open.

This shows that $\pi(U)$ is open. Since the map

$$(x^{k+1},\ldots,x^d) : U \to \mathbb{R}^{d-k}$$

is both continuous and open, it follows that the induced map

$$(\bar{x}^{k+1},\ldots,\bar{x}^d) : \pi(U) \to \mathbb{R}^{d-k}$$

is continuous, open and injective, i.e., is a homeomorphism onto its image.

Now we show that:

Lemma 8.6. *The family $\mathcal{C} = \{(\pi(U),\bar{x}^{k+1},\ldots,\bar{x}^d)\}$ is an atlas generating a smooth structure for $M/\sim$ such that $\pi : M \to M/\sim$ is a submersion.*

Take two pairs in $\mathcal{C}$:

$$(\pi(U),\bar{\phi}) = (\pi(U),\bar{x}^{k+1},\ldots,\bar{x}^d),$$
$$(\pi(V),\bar{\psi}) = (\pi(V),\bar{y}^{k+1},\ldots,\bar{y}^d),$$

which correspond to two charts in M:

$$(U,\phi) = (U,x^1,\ldots,x^d),$$
$$(V,\psi) = (V,y^1,\ldots,y^d).$$

The corresponding transition function:

$$\bar{\psi}\circ\bar{\phi}^{-1}:\mathbb{R}^{d-k}\to\mathbb{R}^{d-k},$$

composed with the projection $p:\mathbb{R}^d\to\mathbb{R}^{d-k}$ in the last $d-k$ components is given by:

$$\bar{\psi}\circ\bar{\phi}^{-1}\circ p = p\circ\psi\circ\phi^{-1}.$$

Since the right-hand side is a smooth map $\mathbb{R}^d\to\mathbb{R}^{d-k}$ it follows that $\bar{\psi}\circ\bar{\phi}^{-1}$ is smooth.

In order to check that $\pi: M\to M/\sim$ is a submersion, it is enough to observe that in the charts $(U,x^1,\ldots,x^d)$ for M and $(\pi(U),\bar{x}^{k+1},\ldots,\bar{x}^d)$ for $M/\sim$, this map corresponds to the projection $p:\mathbb{R}^d\to\mathbb{R}^{d-k}$.

To finish the proof of Theorem 8.3, we check that

Lemma 8.7. *The quotient topology $M/\sim$ is Hausdorff and second countable.*

It is obvious that if M has a countable basis, then the quotient topology also has a countable basis. Since the graph R of $\sim$ is closed in $M\times M$, M is Hausdorff and π is an open map, it follows form Lemma 8.1 that $M/\sim$ is Hausdorff.

□

Remark 8.8. The proof above shows that if we assume that R is embedded, not closed, and $p_1|_R: R\to M$ is a submersion, then the quotient $M/\sim$ is a smooth manifold, second countable, but not Hausdorff (see Exercise 3 for an example).

The proof above also shows that, under the assumptions of the theorem, the equivalence classes of R form a foliation of M of codimension R which is a **simple foliation** (see Exercise 7).

A very important class of equivalence relations on manifolds is given by actions of groups of diffeomorphisms. If G is a group, we recall that an **action** of G on a set M is a group homomorphism $\widehat{\Psi}$ from G to the group of bijections of M. One can also view an action as a map $\Psi: G\times M\to M$, which we write as $(g,p)\mapsto g\cdot p$, if one sets:

$$g\cdot p\equiv\widehat{\Psi}(g)(p).$$

Since $\widehat{\Psi}$ is a group homomorphism, it follows that:

(a) $e \cdot p = p$, for all $p \in M$;
(b) $g \cdot (h \cdot p) = (gh) \cdot p$, for all $g, h \in G$ and $p \in M$.

Conversely, any map $\Psi : G \times M \to M$ satisfying (a) and (b), determines a homomorphism $\widehat{\Psi}$. From now on, we will denote an action by $\Psi : G \times M \to M$, and for each $g \in G$ we denote by Ψ_g the bijection:

$$\Psi_g : M \to M, \quad p \mapsto g \cdot p$$

Assume now that M is a manifold. We say that that a group G *acts on M by diffeomorphims* if, for each $g \in G$, $\Psi_g : M \to M$ is a diffeomorphism. This means that we have a group homomorfismo $\widehat{\Psi} : G \to \mathrm{Diff}(M)$, where $\mathrm{Diff}(M)$ is the group of all diffeomorphisms of M. We can also express this condition by saying that the map $\Psi : G \times M \to M$ is smooth, where G is viewed as a smooth 0-dimensional manifold with the discrete topology. So we will also say in this case that the *discrete group G acts smoothly on M*.

Given any action of G on M the quotient $G \backslash M$ is, by definition, the set of equivalence classes determined by the **orbit equivalence relation**:

$$p \sim q \iff \exists g \in G : \; q = g \cdot p.$$

Let us see conditions on an action by diffeomorphisms for the quotient $G \backslash M$ to be a manifold.

We recall that a **free action** is an action $G \times M \to M$ such that each $g \in G - \{e\}$ acts without fixed points, i.e.,

$$g \cdot p = p \text{ for some } p \in M \quad \Longrightarrow \quad g = e.$$

Denoting by G_p the **isotropy subgroup** of $p \in M$, i.e.,

$$G_p = \{g \in G : g \cdot p = p\},$$

an action is free if and only if $G_p = \{e\}$, for all $p \in M$.

Definition 8.9. *A smooth action $\Psi : G \times M \to M$ of a discrete group G on a smooth manifold M is said to be **proper** if the map:*

$$G \times M \to M \times M, \quad (g, p) \mapsto (g \cdot p, p),$$

is a proper map.

For example, actions of finite groups are always proper.

Corollary 8.10. *Let $\Psi : G \times M \to M$ be a free and proper smooth action of a discrete group G on M. There exists a unique smooth structure on $G \backslash M$ such that $\pi : M \to G \backslash M$ is a local diffeomorphism.*

Proof. We check that condition (ii) of Theorem 8.3 holds.

We claim that $R \subset M \times M$ is a proper submanifold. Since the action if free and proper, it follows (see Exercise 6) that for each $(p_0, g_0 \cdot p_0) \in R$, there exists an open set $p_0 \in U$, such that:

$$g \cdot U \cap U = \emptyset, \quad \forall g \in G - \{e\}.$$

It follows that
$$(U \times g_0 \cdot U) \cap R = \{(q, g_0 \cdot q) : q \in U\}.$$
so the map
$$U \to (U \times g_0 \cdot U) \cap R, \quad q \mapsto (q, g_0 \cdot q),$$
is a parameterization of $O \cap R$, with $O \subset M \times M$ open. It follows that R can be covered by open sets $O \cap R$ embedded in $M \times M$, so that R is an embedded submanifold. Also, the action being proper, the inclusion
$$R = \{(p, g \cdot p) : p \in M, g \in G\} \hookrightarrow M \times M$$
is a proper map.

Now we observe that the projection $p_1 : M \times M \to M$ restricted to R is an inverse to the parameterizations of R constructed above, hence $p_1|_R$ is a local diffeomorphism. $\square$

Under the conditions of this corollary, it is easy to check that the projection $\pi : M \to G\backslash M$ is in fact a covering map. Therefore, if M is simply connected, them M is the universal covering space of $G\backslash M$ and we conclude that $\pi_1(G\backslash M) \simeq G$.

Examples 8.11.

1. *Let $M = \mathbb{S}^d$, with $d > 1$. Consider the action of $\mathbb{Z}_2 \times \mathbb{S}^d \to \mathbb{S}^d$ defined by:*
$$\pm 1 \cdot (x^1, \dots, x^{d+1}) = \pm(x^1, \dots, x^{d+1}).$$
This action is free and proper. We conclude that the quotient $\mathbb{S}^d/\mathbb{Z}_2$ is a manifold. This manifold is diffeomorphic to $\mathbb{P}^d$: the map $\mathbb{S}^d \to \mathbb{P}^d$ given by $(x^1, \dots, x^d) \mapsto [x^1 : \dots : x^d]$ induces a diffeomorphism $\mathbb{S}^d/\mathbb{Z}_2 \to \mathbb{P}^d$ such that the following diagram commutes:
$$\begin{tikzcd} \mathbb{S}^d \arrow[d] \arrow[dr] & \\ \mathbb{S}^d/\mathbb{Z}_2 \arrow[r] & \mathbb{P}^d \end{tikzcd}$$
For $d > 1$, $\mathbb{S}^d$ is simply connected, so we conclude also that the quotient map is a covering map and that $\pi_1(\mathbb{P}^d) = \mathbb{Z}_2$.

2. *Let $\mathbb{Z}^d$ act on $\mathbb{R}^d$ by translations:*
$$(n_1, \dots, n_d) \cdot (x^1, \dots, x^d) = (x^1 + n_1, \dots, x^d + n_d).$$
This action is free and proper. It follows that the d-torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is a smooth manifold. This manifold is diffeomorphic to $\mathbb{T}^d$: the map $\mathbb{R}^d \to \mathbb{T}^d$ given by $(x^1, \dots, x^d) \to (e^{2\pi i x^1}, \dots, e^{2\pi i x^1})$ induces a diffeomorphism $\mathbb{R}^d/\mathbb{Z}^d \to \mathbb{T}^d$ such that the following diagram commutes:
$$\begin{tikzcd} \mathbb{R}^d \arrow[d] \arrow[dr] & \\ \mathbb{R}^d/\mathbb{Z}^d \arrow[r] & \mathbb{T}^d \end{tikzcd}$$

Since $\mathbb{R}^d$ is simply connected, we conclude also that the quotient map is a covering map and that $\pi_1(\mathbb{T}^d) = \mathbb{Z}^n$.

3. *Let $\mathbb{R}$ act on $\mathbb{R}^2$ by translations in the x-direction:*

$$\lambda \cdot (x^1, x^2) = (x^1 + \lambda, x^2).$$

This is not a free and proper action of a discrete group. However, we leave it as an exercise to check that the orbit equivalence relation $\sim$ determined by this action satisfies condition (ii) of Theorem 8.3, so that $\mathbb{R}^2/\sim$ inherits a smooth structure. The quotient $\mathbb{R}^2/\sim$ is diffeomorphic to $\mathbb{R}$ with its usual smooth structure.

The last example suggests that actions of non-discrete groups are also interesting. We will study later *Lie groups* where the group itself carries a smooth structure of positive dimension. These will give rise, as we will see later, to many other examples of quotients.

Homework.

1. Let X be a Hausdorff topological space and $\sim$ an equivalence relation in X such that $\pi : X \to X/\sim$ is an open map, for the quotient topology. Show that $X/\sim$ with the quotient topology is Hausdorff if and only if the graph of $\sim$ is closed in $X \times X$.

2. Let $\mathbb{R}$ act on $\mathbb{R}^2$ by translations in the x-direction:

$$\lambda \cdot (x^1, x^2) = (x^1 + \lambda, x^2).$$

Show that the equivalence relation $\sim$ determined by this action satisfies condition (ii) of Theorem 8.3, so that $\mathbb{R}^2/\sim$ inherits a smooth structure. Check that $\mathbb{R}^2/\sim$ is diffeomorphic to $\mathbb{R}$ with its usual smooth structure.

3. Let $\mathbb{R}^2 - \{0\}$ with the equivalence relation $\sim$ for which the equivalence classes are the connected components of the horizontal lines $y = \text{const}$. Show that the quotient space $\mathbb{R}^2 - \{0\}/\sim$ has a non-Hausdorff smooth structure (this manifold is sometimes called the **line with two origins**).

4. Let $\pi : M \to Q$ be a surjective submersion, $\Phi : M \to N$ and $\Psi : Q \to N$ any maps into a smooth submanifold N such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "\Phi"] & \\ Q \arrow[r, "\Psi"'] & N \end{tikzcd}$$

Show that Φ is smooth if and only if Ψ is smooth. Use this to conclude that if M is a manifold, $\sim$ is an equivalence relation satisfying any of the conditions of Theorem 8.3, and $\Phi : M \to N$ is a smooth map such that $\Phi(x) = \Phi(y)$ whenever $x \sim y$, then there is an induced smooth map $\overline{\Phi} : M/\sim \to N$ such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "\Phi"] & \\ M/\sim \arrow[r, dashed, "\overline{\Phi}"'] & N \end{tikzcd}$$

5. Show that any smooth action $G \times M \to M$ of a finite group G on a manifold M is proper.

6. A smooth action $\Psi : G \times M \to M$ of a discrete group G is said to be **properly discontinuous** if the following two conditions are satisfied:
   (a) For every $p \in M$, there exists a neighborhood U of p, such that:
   $$g \cdot U \cap U = \emptyset, \quad \forall g \in G - G_p.$$
   (b) If $p, q \in M$ do not belong to the same orbit, then there are open neighborhoods U of p and V of q, such that
   $$g \cdot U \cap V = \emptyset, \quad \forall g \in G.$$

   Show that a free action of a discrete group is proper if and only if it is properly discontinuous.

7. Show that for a proper a free action of a discrete group $G \times M \to M$, the projection $\pi : M \to G\backslash M$ is a local diffeomorphism, so that π is a covering map.

8. Let $\mathcal{F}$ be a foliation of M and denote by $M/\mathcal{F}$ its leaf space. One calls $\mathcal{F}$ a **simple foliation** if for each $p \in M$ there exists a foliated chart (U, ϕ) with the property that every leaf L intersects U at most in one plaque. Show that a foliation $\mathcal{F}$ is simple if and only if there exists a smooth structure $M/\mathcal{F}$, in general non-Hausdorff, for which the quotient map $\pi : M \to M/\mathcal{F}$ is a submersion.

Part 2. Lie Theory

In the first part of these lectures we have introduced and study some elementary concepts about manifolds. We will now initiate the study of the local differential geometry of smooth manifolds. The main concept and ideas that we will introduce in this round of lectures are the following:

- Lecture 9: the notion of *vector field* and the related concepts of *integral curve* and *flow* of a vector field.
- Lecture 10: the *Lie bracket* of vector fields and the *Lie derivative*, which allows to differentiate along vector fields.
- Lecture 11: an important generalization of vector fields, called *distributions*. The *Frobenius Theorem* says that foliations are the global objects associated with *involutive distributions*.
- Lecture 12: *Lie groups* and their infinitesimal versions called *Lie algebras*.
- Lecture 11: how to *integrate* Lie algebras to Lie groups.
- Lecture 12: *transformation groups* which are concrete realizations of Lie groups.

Lecture 9. Vector Fields and Flows

Definition 9.1. *A* ***vector field*** *on a manifold M is a section of the tangent bundle $\pi : TM \to M$, i.e., a map $X : M \to TM$ such that $\pi \circ X = I$. We say that the vector field X is smooth or C^∞, if the map $X : M \to TM$ is smooth. We will denote by $\mathfrak{X}(M)$ the set of smooth vector fields on a manifold M.*

If X is a vector field on M, we denote by X_p, rather than $X(p)$, the value of X at $p \in M$. For each $p \in M$, X_p is a derivation, hence, given any $f \in C^\infty(M)$ we can define a new function $X(f) : M \to \mathbb{R}$ by setting

$$X(f)(p) \equiv X_p(f).$$

If you recall the definition of the differential of a function, you see immediately that this definition is equivalent to:

$$X(f) = \mathrm{d}f(X).$$

Also, from the definition of a tangent vector as a derivation, we see that $f \mapsto X(f)$ satisfies:

(i) $X(f + \lambda g) = X(f) + \lambda X(g)$;
(ii) $X(fg) = X(f)g + fX(g)$;

Let $(U, x^1, \ldots, x^d)$ be a coordinate system on M. Then we have the vector fields $\frac{\partial}{\partial x^i} \in \mathfrak{X}(U)$ defined by:

$$\frac{\partial}{\partial x^i}(p) \equiv \left.\frac{\partial}{\partial x^i}\right|_p, \qquad (i = 1, \ldots, d).$$

At each $p \in U$ these vector fields yield a basis of T_pM, so if $X \in \mathfrak{X}(M)$ is any vector field on M, its restriction to the open set U, denoted by $X|_U$, can be written in the form:

$$X|_U = \sum_{i=1}^{d} X^i \frac{\partial}{\partial x^i},$$

where $X^i : U \to \mathbb{R}$ are certain functions which we call the **components of the vector field** X with respect to the chart $(x^1, \ldots, x^d)$.

Lemma 9.2. *Let X be a vector field on M. The following statements are equivalent:*

(i) The vector field X is C^∞;
(ii) For any chart $(U, x^1, \ldots, x^d)$, the components X^i of X with respect to this chart are C^∞;
(iii) For any $f \in C^\infty(M)$, the function $X(f)$ is C^∞.

Proof. We show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

To show that (i) $\Rightarrow$ (ii), note that if X is C^∞ and U is an open set, the restriction $X|_U$ is also C^∞. Hence, if $(U, x^1, \ldots, x^d)$ is any chart, we have that $\mathrm{d}x^i(X|_U) := \mathrm{d}x^i \circ X|_U$ is C^∞. But:

$$\mathrm{d}x^i(X|_U) = \mathrm{d}x^i(\sum_{j=1}^{d} X^j \frac{\partial}{\partial x^j}) = X^i.$$

To show that (ii) $\Rightarrow$ (iii), note that $f \in C^\infty(M)$ if and only if $f|_U \in C^\infty(U)$, for every domain U of a chart. But:

$$X(f)|_U = \sum_{i=1}^{d} X^i \frac{\partial f}{\partial x^i} \in C^\infty(U).$$

To show that (iii) $\Rightarrow$ (i), it is enough to show that $X|_U$ is C^∞, for every domain U of a chart. Recall that if $(U, x^1, \dots, x^d)$ is a chart then

$$(\pi^{-1}(U), (x^1 \circ \pi, \dots, x^d \circ \pi, \mathrm{d}x^1, \dots, \mathrm{d}x^d))$$

is a coordinate systems in TM. Since:

$$\begin{aligned} x^i \circ \pi \circ X|_U &= x^i \in C^\infty(U), \\ \mathrm{d}x^i \circ X|_U &= X(x^i) \in C^\infty(U), \end{aligned}$$

we conclude that $X|_U$ is C^∞. □

We conclude from this lemma, that a vector field $X \in \mathfrak{X}(M)$ defines a linear derivation $D_X : C^\infty(M) \to C^\infty(M)$, $f \mapsto X(f)$. Conversely, we have:

Lemma 9.3. *Every linear derivation $D : C^\infty(M) \to C^\infty(M)$ determines a vector field $X \in \mathfrak{X}(M)$ through the formula:*

$$X_p(f) = D(f)(p).$$

Proof. We only need to show that $X_p(f)$ only depends on the germ $[f] \in \mathcal{G}_p$, i.e., if $f, g \in \mathcal{C}^\infty(M)$ are two function which agree in some neighborhood U of p, then $D(f)(p) = D(g)(p)$. This follows from the fact that derivations are local: if D is a derivation and $f \in C^\infty(M)$ is zero on some open set $U \subset M$, then $D(f)$ is also zero in U. To see this, let $p \in U$ and choose $g \in C^\infty(M)$ such that $g(p) > 0$ and $\operatorname{supp} g \subset U$. Since $gf \equiv 0$, we have that:

$$0 = D(gf) = D(g)f + gD(f).$$

If we evaluate both sides at p, we obtain $D(f)(p) = 0$. Hence, $D(f)|_U = 0$ as claimed. □

From now on we will not distinguish between a vector field and the associated derivation of $C^\infty(M)$, so we will the same letter to denote them.

Recall that a **path** in a manifold M is a continuous map $\gamma : I \to M$, where $I \subset \mathbb{R}$ is an interval. A **smooth path** is a path for which γ is C^∞. Note that if $\partial I \neq \emptyset$, i.e., is not an open interval, then γ is smooth if and only if it has a smooth extension to a smooth path defined in an open interval $J \supset I$. If $\gamma : I \to M$ is a smooth path, its **derivative** is:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) \equiv \mathrm{d}\gamma \cdot \left.\frac{\partial}{\partial t}\right|_t \in T_{\gamma(t)}M, \quad (t \in I)$$

We often abbreviate writing $\dot{\gamma}(t)$ instead of $\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t)$. The derivative $t \mapsto \dot{\gamma}(t)$ is a smooth path in the manifold TM.

Definition 9.4. *Let $X \in \mathfrak{X}(M)$ be a vector field. A smooth path $\gamma : I \to M$ is called an **integral curve** of X if*

$$\dot{\gamma}(t) = X_{\gamma(t)}, \quad \forall t \in I. \tag{9.1}$$

In a chart $(U, (x^1, \dots, x^d))$, a path $\gamma(t)$ is determined by its components $\gamma^i(t) = x^i(\gamma(t))$. Its derivative is then given by

$$\dot{\gamma}(t) = \mathrm{d}\gamma \cdot \frac{\partial}{\partial t} = \sum_{i=1}^{d} \frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\frac{\partial}{\partial x^i}.$$

It follows that the integral curves of a vector field X, which has components X^i in the local chart $(x^1, \dots, x^d)$, are the solutions of the system of o.d.e.'s:

$$\frac{\mathrm{d}\gamma^i}{\mathrm{d}t} = X^i(\gamma^1(t), \dots, \gamma^d(t)), \quad (i = 1, \dots, d). \tag{9.2}$$

This system is the local form of the equation (9.1). Note that it is common to write $x^i(t)$ for the components $\gamma^i(t) = x^i(\gamma(t))$ so that this system of equations becomes:

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = X^i(x^1(t), \dots, x^d(t)), \quad (i = 1, \dots, d).$$

Example 9.5.
In $\mathbb{R}^2$ consider the vector field $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$. The equations for the integral curves (9.2) are:

$$\begin{cases} \dot{x}(t) = -y(t), \\ \dot{y}(t) = x(t). \end{cases}$$

Hence, the curves $\gamma(t) = (R\cos t, R\sin t)$ are integral curves of this vector field.

This vector field is tangent to the submanifold $\mathbb{S}^1 = \{(x, y) : x^2 + y^2 = 1\}$, so defines a vector field on the circle: $Y = X|_{\mathbb{S}^1}$. If we consider the angle coordinate θ on the circle, the smooth functions $C^\infty(\mathbb{S}^1)$ can be identified with the 2π-periodic smooth functions $f(\theta) = f(\theta + 2\pi)$. It is easy to see that the vector field Y as a derivation is given by:

$$Y(f)(\theta) = f'(\theta).$$

Hence we will write this vector field as:

$$Y = \frac{\partial}{\partial \theta},$$

although the function θ is not a globally defined smooth coordinate on $\mathbb{S}^1$.

Now consider the cylinder $M = \mathbb{S}^1 \times \mathbb{R}$, with coordinates (θ, x). We have a well defined vector field:

$$Z = \frac{\partial}{\partial\theta} + x\frac{\partial}{\partial x}.$$

You should try to plot this vector field on a cylinder and verify that the integral curve of Z through a point (θ_0, x_0) is given by

$$\gamma(t) = (\theta_0 + t, x_0 e^t).$$

If $x_0 = 0$, this is just a circle around the cylinder. If $x_0 \neq 0$ this is a spiral that approaches the circle when $t \to -\infty$ and goes to infinity when $t \to +\infty$.

Standard results about existence, uniqueness and maximal interval of definition of solutions a system of o.d.e.'s lead to the following proposition:

Proposition 9.6. *Let $X \in \mathfrak{X}(M)$ be a vector field. For each $p \in M$, there exist real numbers $a_p, b_p \in \mathbb{R} \cup \{\pm\infty\}$ and a smooth path $\gamma_p :]a_p, b_p[\to M$, such that:*

(i) $0 \in]a_p, b_p[$ *and* $\gamma_p(0) = p$*;*
(ii) γ_p *is an integral curve of* X*;*
(iii) If $\eta :]c, d[\to M$ *is any integral curve of* X *with* $\eta(0) = p$*, then* $]c, d[\subset]a_p, b_p[$ *and* $\gamma_p|_{]c,d[} = \eta$*.*

We call the integral curve γ_p given by this proposition the **maximal integral curve** of X through p. For each $t \in \mathbb{R}$, we define the domain $D_t(X)$ consisting of those points for which the integral curve through p exists at least until time t:

$$D_t(X) = \{p \in M : t \in]a_p, b_p[\}.$$

If it is clear the vector field we are referring to, we will write D_t instead of $D_t(X)$. The **flow of the vector field** $X \in \mathfrak{X}(M)$ is the map $\phi_X^t : D_t \to M$ given by

$$\phi_X^t(p) \equiv \gamma_p(t).$$

Proposition 9.7. *Let $X \in \mathfrak{X}(M)$ be a vector field with flow ϕ_X^t. Then:*

(i) For each $p \in M$, there exists a neighborhood U of p and $\varepsilon > 0$, such that the map $(-\varepsilon, \varepsilon) \times U \to M$ given by:

$$(t, q) \mapsto \phi_X^t(q),$$

is well defined and smooth;

(ii) For each $t \in \mathbb{R}$, D_t is open and $\bigcup_{t>0} D_t = M$;

(iii) For each $t \in \mathbb{R}$, $\phi_X^t : D_t \to D_{-t}$ is a diffeomorphism and:

$$(\phi_X^t)^{-1} = \phi_X^{-t};$$

(iv) For each $s, t \in \mathbb{R}$, the domain of $\phi_X^t \circ \phi_X^s$ is contained in D_{t+s} and:

$$\phi_X^{t+s} = \phi_X^t \circ \phi_X^s.$$

Proof. Exercise. □

One calls a vector field X **complete** if $D_t(X) = M$, for every $t \in \mathbb{R}$, i.e., if the maximal integral curve through any $p \in M$ is defined for all $t \in]-\infty, +\infty[$. In this case the flow of X is a map:

$$\mathbb{R} \times M \to M, \quad (t,p) \mapsto \phi_X^t(p).$$

The properties above then say that this map gives an action of the group $(\mathbb{R},+)$ in M. In other words, the map

$$\mathbb{R} \to \mathrm{Diff}(M), \quad t \mapsto \phi_X^t,$$

is a group homomorphism from $(\mathbb{R},+)$ to the group $(\mathrm{Diff}(M), \circ)$ of diffeomorphisms of M. One often says that ϕ_X^t is a *1-parameter group of transformations* of M. In the non-complete case, one also says that ϕ_X^t is a 1-parameter group of *local* transformations of M.

Examples 9.8.

1. *The vector field $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ in $\mathbb{R}^2$ is complete (see Example 9.5) and is flow is given by:*

$$\phi_X^t(x,y) = (x\cos t - y\sin t, x\sin t + y\cos t).$$

2. *The vector field $Y = -x^2\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ in $\mathbb{R}^2$ is not complete: the integral curve through a point (x_0, y_0) is the solution to the system of odes:*

$$\begin{cases} \dot{x}(t) = -x^2, & x(0) = x_0, \\ \dot{y}(t) = -y, & y(0) = y_0. \end{cases}$$

After solving this system, ones obtains the flow of Y:

$$\phi_X^t(x,y) = \left(\frac{x}{xt+1}, ye^{-t}\right).$$

It follows that the flow through points $(0,y)$ exist for all t. But for points (x,y), with $x \neq 0$, the flow exists only for $t \in]-1/x, +\infty[$ if $x > 0$ and for $t \in]-\infty, -1/x[$ if $x > 0$. The domain of the flow is then given by:

$$D_t(Y) = \begin{cases} \{(x,y) \in \mathbb{R}^2 : x > -1/t\} & \text{if } t > 0, \\ \\ \mathbb{R}, & \text{if } t = 0, \\ \\ \{(x,y) \in \mathbb{R}^2 : x < -1/t\} & \text{if } t < 0 \end{cases}$$

Let $\Phi : M \to N$ be a smooth map. In general, given a vector field X in M, it is not possible to use Φ to map X to obtain a vector field Y in N. However, given two vector fields, one in M and one in N, we can say when they are related by this map:

Definition 9.9. *Let $\Phi : M \to N$ be a smooth map. A vector field $X \in \mathfrak{X}(M)$ is said to be **Φ-related** to a vector field $Y \in \mathfrak{X}(N)$ if*

$$Y_{\Phi(p)} = \mathrm{d}\Phi(X_p), \quad \forall p \in M.$$

If X and Y are Φ-related vector fields then, as derivations of $C^\infty(M)$:

$$Y(f) \circ \Phi = X(f \circ \Phi), \quad \forall f \in C^\infty(N).$$

When Y is determined from X via Φ we write $Y = \Phi_*(X)$, and call $\Phi_*(X)$ the **push forward** of X by Φ. This is the case, for example, when Φ is a diffeomorphism, in which case:

$$\Phi_*(X)(f) = X(f \circ \Phi) \circ \Phi^{-1}, \quad \forall f \in C^\infty(N).$$

The integral curves of vector fields which are Φ-related are also Φ-related. The proof is a simple exercise applying the chain rule:

Proposition 9.10. *Let $\Phi : M \to N$ be a smooth map and let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ be Φ-related vector fields. If $\gamma : I \to M$ is an integral curve of X, then $\Phi \circ \gamma : I \to N$ is an integral curve of Y. In particular, we have that $\Phi(D_t(X)) \subset D_t(Y)$ and that the flows of X and Y are related by:*

$$\begin{array}{ccc} D_t(X) & \xrightarrow{\Phi} & D_t(Y) \\ {\scriptstyle \phi_X^t}\downarrow & & \downarrow{\scriptstyle \phi_Y^t} \\ D_{-t}(X) & \xrightarrow{\Phi} & D_{-t}(Y) \end{array}$$

If $X \in \mathfrak{X}(M)$ is a vector field and $f \in C^\infty(M)$, we already know that $X(f) \in C^\infty(M)$. The expression for $X(f)$ is local coordinates show that X is a first order differential operator. If we iterate, we obtain the powers X^k, which are kth-order differential operators:

$$X^{k+1}(f) \equiv X(X^k(f)).$$

Proposition 9.11 (Taylor Formula)**.** *Let $X \in \mathfrak{X}(M)$ be a vector field and $f \in C^\infty(M)$. For each positive integer k, one has the expansion*

$$f \circ \phi_X^t = f + tX(f) + \frac{t^2}{2!}X^2(f) + \cdots + \frac{t^k}{k!}X^k(f) + 0(t^{k+1}),$$

where for each $p \in M$, $t \mapsto 0(t^{k+1})(p)$ denotes a real smooth function defined in a neighborhood of the origin whose terms of order $\leq k$ all vanish.

Proof. By the usual Taylor formula for real functions applied to $t \mapsto f(\phi_X^t(p))$, it is enough to show that:

$$\left.\frac{\mathrm{d}^k}{\mathrm{d}t^k} f(\phi_X^t(p))\right|_{t=0} = X^k(f)(p).$$

To prove this, we show by induction that:

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k} f(\phi_X^t(p)) = X^k(f)(\phi_X^t(p)).$$

When $k = 1$, this follows because:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} f(\phi_X^t(p)) &= \mathrm{d}_p f \cdot X_{\phi_X^t(p)} \\ &= X_{\phi_X^t(p)}(f) = X(f)(\phi_X^t(p)). \end{aligned}$$

On the other hand, if we assume that the formula is valid for $k-1$, we obtain:

$$\begin{aligned}\frac{\mathrm{d}^k}{\mathrm{d}t^k} f(\phi_X^t(p)) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}} f(\phi_X^t(p))\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}t} X^{k-1}(f)(\phi_X^t(p))\\ &= X(X^{k-1}(f))(\phi_X^t(p)) = X^k(f)(\phi_X^t(p)).\end{aligned}$$

□

Another common notation for the flow of a vector field, which is justified by the previous result, is the exponential notation:

$$\exp(tX) \equiv \phi_X^t.$$

In this notation, the properties of the flow are written as:

$$\exp(tX)^{-1} = \exp(-tX), \quad \exp((t+s)X) = \exp(tX) \circ \exp(sX),$$

while the Taylor expansion takes the following suggestive form:

$$f(\exp(tX)) = f + tX(f) + \frac{t^2}{2!}X^2(f) + \cdots + \frac{t^k}{k!}X^k(f) + 0(t^{k+1}).$$

We will not use this notation in these lectures.

If $X \in \mathfrak{X}(M)$ is a vector field, a point $p \in M$ is called a **singular point** or an **equilibrium point** of X if $X_p = 0$. It should be obvious that the integral curve through a singular point of X is the constant path: $\phi_X^t(p) = p$, para todo o $t \in \mathbb{R}$. On the other hand, for non-singular points we have a unique local canonical form X:

Theorem 9.12 (Flow Box Theorem)**.** *Let $X \in \mathfrak{X}(M)$ be a vector field and $p \in M$ a non-singular point: $X_p \neq 0$. There are local coordinates $(U, (x^1, \dots, x^d))$ centered at p, such that:*

$$X|_U = \frac{\partial}{\partial x^1}.$$

Proof. First we choose a chart $(V, (y^1, \dots, y^d)) = (V, \psi)$, centered at p, such that:

$$X|_p = \left.\frac{\partial}{\partial y^1}\right|_p.$$

The map $\sigma : \mathbb{R}^d \to M$ given by

$$\sigma(t_1, \dots, t_d) = \phi_X^{t_1}(\psi^{-1}(0, t_2, \dots, t_d)),$$

is well defined and C^∞ in a neighborhood of the origin. Its differential at the origin is given by:

$$\begin{aligned}
\mathrm{d}_0\sigma \cdot \left.\frac{\partial}{\partial t_1}\right|_0 &= \left.\frac{\mathrm{d}}{\mathrm{d}t_1}\phi_X^{t_1}(\psi^{-1}(0,0,\dots,0))\right|_{t_1=0} = X_p = \left.\frac{\partial}{\partial y^1}\right|_p, \\
\mathrm{d}_0\sigma \cdot \left.\frac{\partial}{\partial t_i}\right|_0 &= \left.\frac{\partial}{\partial t_i}\psi^{-1}(0,t_2,\dots,t_d))\right|_0 = \left.\frac{\partial}{\partial y^i}\right|_p.
\end{aligned}$$

We conclude that σ is a local diffeomorphism in a neighborhood of the origin. Hence, there exists an open set U containing p such that $\phi = \sigma^{-1} : U \to \mathbb{R}^d$ is a chart. If we write $(U,\phi) = (U,(x^1,\dots,x^d))$, we have:

$$\begin{aligned}
\left.\frac{\partial}{\partial x^1}\right|_{\sigma(t_1,\dots,t_d)} &= \mathrm{d}\sigma \cdot \left.\frac{\partial}{\partial t_1}\right|_{(t_1,\dots,t_d)} \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\phi_X^t(\psi^{-1}(0,t_2,\dots,t_d))\right|_{t=t_1} \\
&= X(\phi_X^{t_1}(\psi^{-1}(0,t_2,\dots,t_d))) = X_{\sigma(t_1,\dots,t_d)}.
\end{aligned}$$

□

Homework.

1. Let M be a connected manifold. Show that for any pair of points $p,q \in M$, with $p \neq q$, there exists a smooth path $\gamma : [0,1] \to M$ such that
   (a) $\gamma(0) = p$ and $\gamma(1) = q$;
   (b) $\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) \neq 0$, for every $t \in [0,1]$;
   (c) γ is simple (i.e., γ is injective).
   Use this to prove that any connected manifold of dimension 1 is diffeomorphic to either $\mathbb{R}$ or $\mathbb{S}^1$.

2. Let $X \in \mathfrak{X}(M)$ be a vector field and $\lambda \in \mathbb{R}$. What is the relationship between the integral curves of X and the integral curves of λX?

3. Verify the properties of the flow of a vector field given by Proposition 9.7.

4. Determine the flow of the vector field $X = y\partial/\partial x - x\partial/\partial y$ in $\mathbb{R}^3$ with coordinates (x,y,z).

5. Is a vector field X in $\mathbb{R}$ necessarily complete? What about in $\mathbb{R}^2$?

6. Show that on a compact manifold M every vector field $X \in \mathfrak{X}(M)$ is complete. Give an example of two vector fields X_1 and X_2 which are complete but for which their sum $X_1 + X_2$ is not complete.

7. Let $A \subset M$. Call X a *vector field along* A if $X : A \to TM$ satisfies $X_p \in T_pM$ for all $p \in A$. We say that X is smooth if every $p \in A$ has a neighborhood U_p and a smooth vector field $\tilde{X} \in \mathfrak{X}(U_p)$ such that $\tilde{X}|_{A \cap U_p} = X$. Show that if $A \subset O \subset M$, with A closed and O open, then every smooth vector

field X along A can be extended to a smooth vector field in M such that $X_p = 0$ for $p \notin O$.

8. Let $X \in \mathfrak{X}(M)$ be a vector field without singular points. Show that the integral curves of X form a foliation $\mathcal{F}$ of M of dimension 1. Conversely, show that locally the leaves of a foliation of dimension 1 are the orbits of a vector field. What about globally?

9. A Riemannian structure on a manifold M is a smooth choice of an inner product $\langle\ ,\ \rangle_p$ in each tangent space T_pM. Here by smooth we mean that for any vector fields $X, Y \in \mathfrak{X}(M)$, the function $p \mapsto \langle X(p), Y(p)\rangle_p$ is C^∞). Show that every smooth manifold admits a Riemannian structure M.

Lecture 10. Lie Bracket and Lie Derivative

Definition 10.1. *Let $X, Y \in \mathfrak{X}(M)$ be smooth vector fields. The **Lie bracket** of X and Y is the vector field $[X, Y] \in \mathfrak{X}(M)$ given by:*

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \quad \forall f \in C^\infty(M).$$

Note that the formula for the Lie bracket $[X, Y]$ shows that it is a differential operator of order ≤ 2. A simple computation shows that $[X, Y]$ is a linear derivation of $C^\infty(M)$:

$$[X, Y](fg) = [X, Y](f)g + f[X, Y](g), \quad \forall f, g \in C^\infty(M).$$

In order words, the terms of 2nd order cancel each other and we have in fact that $[X, Y] \in \mathfrak{X}(M)$.

In a local chart we can compute the Lie bracket in a straightforward way if we think of vector fields as differential operators. This is illustrated in the next example.

Example 10.2.
Let $M = \mathbb{R}^3$ with coordinates (x, y, z), and consider the vector fields:

$$\begin{aligned} X &= z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \\ Y &= x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \\ Z &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \end{aligned}$$

Then we compute:

$$\begin{aligned} [X, Y] &= \left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)\left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right) - \left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right)\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right) \\ &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = Z. \end{aligned}$$

We leave it as an exercise the computation of the other Lie brackets:

$$[Y, Z] = X, \quad [Z, X] = Y.$$

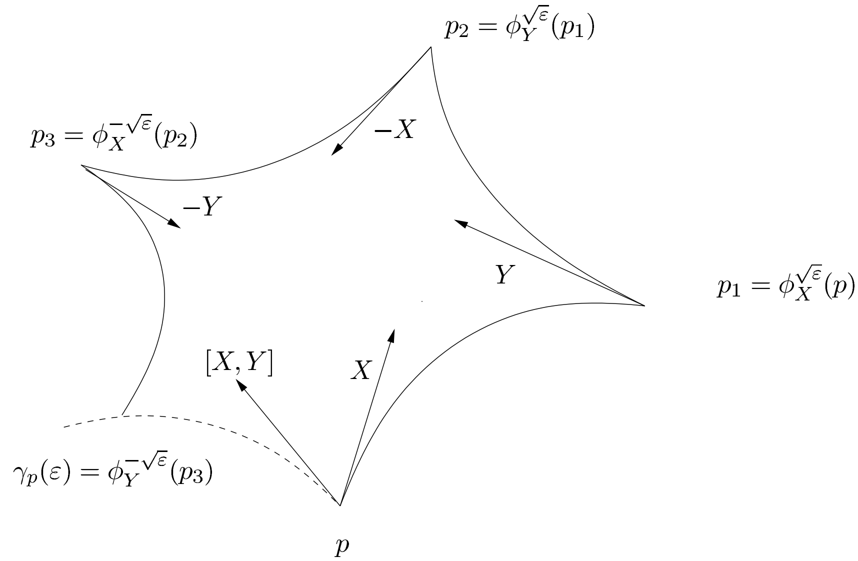
Our next result shows that the Lie bracket $[X,Y]$ measures the failure in the commutativity of the flows of X and Y.

Proposition 10.3. *Let $X,Y\in\mathfrak{X}(M)$ be vector fields. For each $p\in M$, the commutator*

$$\gamma_p(\varepsilon)\equiv\phi_Y^{-\sqrt{\varepsilon}}\circ\phi_X^{-\sqrt{\varepsilon}}\circ\phi_Y^{\sqrt{\varepsilon}}\circ\phi_X^{\sqrt{\varepsilon}}(p)$$

is well defined for a small enough $\varepsilon\geq 0$, and we have:

$$[X,Y]_p=\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\gamma_p(\varepsilon)\bigg|_{\varepsilon=0^+}.$$



Proof. Fix a local chart $(U,x^1,\ldots,x^d)$, centered at p, and write:

$$X=\sum_{i=1}^d X^i\frac{\partial}{\partial x^i},\quad Y=\sum_{i=1}^d Y^i\frac{\partial}{\partial x^i}.$$

The Lie bracket of X and Y is given by:

$$[X,Y](x^i)=X(Y^i)-X(Y^i).$$

Consider the points p_1, p_2 and p_3 defined by (see figure above):

$$p_1=\phi_X^{\sqrt{\varepsilon}}(p),\quad p_2=\phi_Y^{\sqrt{\varepsilon}}(p_1),\quad p_3=\phi_X^{-\sqrt{\varepsilon}}(p_2),$$

Then $\gamma_p(\varepsilon)=\phi_Y^{-\sqrt{\varepsilon}}(p_3)$, and Taylor's formula (see Proposition 9.11), applied to each coordinate x^i, yields:

$$x^i(p_1)=x^i(p)+\sqrt{\varepsilon}X^i(p)+\frac{1}{2}\varepsilon X^2(x^i)(p)+O(\varepsilon^{\frac{3}{2}})$$

Similarly, we have:

$$\begin{aligned}x^i(p_2) &= x^i(p_1) + \sqrt{\varepsilon}Y^i(p_1) + \frac{1}{2}\varepsilon Y^2(x^i)(p_1) + O(\varepsilon^{\frac{3}{2}}) = \\ &= x^i(p) + \sqrt{\varepsilon}X^i(p) + \frac{1}{2}\varepsilon X^2(x^i)(p) + \\ &\quad + \sqrt{\varepsilon}Y^i(p_1) + \frac{1}{2}\varepsilon Y^2(x^i)(p_1) + O(\varepsilon^{\frac{3}{2}})\end{aligned}$$

The last two terms can also be estimated using again Taylor's formula:

$$\begin{aligned}Y^i(p_1) &= Y^i(\phi_X^{\sqrt{\varepsilon}}(p)) = Y^i(p) + \sqrt{\varepsilon}X(Y^j)(p) + O(\varepsilon) \\ Y^2(x^i)(p_1) &= Y^2(x^i)(\phi_X^{\sqrt{\varepsilon}}(p)) = Y^2(x^i)(p) + \sqrt{\varepsilon}Y^2(x^i)(p) + O(\varepsilon)\end{aligned}$$

hence, we have:

$$\begin{aligned}x^i(p_2) =& x^i(p) + \sqrt{\varepsilon}(Y^i(p) + X^i(p)) + \\ &+ \varepsilon\left(\frac{1}{2}Y^2(x^i)(p) + X(Y^i)(p) + \frac{1}{2}X^2(x^i)(p)\right) + O(\varepsilon^{\frac{3}{2}})\end{aligned}$$

Proceeding in a similar fashion, we can estimate $x^i(p_3)$ and $x^i(\gamma_p(\varepsilon))$, obtaining:

$$\begin{aligned}x^i(p_3) &= x^i(p_2) - \sqrt{\varepsilon}X^i(p_2) + \frac{1}{2}\varepsilon X^2(x^i)(p_2) + O(\varepsilon^{\frac{3}{2}}) \\ &= x^i(p) + \sqrt{\varepsilon}Y^i(p) + \varepsilon\left(X(Y^i)(p) - Y(X^i)(p) + \frac{1}{2}Y^2(x^i)(p)\right) + O(\varepsilon^{\frac{3}{2}}) \\ x^i(\gamma_p(\varepsilon)) &= x^i(p_3) - \sqrt{\varepsilon}Y^i(p_3) + \frac{1}{2}\varepsilon Y^2(x^i)(p_3) + O(\varepsilon^{\frac{3}{2}}) \\ &= x^i(p) + \varepsilon\left(X(Y^i)(p) - Y(X^i)(p)\right) + O(\varepsilon^{\frac{3}{2}})\end{aligned}$$

Therefore:

$$\lim_{\varepsilon\to 0^+} \frac{x^i(\gamma_p(\varepsilon)) - x^i(p)}{\varepsilon} = X(Y^i)(p) - Y(X^i)(p) = [X,Y]_p(x^i).$$

□

The following proposition gives the most basic properties of the Lie bracket of vector fields. The proof is elementary and is left as an exercise.

Proposition 10.4. *The Lie bracket satisfies the following properties:*

(i) Skew-symmetry: $[X,Y] = -[Y,X]$;
(ii) Bi-linearity: $[aX + bY, Z] = a[X,Z] + b[Y,Z]$, $\forall a,b \in \mathbb{R}$;
(iii) Jacobi identity: $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$;
(iv) Leibniz identity: $[X, fY] = X(f)Y + f[X,Y]$, $\forall f \in C^\infty(M)$.

Moreover, if $\Phi : M \to N$ *is a smooth map,* X *and* $Y \in \mathfrak{X}(M)$ *are* Φ*-related with, respectively,* Z *and* $W \in \mathfrak{X}(N)$*, then* $[X,Y]$ *is* Φ*-related with* $[Z,W]$.

The geometric interpretation of the Lie bracket given by Proposition 10.3 shows that the Lie bracket and the flow of vector fields are intimately related. There is another form of this relationship which we now explain. For that, we need the following definition:

Definition 10.5. *Let $X \in \mathfrak{X}(M)$ be a vector field.*

*(i) The **Lie derivative of a function** $f \in C^\infty(M)$ along X is the smooth function $\mathcal{L}_X f$ given by:*

$$(\mathcal{L}_X f)_p = \lim_{t\to 0} \frac{1}{t}\left(f(\phi_X^t(p) - f(p)\right).$$

*(ii) The **Lie derivative of a vector field** $Y \in \mathfrak{X}(M)$ along X is the smooth vector field $\mathcal{L}_X Y$ given by:*

$$(\mathcal{L}_X Y)_p = \lim_{t\to 0} \frac{1}{t}\left(\mathrm{d}\phi_X^{-t} \cdot Y_{\phi_X^t(p)} - Y_p\right).$$

One can "unify" these two definitions observing that a diffeomorphism $\Phi : M \to M$ acts on functions $C^\infty(M)$ by:

$$(\Phi^* f)(p) = f(\Phi(p)),$$

and it acts on vector fields $Y \in \mathfrak{X}(M)$:

$$(\Phi^* Y)_p = \mathrm{d}\Phi^{-1} \cdot Y_{\Phi(p)}.$$

Note that $\Phi^* Y = (\Phi^{-1})_* Y$, so the two operations are related by:

$$\Phi^* Y(f) = Y((\Phi^{-1})^* f).$$

It follows that the Lie derivative of an object P (a function or a vector field) is given by:

$$\mathcal{L}_X P = \lim_{t\to 0} \frac{1}{t}\left((\phi_X^t)^* P - P\right). \tag{10.1}$$

We will see later that one can take Lie derivatives of other objects using precisely this definition.

Theorem 10.6. *Let $X \in \mathfrak{X}(M)$ be a vector field.*

(i) For any functions $f \in C^\infty(M)$: $\mathcal{L}_X f = X(f)$.

(ii) For any vector field $Y \in \mathfrak{X}(M)$: $\mathcal{L}_X Y = [X, Y]$.

Proof. To prove (i), we simply observe that:

$$\mathcal{L}_X f = \frac{\mathrm{d}}{\mathrm{d}t} f \circ \phi_X^t \bigg|_{t=0} = \mathrm{d}f \cdot X = X(f).$$

To prove (ii), we note first that:

$$\begin{aligned}(\mathcal{L}_X Y)(f)(p) &= \lim_{t\to 0} \frac{1}{t}\left(\mathrm{d}\phi_X^{-t} \cdot Y_{\phi_X^t(p)} - Y_p\right)(f)\\ &= \lim_{t\to 0} \frac{1}{t}\left(Y_{\phi_X^t(p)}(f \circ \phi_X^{-t}) - Y_p(f)\right).\end{aligned}$$

On the other hand, Taylor's formula gives::

$$f \circ \phi_X^{-t} = f - tX(f) + O(t^2),$$

hence:

$$\begin{aligned}(\mathcal{L}_X Y)(f)(p) &= \lim_{t\to 0} \frac{1}{t}\left(Y_{\phi_X^t(p)}(f) - tY_{\phi_X^t(p)}(X(f)) - Y_p(f)\right)\\ &= \lim_{t\to 0} \frac{1}{t}\left(Y_{\phi_X^t(p)}(f) - Y_p(f)\right) - Y_p(X(f))\\ \\ &= X_p(Y(f)) - Y_p(X(f)) = [X,Y](f)(p).\end{aligned}$$

□

Homework.

1. Complete the computation of the Lie brackets in Example 10.2 and show that all 3 vector fields X,Y and Z are tangent to the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. Show that there are unique vector fields $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ on $\mathbb{P}^2$ such that $\pi_* X = \tilde{X}$, $\pi_* Y = \tilde{Y}$ and $\pi_* Z = \tilde{Z}$ where $\pi : \mathbb{S}^2 \to \mathbb{P}^2$ is the projection. What are the Lie brackets between $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$?

2. Find 3 everywhere linearly independent vector fields X,Y and Z on the sphere $\mathbb{S}^3$ such that $[X,Y] = Z$, $[Y,Z] = X$ and $[Z,X] = Y$.

   Hint: Recall that $\mathbb{S}^3$ can be identified with the unit quaternions.

3. Check the properties of the Lie bracket given in Proposition 10.4.

4. In $\mathbb{R}^2$ consider the vector fields $X = \frac{\partial}{\partial x}$ and $Y = x\frac{\partial}{\partial y}$. Compute the Lie bracket $[X,Y]$ in two distinct ways: (i) using the definition and (ii) using the flows of X and Y, as in Proposition 10.3.

5. Let $X, Y \in \mathfrak{X}(M)$ be complete vector fields with flows ϕ_X^t and ϕ_Y^s. Show that $\phi_X^t \circ \phi_Y^s = \phi_Y^s \circ \phi_X^t$ for all s and t if and only if $[X,Y] = 0$.

6. Let $X_1, \ldots, X_k \in \mathfrak{X}(M)$ be vector fields such that:
   (a) $\{X_1|_p, \ldots, X_k|_p\}$ are linearly independent, for all $p \in M$;
   (b) $[X_i, X_j] = 0$, for all $i, j = 1, \ldots, k$.
   Show that there exists a unique k-dimensional foliation $\mathcal{F}$ of M such that for all $p \in M$:

   $$T_p L = \langle X_1|_p, \ldots, X_k|_p \rangle,$$

   where $L \in \mathcal{F}$ is the leaf containing p.

   Hint: Use the previous exercise and show that

   $$L = \{\phi_{X_1}^{t_1} \circ \phi_{X_2}^{t_2} \circ \cdots \circ \phi_{X_k}^{t_k}(p) : t_1, t_2, \ldots, t_k \in \mathbb{R}\}.$$

Lecture 11. Distributions and the Frobenius Theorem

A vector field $X \in \mathfrak{X}(M)$ which is nowhere vanishing determines a subspace $\langle X_p \rangle \subset T_pM$, for each $p \in M$. These subspaces depend smoothly on p and our next definition generalizes this situation:

Definition 11.1. *Let M be a smooth manifold of dimension d and let $1 \leq k \leq d$ be an integer. A k-dimensional* ***distribution*** *D in M is a map*

$$M \ni p \mapsto D_p \subset T_pM,$$

which associates to each $p \in M$ a subspace $D_p \subset T_pM$ of dimension k. We say that a distribution D is of class C^∞ if for each $p \in M$ there exists a neighborhood U of p and smooth vector fields $X_1, \dots, X_k \in \mathfrak{X}(U)$, such that:

$$D_q = \langle X_1|_q, \dots, X_k|_q \rangle, \quad \forall q \in U.$$

If D is a distribution in M we consider the set of **vector fields tangent to** D:

$$\mathfrak{X}(D) := \{X \in \mathfrak{X}(M) : X_p \in D_p, \forall p \in M\}.$$

Note that $\mathfrak{X}(D)$ is a module over the ring $C^\infty(M)$: if $f \in C^\infty(M)$ and $X \in \mathfrak{X}(D)$ then $fX \in \mathfrak{X}(D)$.

Examples 11.2.

1. *Every nowhere vanishing smooth vector field X defines a 1-dimensional smooth distribution by:*

$$D_p := \langle X_p \rangle = \{\lambda X_p : \lambda \in \mathbb{R}\}.$$

We have that $Y \in \mathfrak{X}(D)$ if and only $Y = fX$ for some uniquely defined smooth function $f \in C^\infty(M)$.

2. *A set of smooth vector fields $X_1, \dots, X_k$ which at each $p \in M$ are linearly independent define a k-dimensional smooth distribution by:*

$$D_p := \langle X_1|_p, \dots, X_k|_p \rangle.$$

We have that a vector field $X \in \mathfrak{X}(D)$ if and only if

$$X = f_1X_1 + \dots + f_kX_k,$$

for uniquely defined functions $f_i \in C^\infty(M)$.

For example, in $M = \mathbb{R}^3$, we have the 2-dimensional smooth distribution $D = \langle X_1, X_2 \rangle$ generated by the vector fields:

$$X_1 = \frac{\partial}{\partial x} + z^2 \frac{\partial}{\partial y},$$
$$X_2 = \frac{\partial}{\partial y} + z^2 \frac{\partial}{\partial z}.$$

and every vector field $X \in \mathfrak{X}(D)$ is a linear combination $aX + bY$, where the smooth functions $a = a(x, y)$ and $b = b(x, y)$ are uniquely determined.

3. *More generally, a set of smooth vector fields $X_1, \dots, X_s$ which at each $p \in M$ span a k-dimensional subspace define a k-dimensional smooth distribution by:*

$$D_p := \langle X_1|_p, \dots, X_s|_p \rangle.$$

We have that $X \in \mathfrak{X}(D)$ if and only if

$$X = f_1 X_1 + \cdots + f_s X_s,$$

for some smooth functions $f_i \in C^\infty(M)$. The difference from the previous example is that the functions f_i are not uniquely defined. Moreover, we may not be able to find k-vector fields tangent to D which globally generate D.

For example, in $M = \mathbb{R}^3 - \{0\}$ consider the vector fields X, Y and Z defined in Example 10.2. The matrix whose columns are the components of the vector fields X, Y and Z relative to the usual coordinates (x, y, z) of $\mathbb{R}^3$ is:

$$\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$$

and has rank 2 everywhere. Hence, we have the 2-dimensional distribution $D = \langle X, Y, Z \rangle$. We leave it as an exercise to check that this distribution is not globally generated by only 2 vector fields.

We can think of a distribution as a generalization of the notion of a vector field. In this sense, the concept of an integral curve of a vector field is replaced by the following:

Definition 11.3. *Let D be a distribution in M. A connected submanifold (N, Φ) of M is called an **integral manifold** of D if:*

$$\mathrm{d}_p\Phi(T_pN) = D_{\Phi(p)}, \forall p \in N.$$

Note that if D is a k-dimensional distribution, its integral manifolds, if they exist, are k-dimensional manifolds.

Examples 11.4.

1. *Consider the 2-distribution of $\mathbb{R}^3$ given in Example 11.2.2. The plane $N = \{z = 0\}$ is an integral manifold of this distribution, since it is a connected submanifold and*

$$D_{(x,y,0)} = \left\langle \left.\frac{\partial}{\partial x}\right|_{(x,y,0)}, \left.\frac{\partial}{\partial y}\right|_{(x,y,0)} \right\rangle = T_{(x,y,0)}N.$$

2. *Consider the 2-distribution D of $\mathbb{R}^3 - \{0\}$ defined by the vector fields X, Y and Z in Example 11.2.3. The spheres*

$$S_c = \{(x, y, z) \in \mathbb{R}^3 - 0 : x^2 + y^2 + z^2 = c\},$$

are integral manifolds of D: each sphere is a connected submanifold and

$$T_{(x,y,z)}S_c = \{\vec{v} \in \mathbb{R}^3 : (x, y, z) \cdot \vec{v} = 0\},$$

hence:

$$T_{(x,y,z)}S_c \subset \langle X, Y, Z \rangle.$$

Since both spaces have dimension 2, we have $T_pS_c = D_p$, for all $p \in S_c$.

In the last example, through each $p \in M$ there is an integral manifold containing p. Moreover, the collection of all these integral manifolds form a foliation of $\mathbb{R}^3 - 0$.

More generally, if $\mathcal{F}$ is a smooth k-dimensional foliation of a manifold M, we denote by $T_p\mathcal{F} \equiv T_pL$ the tangent space to L that contains p. The assignment $p \mapsto T_p\mathcal{F}$ gives a smooth k-dimensional distribution in M and a vector field is tangent to $T\mathcal{F}$ if and only if it is tangent to the foliation, i.e., if and only if every integral curve of X is contained in a leaf of $\mathcal{F}$.

Definition 11.5. *A smooth distribution D in M is called **integrable** if there exists a foliation $\mathcal{F}$ in M such that $D = T\mathcal{F}$.*

A distribution D in M may fail to be integrable. In fact, there may not even exist integral manifolds through each point of M. The following proposition gives a necessary condition for this to happen:

Proposition 11.6. *Let D be a smooth distribution in M. If there exists an integral manifold of D through $p \in M$, then for any $X, Y \in \mathfrak{X}(D)$ we must have that $[X, Y]_p \in D_p$.*

Proof. Let $X, Y \in \mathfrak{X}(D)$ and fix $p \in M$. Assume there exists an integral manifold (N, Φ) of D through p and choose $q \in N$, such that $\Phi(q) = p$. For any $q' \in N$, the map $\mathrm{d}_{q'}\Phi : T_{q'}N \to T_{\Phi(q')}M$ is injective and its image is $D_{\Phi(q')}$. Hence, there exist smooth vector fields $\tilde{X}, \tilde{Y} \in \mathfrak{X}(N)$ which are Φ-related with X and Y, respectively. It follows that $[\tilde{X}, \tilde{Y}]$ is also Φ-related with $[X, Y]$ and we must have

$$[X, Y]_p = \mathrm{d}_{q_0}\Phi([\tilde{X}, \tilde{Y}]_q) \in \mathrm{d}_q\Phi(T_qN) = D_p.$$

□

In particular, if $D = T\mathcal{F}$ for some foliation $\mathcal{F}$, then for any pair of vector fields $X, Y \in \mathfrak{X}(D)$ we have that $[X, Y] \in \mathfrak{X}(D)$.

Definition 11.7. *A smooth distribution D in M is called **involutive** if for any $X, Y \in \mathfrak{X}(D)$ one has $[X, Y] \in \mathfrak{X}(D)$.*

The following result says that the lack of involutivity is the only obstruction to integrability of a distribution:

Theorem 11.8 (Frobenius)**.** *A smooth distribution D is integrable if and only if it is involutive. In this case, the integral foliation tangent to D is unique.*

Proof. Proposition 11.6 show that one of the implications hold. To check the other implication we assume that D is an involutive distribution.

We claim that, for each $p \in M$, there exist vector fields $X_1, \dots, X_k \in \mathfrak{X}(U)$, defined in an open neighborhood U de p, such that:

(a) $D|_U = \langle X_1, \dots, X_k \rangle$;
(b) $[X_i, X_j] = 0$, for every $i, j = 1, \dots, k$.

Then, by Exercise 6 in Lecture 10, we obtain an open cover $\{U_i\}_{i\in I}$ of M, such that for each $i \in I$ there exists a unique foliation $\mathcal{F}_i$ in U_i which satisfies $T\mathcal{F}_i = D|_{U_i}$. By uniqueness, whenever $U_i \cap U_j \neq 0$, we obtain $\mathcal{F}_i|_{U_i\cap U_j} = \mathcal{F}_j|_{U_i\cap U_j}$. Hence, there exists a unique foliation $\mathcal{F}$ of M such that $\mathcal{F}|_{U_i} = \mathcal{F}_i$.

To prove the claim, fix $p \in M$. Since D is smooth, there exist vector fields $Y_1, \dots, Y_k$ defined in some neighborhood V of p, such that $D|_V = \langle Y_1, \dots, Y_k\rangle$. We can also assume that V is the domain of some coordinate system $(x^1, \dots, x^d)$ of M, so that

$$Y_i = \sum_{l=1}^{d} a_{il}\frac{\partial}{\partial x^l}, \quad (i = 1, \dots, k),$$

where $a_{il} \in C^\infty(V)$. The matrix $A(q) = [a_{il}(q)]_{i,l=1}^{k,d}$ has rank k at p and we can assume, eventually after some relabeling of the the coordinates, that the $k \times k$ minor formed by the first k rows and k columns of A has non-zero determinant in a smaller open neighborhood U of p. Let B be the $k \times k$ inverse matrix of this minor, and define new vector fields $X_1, \dots, X_k \in \mathfrak{X}(U)$ by:

$$\begin{aligned} X_i &= \sum_{j,l=1}^{k,d} b_{ij}a_{jl}\frac{\partial}{\partial x^l} \\ &= \frac{\partial}{\partial x^i} + \sum_{l=k+1}^{d} c_{il}\frac{\partial}{\partial x^l}, \quad (i = 1, \dots, k), \end{aligned}$$

where $c_{il} \in C^\infty(U)$. On the one hand, we have that

$$D|_U = \langle Y_1, \dots, Y_k\rangle = \langle X_1, \dots, X_k\rangle,$$

so (a) is satisfied. On the other hand, a simple computation shows that:

$$[X_i, X_j] = \sum_{l=k+1}^{d} d_l^{ij}\frac{\partial}{\partial x^l}, \quad (i, j = 1, \dots, k),$$

for certain functions $d_l^{ij} \in C^\infty(U)$. Since D is involutive, this commutator must be a $C^\infty(M)$-linear combination of $X_1, \dots, X_k$. Therefore, the functions d_l^{ij} must be identically zero, so (b) also holds. $\square$

Homework.

1. Give an example of a smooth distribution D of dimension 1 which is not globally generated by only one vector field.

2. Show that the 2-dimensional distribution D in Example 11.2.3 is not globally generated by only 2 vector fields.

3. For the distribution given in Example 11.2.2, show that the only points through which there exist integral manifolds are the points in the plane $z = 0$.

4. Show that the 2-dimensional distribution in $\mathbb{R}^3$ defined by the vector fields
$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = e^{-x}\frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$
has no integral manifolds.

5. Consider the distribution D in $\mathbb{R}^3$ generated by the vector fields:
$$\frac{\partial}{\partial x} + \cos x \cos y \frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial y} - \sin x \sin y \frac{\partial}{\partial z}.$$
Check that D is involutive and determine the foliation $\mathcal{F}$ that integrates it.

6. In the 3-sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ consider the 1-dimensional distribution defined by
$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - w\frac{\partial}{\partial z} + z\frac{\partial}{\partial w}.$$
Determine the foliation $\mathcal{F}$ integrating this distribution.

Lecture 12. Lie Groups and Lie Algebras

The next definition axiomatizes some of the properties of the Lie bracket of vector fields (see Proposition 10.4).

Definition 12.1. *A **Lie algebra** is a vector space $\mathfrak{g}$ with a binary operation $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the **Lie bracket**, which satisfies:*

(i) Skew-symmetry: $[X, Y] = -[Y, X]$;
(ii) Bilinearity: $[aX + bY, Z] = a[X, Z] + b[Y, Z]$, $\forall a, b \in \mathbb{R}$;
(iii) Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

We can also define Lie algebras over the complex numbers ($\mathfrak{g}$ is a complex vector space) or over other fields. Note also, that $\mathfrak{g}$ can have infinite dimension, but we will be mainly interested in finite dimensional Lie algebras.

Examples 12.2.

*1. $\mathbb{R}^d$ with the zero Lie bracket $[\ ,\ ] \equiv 0$ is a Lie algebra, called the **abelian Lie algebra** of dimension d.*

2. In $\mathbb{R}^3$, we can define a Lie algebra structure where the Lie bracket is the vector product:
$$[\vec{v}, \vec{w}] = \vec{v} \times \vec{w}.$$

3. If V is any vector space, the vector space of all linear transformations $T : V \to V$ is a Lie algebra with Lie bracket the commutator:
$$[T, S] = T \circ S - S \circ T.$$
*This Lie algebra is called the **general linear Lie algebra** and denoted $\mathfrak{gl}(V)$. When $V = \mathbb{R}^n$, we denote it by $\mathfrak{gl}(n)$. After fixing a basis, we can identitify $\mathfrak{gl}(n)$ with the space of all $n \times n$ matrices, and the Lie bracket becomes the commutator of matrices.*

4. If $\mathfrak{g}_1, \dots, \mathfrak{g}_k$ are Lie algebras, their cartesian product $\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$ is a Lie algebra with Lie bracket:

$$[(X_1, \dots, X_k), (Y_1, \dots, Y_k)] = ([X_1, Y_1]_{\mathfrak{g}_1}, \dots, [X_k, Y_k]_{\mathfrak{g}_k}).$$

We shall see shortly that Lie algebras are the "infinitesimal versions" of groups with a smooth structure:

Definition 12.3. *A* ***Lie group*** *is a group G with a smooth structure such that the following maps are smooth:*

$$\mu : G \times G \to G,\ (g, h) \mapsto gh \quad \text{(multiplication)},$$
$$\iota : G \to G,\ g \mapsto g^{-1} \quad \text{(inverse)}.$$

One can also define topological groups, analytic groups, etc.

EXAMPLES 12.4.

1. *Any countable group with the discrete topology is a Lie group of dimension 0 (we need it to be countable so that the discrete topology is second countable).*
2. *$\mathbb{R}^d$ with the usual addition of vectors is an abelian Lie group. The groups of all non-zero real numbers $\mathbb{R}^*$ and all non-zero complex numbers $\mathbb{C}^*$, with the usual multiplication operations, are also abelian Lie groups. Note that $\mathbb{C}^*$ is also a complex Lie group (thinking of $\mathbb{C}^*$ as a complex manifold), but we will restrict ourselves always to real Lie groups (i.e., all our manifolds are real manifolds).*
3. *The circle $\mathbb{S}^1 = \{z \in \mathbb{C} : ||z|| = 1\} \subset \mathbb{C}^*$ with the usual complex multiplication is also an abelian Lie group. The unit quaternions $\mathbb{S}^3$, with quaternionic multiplication, is a non-abelian Lie group. It can be shown that the only spheres $\mathbb{S}^d$ that admit Lie group structures are $d = 0, 1, 3$.*
4. *The set of all invertible linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$ is a Lie group, called the* ***general linear group*** *and denoted by $GL(n)$. After fixing a basis we can identify $GL(n)$ with the group of all invertible $n \times n$ matrices.*
5. *If $G_1, \dots, G_k$ are Lie groups their cartesian product $G \times \cdots \times G_k$ is also a Lie group. For example, the torus $\mathbb{T}^d = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ is a (abelian) Lie group.*
6. *If G is a Lie group, its connected component of the identity, is a Lie group denoted by G^0. For example, the connected component of the identity of the Lie group $(\mathbb{R}^*, \times)$ is group of positive real numbers $(\mathbb{R}_+, \times)$.*

In a Lie group G, a **left invariant vector field** is a vector field X such that:

$$(L_g)_* X = X, \quad \forall g \in G,$$

where $L_g : G \to G$, $h \mapsto gh$ denotes the **left translation** by g. One defines analogously a **right invariant vector field** using the **right translation** $R_g : G \to G$, $h \mapsto hg$.

Proposition 12.5. *Let G be a Lie group.*

(i) Every left invariant vector field if smooth.
(ii) If $X, Y \in \mathfrak{X}(G)$ are left invariant vector fields then $[X, Y]$ is also left invariant.
(iii) The set of all left invariant vector fields is a finite dimensional subspace of $\mathfrak{X}(G)$ of dimension $\dim G$.

Proof. We leave the prof of (i) as an exercise. To check (ii), it is enough to observe that if X and Y are left invariant vector fields then:

$$(L_g)_*[X, Y] = [(L_g)_*X, (L_g)_*Y] = [X, Y], \quad \forall g \in G.$$

Hence, $[X, Y]$ is also a left invariant vector field.

Now to see that (iii) holds, let $\mathfrak{X}_{\mathrm{inv}}(G)$ be the set of all left invariant vector fields. It is clear from the definition of a left invariant vector field that $\mathfrak{X}_{\mathrm{inv}}(G) \subset \mathfrak{X}(G)$ is a linear subspace. O the other hand, the restriction map

$$\mathfrak{X}_{\mathrm{inv}}(G) \to T_eG, \; X \mapsto X_e,$$

is a linear isomorphism: if $\mathbf{v} \in T_eG$ we can define a left invariant vector field X in G with $X_e = \mathbf{v}$ by setting

$$X_g = \mathrm{d}L_g \cdot \mathbf{v}.$$

Hence, the restriction $\mathfrak{X}_{\mathrm{inv}}(G) \to T_eG$ is invertible. We conclude that:

$$\dim \mathfrak{X}_{\mathrm{inv}}(G) = \dim T_eG = \dim G.$$

□

This proposition show that for a Lie group G the set of all left invariant vector fields forms a Lie algebra, called the **Lie algebra of the Lie group** G, and denoted by $\mathfrak{g}$. The proof also shows that $\mathfrak{g}$ can be identified with T_eG.

Examples 12.6.

1. *The Lie algebra of a discrete Lie group G is the zero dimensional vector space $\mathfrak{g} = \mathbb{R}^0 = \{0\}$.*
2. *Let $G = (\mathbb{R}^d, +)$. A vector field in $\mathbb{R}^d$ is left invariant if and only if it is constant: $X = \sum_{i=1}^d a_i \frac{\partial}{\partial x^i}$, with $a_i \in \mathbb{R}$. The Lie bracket of any two such constant vector fields is zero, hence the Lie algebra of G is the abelian Lie algebra of dimension d.*
3. *The Lie algebra of the cartesian product $G \times H$ of two Lie groups, is the cartesian product $\mathfrak{g} \times \mathfrak{h}$ of their Lie algebras. For example, the Lie algebra of $\mathbb{S}^1$ has dimension 1, hence it is abelian. It follows that the Lie algebra of the torus $\mathbb{T}^d$ is also the abelian Lie algebra of dimension d.*
4. *The tangent space at the identity to the general linear group $G = GL(n)$ can be identified with $\mathfrak{gl}(n)$. The restriction map $\mathfrak{g} \to \mathfrak{gl}(n)$, maps the commutator of left invariant vector fields in the commutator of matrices (exercise). Hence, we can identify the Lie algebra of $GL(n)$ with $\mathfrak{gl}(n)$.*

Remark 12.7. The space $\mathfrak{X}(M)$ formed by all vector fields in a manifold M is a Lie algebra. One may wonder if the Lie algebra $\mathfrak{X}(M)$ is associated with some Lie group. Since this Lie algebra is infinite dimensional (if $\dim M > 0$), this Lie group must be infinite dimension. This group exists: it is the group $\mathrm{Diff}(M)$ of all diffeomorphisms of M under composition. The study of such infinite dimensional Lie groups is an important topic which is beyond the scope of this course.

We have seen that to each Lie groups there is associated a Lie algebra. Similarly, to each homomorphism of Lie groups there is associated a homomorphism of their Lie algebras.

Definition 12.8.

(i) A ***homomorphism of Lie algebras*** *is a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ between two Lie algebras which preserves the Lie brackets:*

$$\phi([X, Y]_{\mathfrak{g}}) = [\phi(X), \phi(Y)]_{\mathfrak{h}}, \quad \forall X, Y \in \mathfrak{g}.$$

(ii) A ***homomorphism of Lie groups*** *is a smooth map $\Phi : G \to H$ between two Lie groups which is also a group homorphism:*

$$\Phi(gh^{-1}) = \Phi(g)\Phi(h)^{-1}, \quad \forall g, h \in G.$$

If $\Phi : G \to H$ is a homomorphism of Lie groups we have an induced map $\Phi_* : \mathfrak{g} \to \mathfrak{h}$: if $X \in \mathfrak{g}$, then $\Phi_*(X) \in \mathfrak{h}$ is the unique left invariant vector field such that $\mathrm{d}_e\Phi \cdot X_e$.

Proposition 12.9. *If $\Phi : G \to H$ is a Lie group homomorphism, then:*

(i) For all $X \in \mathfrak{g}$, $\Phi_* X$ is Φ-related with X;

(ii) $\Phi_* : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

Proof. Part (ii) follows from (i), since the Lie bracket of Φ-related vector fields is preserved. In order to show that (i) holds, we observe that since Φ is a group homomorphism, $\Phi \circ L_g = L_{\Phi(g)} \circ \Phi$. Hence:

$$\begin{aligned}
\Phi_*(X)_{\Phi(g)} &= \mathrm{d}_e L_{\Phi(g)} \cdot \mathrm{d}_e\Phi \cdot X_e \\
&= \mathrm{d}_e(L_{\Phi(g)} \circ \Phi) \cdot X_e \\
&= \mathrm{d}_e(\Phi \circ L_g) \cdot X_e \\
&= \mathrm{d}_g\Phi \cdot \mathrm{d}_e L_g \cdot X_e = \mathrm{d}_g\Phi \cdot X_g.
\end{aligned}$$

□

Examples 12.10.

1. *Let $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$. For each $a \in \mathbb{R}$ we have the Lie group homomorphism $\Phi_a : \mathbb{R} \to \mathbb{T}^2$ given by:*

$$\Phi_a(t) = (e^{it}, e^{iat}).$$

If a is rational, the image Φ_a is a closed curve, while if a is irrational the image is dense curve in the torus. The induced Lie algebra homomorphism $(\Phi_a)_* : \mathbb{R} \to \mathbb{R}^2$ is given by:

$$(\Phi_a)_* = (X, aX).$$

2. *The determinant defines a Lie group homomorphim* $\det : GL(n) \to \mathbb{R}^*$*. The induced Lie algebra homomorphism is the trace* $\mathrm{tr} = (\det)_* : \mathfrak{gl}(n) \to \mathbb{R}$.
3. *Each invertible matrix* $A \in GL(n)$ *determines a Lie group automorphism* $\Phi_A : GL(n) \to GL(n)$ *given by conjugation:*
$$\Phi_A(B) = ABA^{-1}.$$
Since this map is linear, the associated Lie algebra automorphism $(\Phi_A)_* : \mathfrak{gl}(n) \to \mathfrak{gl}(n)$ *is also given by:*
$$(\Phi_A)_*(X) = AXA^{-1}.$$
4. *More generally, for any Lie group* G *we can consider conjugation by a fix* $g \in G$*:* $i_g : G \to G$, $h \mapsto ghg^{-1}$*. This is a Lie group automorphism and the induced Lie algebra automorphism is denoted by* $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$*:*
$$\mathrm{Ad}(g)(X) = (i_g)_* X.$$

Let us continue our study of the Lie group/algebra correspondence. We show now that to each subgroup of a Lie group G corresponds a Lie sub algebra of the Lie algebra $\mathfrak{g}$ of G.

Definition 12.11. *A subspace* $\mathfrak{h} \subset \mathfrak{g}$ *is called a* ***Lie subalgebra*** *if, for all* $X, Y \in \mathfrak{h}$*, we have* $[X, Y] \in \mathfrak{h}$*.*

Examples 12.12.

1. *Any subspace of the abelian Lie algebra* $\mathbb{R}^d$ *is a Lie subalgebra.*
2. *In the Lie algebra* $\mathfrak{gl}(n)$ *we have the Lie subalgebra formed by all matrices of zero trace:*
$$\mathfrak{sl}(n) = \{X \in \mathfrak{gl}(n) : \mathrm{tr}\, X = 0\},$$
and also the Lie subalgebra formed by all skew-symmetric matrices:
$$\mathfrak{o}(n) = \{X \in \mathfrak{gl}(n) : X + X^T = 0\}.$$
3. *The complex* $n \times n$ *matrices, denoted by* $\mathfrak{gl}(n, \mathbb{C})$*, can be seen as a real Lie algebra. It has the Lie subalgebra of all skew-Hermitean matrices:*
$$\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) : X + \bar{X}^T = 0\},$$
and the Lie subalgebra of all skew-Hermitean matrices of trace zero:
$$\mathfrak{su}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) : X + \bar{X}^T = 0, \mathrm{tr}\, X = 0\}.$$
4. *If* $\phi : \mathfrak{g} \to \mathfrak{h}$ *is a homomorphism of Lie algebras de Lie, then its kernel is a Lie subalgebra of* $\mathfrak{g}$ *and its image is a Lie subalgebra of* $\mathfrak{h}$*.*

A notion of a Lie subgroup is defined similarly:

Definition 12.13. *A* ***Lie subgroup*** *of* G *is a submanifold* (H, Φ) *of* G *such that:*

(i) H *is Lie group;*
(ii) $\Phi : H \to G$ *is a Lie group homomorphism.*

As we discussed in Lecture 5, we can always replace the submanifold (H, Φ) by the subset $\Phi(G) \subset G$, and the immersion Φ by the inclusion i. Since $\Phi(G)$ is a subgroup of G, in the definition of a Lie subgroup we can assume that $H \subset G$ is a a subgroup and that Φ is the inclusion. On the other hand, since the induced map $\Phi_* : \mathfrak{h} \to \mathfrak{g}$ is injective, we can assume that the Lie algebra of a Lie subgroup $H \subset G$ is a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

Examples 12.14.

1. *In Example 12.10.1, for each $a \in \mathbb{R}$ we have a Lie subgroup $\Phi_a(\mathbb{R})$ of $\mathbb{T}^2$. If a is rational, this Lie subgroup is embedded, while if a is irrational this Lie subgroup is only immersed.*

2. *The general linear group $GL(n)$ has the following (embedded) subgroups:*
   (i) *The **special linear group** of all matrices of determinant 1:*
   $$SL(n) = \{A \in GL(n) : \det A = 1\}.$$
   To this subgroup corresponds the Lie subalgebra $\mathfrak{sl}(n)$.
   (ii) *The **orthogonal group** of all orthogonal matrices:*
   $$O(n) = \{A \in GL(n) : AA^T = I\}.$$
   To this subgroup corresponds the Lie subalgebra $\mathfrak{o}(n)$.
   (iii) *The **special orthogonal group** of all orthogonal matrices of positive determinant:*
   $$SO(n) = \{A \in O(n) : \det A = 1\}.$$
   To this subgroup corresponds the Lie subalgebra $\mathfrak{so}(n) = \mathfrak{o}(n)$.

3. *The (real) Lie group $GL(n, \mathbb{C})$ has the following (embedded) subgroups:*
   (i) *The **unitary group** of all unitary matrices:*
   $$U(n) = \{A \in GL(n, \mathbb{C}) : A\bar{A}^T = I\}.$$
   To this subgroup corresponds the Lie subalgebra $\mathfrak{u}(n)$.
   (ii) *The **special unitary group** of all unitary matrices of determinant 1:*
   $$SU(n) = \{A \in U(n) : \det A = 1\}.$$
   To this subgroup corresponds the Lie subalgebra $\mathfrak{su}(n)$.

4. *Let $\Phi : G \to H$ is a Lie group homomorphism and let $(\Phi)_* : \mathfrak{g} \to \mathfrak{h}$ the induced Lie algebra homomorphism. Then $\operatorname{Ker}\Phi \subset G$ and $\operatorname{Im}\Phi \subset H$ are Lie subgroups whose Lie algebras coincide with $\operatorname{Ker}(\Phi)_* \subset \mathfrak{g}$ and $\operatorname{Im}(\Phi)_* \subset \mathfrak{h}$, respectively.*

Homework.

1. Show that in the definition of a Lie group, it is enough to assume that:
   (a) The inverse map $G \to G$, $g \mapsto g^{-1}$ is smooth, or that
   (b) The map $G \times G \to G$, $(g, h) \mapsto gh^{-1}$, is smooth.

2. Show that every left invariant vector field in a Lie group G is smooth and complete.

3. Show that the tangent space at the identity of $GL(n)$ can be identified with $\mathfrak{gl}(n)$. Show also that, under this identification, the linear isomorphism $\mathfrak{g} \to \mathfrak{gl}(n)$ takes the Lie bracket of left invariant vector fields to the commutator of matrices.

4. Show that the the tangent bundle TG of a Lie group G is trivial, i.e., there exist vector fields $X_1, \ldots, X_d \in \mathfrak{X}(G)$ which at each $g \in G$ give a basis for T_gG. Conclude that an even dimension sphere $\mathbb{S}^{2n}$ does not admit the structure of a Lie group.

5. Show that the Lie algebra homomorphism induced by the determinant $\det : GL(n) \to \mathbb{R}^*$ is the trace: $\mathrm{tr} = (\det)_* : \mathfrak{gl}(n) \to \mathbb{R}$.

6. Consider $\mathbb{S}^3 \subset \mathbb{H}$ as the set of quaternions of norm 1. Show that $\mathbb{S}^3$, with the product of quaternions, is a Lie group and determine its Lie algebra.

7. Show that $\mathbb{S}^3$ and $SU(2)$ are isomorphic Lie groups.

HINT: For any pair of complex numbers $z, w \in \mathbb{C}$ with $|z|^2 + |w|^2 = 1$, the matrix:

$$\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$$

is an element in $SU(2)$.

8. Identify the vectors $v \in \mathbb{R}^3$ with the purely imaginary quaternions. For each quaternion $q \in \mathbb{S}^3$ of norm 1 define a linear map $T_q : \mathbb{R}^3 \to \mathbb{R}^3$ by $v \mapsto qvq^{-1}$. Show that T_q is a special orthogonal transformation and that the map $\mathbb{S}^3 \to SO(3)$, $q \mapsto T_q$, is a Lie group homomorphism. Is this map surjective? Injective?

9. Let G be a Lie group. Show that the connected component of the identity is a Lie group G^0 whose Lie algebra is isomorphic to the Lie algebra of G.

10. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Show that G is abelian if and only if $\mathfrak{g}$ is abelian. What can you say if G is not connected?

11. Show that a compact connected abelian Lie group G is isomorphic to a torus $\mathbb{T}^d$.

12. Let (H, Φ) be a Lie subgroup of G. Show that Φ is an embedding if and only if $\Phi(H)$ is closed in G.

13. Let $A \subset G$ be a subgroup of a Lie group G. Show that if (A, i) has a smooth structure making it into a submanifold of G, then this smooth structure is unique and that for that smooth structure A is a Lie group and (A, i) a Lie subgroup.

HINT: Show that (A, i) is a regularly embedded submanifold.

We saw in the previous lecture that:

- To each Lie group corresponds a Lie algebra;
- To each Lie group homomorphism corresponds a Lie algebra homomorphism;
- To each Lie subgroup corresponds a Lie subalgebra.

It is natural to wonder about the inverse to these correspondences. We have seen that two distinct Lie groups can have isomorphic Lie algebras (e.g., $\mathbb{R}^n$ and $\mathbb{T}^n$,$O(n)$ and $SO(n)$, $SU(2)$ and $SO(3)$). There are indeed topological issues that one must take care of when studying the inverse correspondences.

We start with the following result that shows that a connected Lie group is determined by a neighborhood of the identity:

Proposition 13.1. *Let G be a connected Lie group and U a neighborhood of the identity $e \in G$. Then,*

$$G = \bigcup_{n=1}^{\infty} U^n,$$

where $U^n = \{g_1 \cdots g_n : g_i \in U, i = 1, \ldots, n\}$.

Proof. If $U^{-1} = \{g^{-1} : g \in U\}$ and $V = U \cap U^{-1}$, then V is a neighborhood of the origin such that $V = V^{-1}$. Let:

$$H = \bigcup_{n=1}^{\infty} V^n \subset \bigcup_{n=1}^{\infty} U^n.$$

To complete the proof it is enough to show that $H = G$. For that observe that:

(i) H is a subgroup: if $g, h \in H$, then $g = g_1 \ldots g_n$ and $h = h_1 \ldots h_m$, with $g_i, h_j \in V$. Hence,
$$gh^{-1} = g_1 \ldots g_n h_m^{-1} \ldots h_1^{-1} \in V^{n+m} \subset H.$$
(ii) H is open: if $g \in H$ then $gV \subset gH = H$ is an open set containing g.
(iii) H is closed: for each $g \in G$, gH is an open set and we have
$$H^c = \bigcup_{g \notin H} gH.$$

Since G is connected and $H \neq \emptyset$ is open and closed, we conclude that $H = G$. □

We can now prove:

Theorem 13.2. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Gien a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, there exists a unique connected Lie subgroup $H \subset G$ with Lie algebra $\mathfrak{h}$.*

Proof. A Lie subalgebra $\mathfrak{h}$ defines a distribution in G by setting:

$$D : g \mapsto D_g \equiv \{X_g : X \in \mathfrak{h}\}.$$

This distribution is smooth and involutive: if $X_1, \dots, X_k$ is a basis for $\mathfrak{h}$, then these vector fields are smooth and generate D everywhere, hence D is smooth C^∞. On the other hand, if $Y, Z \in \mathfrak{X}(D)$, then

$$Y = \sum_{i=1}^{k} a_i X_i, \quad Z = \sum_{j=1}^{k} b_j X_j.$$

so using that $\mathfrak{h}$ is a Lie subalgebra it follows that:

$$[Y, Z] = \sum_{i,j=1}^{k} a_i a_j [X_i, X_j] + a_i X_i(b_j) X_j - b_j X_j(a_i) X_i \in \mathfrak{X}(D),$$

so this distribution is involutive.

Let (H, Φ) be the leaf of this distribution that contains the identity $e \in G$. If $g \in \Phi(H)$, then $(H, L_{g^{-1}} \circ \Phi)$ is also an integral manifold of D which contains e. Hence, $L_{g^{-1}} \circ \Phi(H) \subset \Phi(H)$. We conclude that for all $g, h \in \Phi(H)$, we have $g^{-1}h \in \Phi(H)$, so $\Phi(H)$ is a subgroup of G. Since $\Phi : H \to \Phi(H)$ is a bijection, it follows that H as unique group structure such that $\Phi : H \to G$ is a group homomorphism. To verify that (H, Φ) is a Lie subgroup, it remains to prove that $\hat{\nu} : H \times H \to H$, $(g, h) \mapsto g^{-1}h$, is smooth. For this we observe that the map $\nu : H \times H \to G$, $(g, h) \mapsto \Phi(g)^{-1}\Phi(h)$ is smooth, being the composition of smooth maps. The following diagram is commutative:

$$\begin{tikzcd}
H \times H \arrow[r, "\nu"] \arrow[dr, "\hat{\nu}"'] & G \\
& H \arrow[u, "\Phi"']
\end{tikzcd}$$

Since the leaves of any foliation are regularly embedded, we conclude that $\hat{\nu} : H \times H \to H$ is smooth.

Uniqueness follows from Proposition 13.1 (exercise). ☐

The question of deciding if every finite dimensional Lie algebra $\mathfrak{g}$ is associated with some Lie group G is a much harder question which is beyond these notes. There are several ways to proceed to prove that this is indeed true. One way, is to first show that any finite dimensional Lie algebra is isomorphic to a matrix Lie algebra. This requires developing the structure theory of Lie algebras and can be stated as follows:

Theorem 13.3 (Ado)**.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. There exists an integer n and an injective Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{gl}(n)$.*

Remark 13.4. A **representation of a Lie algebra** $\mathfrak{g}$ in a vector space V is a Lia algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. A representation (V, ρ) is called **faithful** if ρ is injective. In this language, Ado's Theorem states that every finite dimensional representation has a faithful representation.

Since $\mathfrak{gl}(n)$ is the Lie algebra of $GL(n)$, as a corollary of Ado's Theorem and Theorem 13.2 we obtain:

Theorem 13.5. *For any finite dimensional Lie algebra $\mathfrak{g}$ there exists a Lie group G with Lie algebra isomorphic to $\mathfrak{g}$.*

The previous theorem gives a matrix group integrating any finite dimensional Lie algebra. Note however, in spite of what Ado's Theorem may suggest, that there are Lie groups which *are not* isomorphic to any matrix group. This happens because, as we know, there can be several Lie groups integrating the same Lie algebra.

In order to clarify the issue of multiple Lie groups integrating the same Lie algebra, recall that if $\pi : N \to M$ is a covering of a manifold M, then there is a unique differentiable structure on N for which the covering map is a local diffeomorphism. In particular, if M is connected then the **universal covering space** of M, which is characterize as a 1-connected (i.e., connected and simply connected) covering of M, is a manifold and is unique up to diffeomorphism. For Lie groups this leads to:

Proposition 13.6. *Let G be a Lie group. Its universal covering space $\widetilde{G}$ has a unique Lie group structure such that the covering map $\pi : \widetilde{G} \to G$ is a Lie group homomorphism and the Lie algebras of G and $\widetilde{G}$ are isomorphic.*

Proof. We can identify the universal covering space $\widetilde{G}$ with the homotopy classes of paths $\gamma : [0,1] \to G$, $\gamma(0) = e$, so that the covering map is $\pi([\gamma]) = \gamma(1)$. We define a group structure in $\widetilde{G}$ as follows:

(a) Multiplication $\mu : \widetilde{G} \times \widetilde{G} \to \widetilde{G}$: the product $[\gamma][\eta]$ is the homotopy class of the path $t \mapsto \gamma(t)\eta(t)$.

(b) Identity $\tilde{e} \in \widetilde{G}$: it is homotopy class of the constant path based at the identity $\gamma(t) = e$.

(c) Inverse $i : \widetilde{G} \to \widetilde{G}$: the inverso of the element $[\gamma]$ is the homotopy class of the path $t \mapsto \gamma(t)^{-1}$.

It is clear that we these choices the covering map $\pi : \widetilde{G} \to G$ is a group homomorphism.

Recall now that there is a unique smooth structure on $\widetilde{G}$ for which the covering map is a local diffeomorphism. To check that $\widetilde{G}$ is a Lie group, we observe that $\tilde{\nu} : \widetilde{G} \times \widetilde{G} \to \widetilde{G}$, $(g,h) \to g^{-1}h$, is smooth since we have a commutative diagram:

$$\begin{tikzcd}
\widetilde{G} \times \widetilde{G} \arrow[r, "\tilde{\nu}"] \arrow[d, "\pi\times\pi"'] & \widetilde{G} \arrow[d, "\pi"] \\
G \times G \arrow[r, "\nu"'] & G
\end{tikzcd}$$

where the vertical arrows are local diffeomorphisms and ν is differentiable. Since $\pi : \widetilde{G} \to G$ is a local diffeomorphism it induces an isomorphism between the Lie algebras of $\widetilde{G}$ and G.

Uniqueness follows, because the condition that $\pi : \widetilde{G} \to G$ induces an isomorphism between the Lie algebras of $\widetilde{G}$ and G implies that π is a local diffeomorphism, so both the smooth structure and the group structure are uniquely determined. $\square$

From the uniqueness of the universal covering space we conclude that:

Corollary 13.7. *Given a finite dimensional Lie algebra $\mathfrak{g}$ there exists, up to isomorphism, a unique 1-connected Lie group G with Lie algebra isomorphic to $\mathfrak{g}$.*

Example 13.8.

The special unitary group $SU(2)$ is formed by the matrices:

$$SU(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}.$$

Therefore $SU(2)$ is isomorphic as a manifold to $\mathbb{S}^3$, hence it is 1-connected. In fact, by an exercise in the previous lecture, $SU(2)$ is isomorphic, as a Lie group, to $\mathbb{S}^3$.

The Lie algebra of $SU(2)$ consists of the skew-hermitean matrices of trace zero:

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} i\alpha & \beta \\ -\bar{\beta} & -i\alpha \end{pmatrix} : \alpha \in \mathbb{R}, \beta \in \mathbb{C} \right\}.$$

We will identify the matrix defined by the elements α and β with the vector $(\alpha, \operatorname{Re}\beta, \operatorname{Im}\beta) \in \mathbb{R}^3$, so we will think of $\mathfrak{su}(2)$ as $\mathbb{R}^3$ with the standard euclidean inner product.

For each $g \in SU(2)$ we have the linear transformation $\operatorname{Ad} g : \mathfrak{su}(2) \to \mathfrak{su}(2)$ (see Example 12.10.3). We leave it as an exercise to check that:

(a) The linear transformation $\operatorname{Ad} g$ determines an element in $SO(3)$.

(b) $\operatorname{Ad} : SU(2) \to SO(3)$ is a surjective group homomorphism with kernel $\{\pm I\}$.

It follows that $\operatorname{Ad} : SU(2) \to SO(3)$ is a covering map. Since $SU(2)$ is 1-connected, we conclude that $SU(2) \simeq \mathbb{S}^3$ is the universal covering space of $SO(3)$. The covering map identifies the antipodal points in the sphere, so we can identify $SO(3)$ with the projective space $\mathbb{P}^3$ and $\pi_1(SO(3)) = \mathbb{Z}_2$.

Let us consider now the question of integrating homomorphisms of Lie algebras to homomorphisms of Lie groups.

We start by remarking that again there are topological obstructions. For example, the identity map $\phi : \mathbb{R} \to \mathbb{R}$ is a Lie algebra isomorphism between the Lie algebras of the Lie groups $\mathbb{S}^1$ and $\mathbb{R}$. However, there are no non-trivial Lie group homomorphisms $\Phi : \mathbb{S}^1 \to \mathbb{R}$: if there was such homomorphism its image $\Phi(\mathbb{S}^1)$ would be a compact, nontrivial, subgroup of $\mathbb{R}$. Therefore, there is no Lie group homomorphism $\Phi : \mathbb{S}^1 \to \mathbb{R}$ with $\Phi_* = \phi$.

The problem in this example is that $\mathbb{S}^1$ is not simply connected. In fact, we have:

Theorem 13.9. *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. If G is 1-connected then for every Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ there exists a unique Lie group homomorphism $\Phi : G \to H$ such that $\Phi_* = \phi$.*

Proof. Let $\mathfrak{k} = \{(X, \phi(X)) : X \in \mathfrak{g}\} \subset \mathfrak{g} \times \mathfrak{h}$ be the graph of ϕ. Since ϕ is a Lie algebra homomorphism, de álgebras $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g} \times \mathfrak{h}$. Hence, there exists a unique connected Lie subgroup $K \subset G \times H$ with Lie algebra $\mathfrak{k}$. Let us consider the restriction to K of the projections on each factor:

$$\begin{tikzcd}
 & K \subset G \times H \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
G & & H
\end{tikzcd}$$

The restriction of the first projection $\pi_1|_K : K \to G$ gives a Lie group homomorphism such that:

$$(\pi_1)_*(X, \phi(X)) = X.$$

Hence, the map $(\pi_1|_K)_* : \mathfrak{k} \to \mathfrak{g}$ is a Lie algebra isomorphism and it follows that $\pi_1|_K : K \to G$ is a covering map (see the Exercises). Since G is 1-connected, we conclude that $\pi_1|_K$ is a Lie isomorphism. Then, the composition

$$\Phi = \pi_2 \circ (\pi_1|_K)^{-1} : G \to H$$

is a Lie group homomorphism and we have that:

$$\begin{aligned}(\Phi)_*(X) &= (\pi_2)_* \circ (\pi_1|_K)_*^{-1}(X) \\ &= (\pi_2)_*(X, \phi(X)) = \phi(X).\end{aligned}$$

We leave the proof of uniqueness as an exercise. □

We saw above that every Lie algebra is isomorphic to a Lie algebra of matrices (Ado's Theorem). As an application of the integration of morphisms we show that there are Lie groups which are not groups of matrices.

Example 13.10.
For the special linear group

$$SL(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}.$$

the Lie algebra $\mathfrak{sl}(2)$ is formed by the space of 2×2 matrices with trace zero. To exhibit the topological structure of $SL(2)$ it is convenient to perform the change of variables $(a, b, c, d) \mapsto (p, q, r, s)$ defined by

$$a = p + q, \quad d = p - q, b = r + s, c = r - s.$$

Then

$$ad - bc = 1 \quad \Longleftrightarrow p^2 + s^2 = q^2 + r^2 + 1.$$

Hence we se that we can also describe $SL(2)$ as:

$$SL(2) = \{(p, q, r, s) \in \mathbb{R}^4 : p^2 + s^2 = q^2 + r^2 + 1\}$$

so we conclude that $SL(2)$ is diffeomorphic to $\mathbb{R}^2 \times \mathbb{S}^1$. In particular,

$$\pi_1(SL(2)) = \pi_1(\mathbb{S}^1) = \mathbb{Z}.$$

Let $\widetilde{SL(2)}$ be the universal covering group of $SL(2)$. We claim that $\widetilde{SL(2)}$ is not isomorphic to any group of matrices. We shall need the following lemma, whose proof we leave as an exercise:

LEMMA 13.11. *Let $\phi : \mathfrak{sl}(2) \to \mathfrak{gl}(n)$ be a Lie algebra morphism. There exists a unique Lie group morphism $\Phi : SL(2) \to GL(n)$ such that $\Phi_* = \phi$.*

Assume that, for some n, there exists an injective Lie group homomorphism:

$$\widetilde{\Phi} : \widetilde{SL(2)} \to GL(n).$$

We claim that this leads to a contradiction. In fact, $\widetilde{\Phi}$ induces a Lie algebra morphism $\phi : \mathfrak{sl}(2) \to \mathfrak{gl}(n)$. By the lemma, there exists a unique Lie group homomorphism $\Phi : SL(2) \to GL(n)$ such that $\Phi_* = \phi$ and we obtain a commutative diagram:

$$\begin{tikzcd}
\widetilde{SL(2)} \arrow[r, "\widetilde{\Phi}"] \arrow[d, "\pi"'] & GL(V) \\
SL(2) \arrow[ur, "\Phi"'] & G
\end{tikzcd}$$

In this diagram the morphism π is not injective, while the morphism $\widetilde{\Phi}$ is injective, which is a contradiction.

As another application of the integration of morphisms, we show how one can construct an *exponential map* for Lie groups/algebras, which generalizes the exponential of matrices. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Given a left invariant vector field $X \in \mathfrak{g}$, the map $\mathbb{R} \to \mathfrak{g}$, $t \mapsto tX$, is a Lie algebra homomorphism. Since $\mathbb{R}$ is 1-connected it follows that there exists a unique Lie group homomorphism $\Phi_X : \mathbb{R} \to G$ with $\Phi_* = \phi$. We note that

$$\begin{aligned}
\Phi_X(0) &= e \\
\Phi_X(t+s) &= \Phi_X(t)\Phi_X(s) = L_{\Phi_X(t)}\Phi_X(s), \\
\frac{\mathrm{d}}{\mathrm{d}t}\Phi_X(t) &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi_X(t+s)\right|_{s=0} \\
&= \mathrm{d}_e L_{\Phi_X(t)} \cdot \left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi_X(s)\right|_{s=0} \\
&= \mathrm{d}_e L_{\Phi_X(t)} \cdot X_e = X_{\Phi_X(t)}.
\end{aligned}$$

This means that $t \mapsto \Phi_X(t)$ is actually the integral curve of X through $e \in G$. Recalling that ϕ_X^t denotes the flow of the vector field X, we have:

Definition 13.12. *The **exponential map** $\exp : \mathfrak{g} \to G$ is the map*

$$\exp(X) = \phi_X^1(e).$$

The following proposition lists the main properties of the exponential map. Its proof is left for the exercises.

Proposition 13.13. *The exponential map* $\exp : \mathfrak{g} \to G$ *satisfies:*

(i) $\exp((t+s)X) = \exp(sX)\exp(tX)$*;*

(ii) $\exp(-tX) = [\exp(tX)]^{-1}$*;*

(iii) $\exp$ *is a smooth map and* $\mathrm{d}_0 \exp = I$*;*

(iv) For any Lie group homomorphism $\Phi : G \to H$ *the following diagram is a commutative:*

$$\begin{tikzcd} G \arrow[r, "\Phi"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\Phi_*"'] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

Property (iii) implies that that the exponential is a diffeomorphism from a neighborhood of $0 \in \mathfrak{g}$ to a neighborhood of $e \in G$. In geral, the exponential $\exp : \mathfrak{g} \to G$ is neither surjective, nor injective. Also, it may fail to be a local diffeomorphism at other points of G. There are however examples of Lie groups/algebras in which some of these properties do hold (see also the exercises).

Example 13.14.

Recall that the Lie algebra of $G = GL(n)$ *can be identified with* $\mathfrak{gl}(n)$*. If* $A \in \mathfrak{gl}(n)$*, the left invariant vector field associated with the Lie algebra* $A = (a_{ij})$ *is:*

$$X_A = \sum_{ijk} a_{ik} x_{kj} \frac{\partial}{\partial x_{ij}}.$$

Hence, the integral curves if this vector field are the solutions of the system of ode's:

$$\dot{x}_{ij} = \sum_k a_{ik} x_{kj},$$

These are given by:

$$(x_{ij})(t) = e^{tA}(x_{ij})(0),$$

where the matrix exponential is defined by:

$$e^A = \sum_{k=0}^{+\infty} \frac{A^n}{n!}.$$

We conclude that the exponential map $\exp : \mathfrak{gl}(n) \to GL(n)$ *coincides with the usual matrix exponential.*

By item (iv) in Proposition 13.13, we conclude that if $\mathfrak{h} \subset \mathfrak{gl}(n)$ *a Lie subalgebra and* $H \subset GL(n)$ *is the associated connected Lie subgroup, then exponential map* $\exp : \mathfrak{h} \to H$ *also coincides with the matrix exponential. For example, if* $\mathfrak{h} = \mathfrak{sl}(n)$ *and* $H = SL(n)$ *the exponential of a matrix of zero trace is a matrix of determinant 1, a fact that also follows from the well-known formula:*

$$\det(e^A) = e^{\operatorname{tr} A}.$$

The exponential map is very useful in the study of Lie groups and Lie algebras since it provides a direct link between the Lie algebra (the infinitesimal object) and the Lie group (the global object). For example, we have the following result whose proof is left as an exercise:

Proposition 13.15. *Let H be a subgroup of a Lie group G and let $\mathfrak{h} \subset \mathfrak{g}$ be a subspace of the Lie algebra of G. If $U \subset \mathfrak{g}$ is some neighborhood of 0 which is diffeomorphic via the exponential map to a neighborhood $V \subset G$ of e, and*

$$\exp(\mathfrak{h} \cap U) = H \cap V,$$

then H, with the relative topology, is a Lie subgroup of G whose Lie algebra is $\mathfrak{h}$.

Using this proposition one can then proof the following important result:

Theorem 13.16. *Let G be a Lie group and $H \subset G$ a closed subgroup. Then H, with relative topology, is a Lie subgroup.*

Homework.

1. Let $\Phi : G \to H$ be a Lie group homomorphism between connected Lie groups G and H such that $(\Phi)_* : \mathfrak{g} \to \mathfrak{h}$ is an isomorphism. Show that Φ is a covering map.

2. Complete the proof of Theorem 13.2 by showing that the integrating Lie subgroup is unique.

3. Let G be a Lie group and let $\pi : H \to G$ be a covering map. Show that H is a Lie group.

4. Let $SL(2,\mathbb{C})$ be the group of complex 2×2 matrices with determinant 1. Show that $SL(2,\mathbb{C})$ is 1-connected.
(Hint: Show that there exists a retraction of $SL(2,\mathbb{C})$ in $SU(2) = \mathbb{S}^3$.)

5. Show that any homomorphism of Lie algebras $\phi : \mathfrak{sl}(2) \to \mathfrak{gl}(n)$ integrates to a unique homomorphism of Lie groups $\Phi : SL(2) \to GL(n)$.
(Hint: Consider the complexification $\phi^c : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(n,\mathbb{C})$ of ϕ and use the previous exercise.)

6. Show that the matrix
$$\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$
is not in the image of $\exp : \mathfrak{gl}(2) \to GL(2)$.

7. Let G be a compact Lie group. Show that $\exp : \mathfrak{g} \to G$ is surjective.
(Hint: Use the fact, to be proved later, that any compact Lie group has a bi-invariant metric, i.e., a metric invariant under both right and left translations.)

8. Let $\Phi : G \to H$ be a Lie group homomorphism, with G connected. Show that if the kernel of Φ is discrete then it is contained in the center of G.

Conclude that the fundamental group of a Lie group is always an abelian group.

9. Let G and H be Lie groups. Show that:
   (a) Every continuous homomorphism $\Phi : \mathbb{R} \to G$ is smooth;
   (b) Every continuous homomorphism $\Phi : G \to H$ is smooth;
   (c) If G and H are isomorphic as topological groups, then G and H are isomorphic as Lie groups.

10. Prove Proposition 13.15.

Lecture 14. Transformation Groups

Let G be a group. Recall (Lecture 8) that we denote an action of G on a set M by a map $\Psi : G \times M \to M$, which we write as $(g, p) \mapsto g \cdot p$, and satisfies:

(a) $e \cdot p = p$, for all $p \in M$;
(b) $g \cdot (h \cdot p) = (gh) \cdot p$, for all $g, h \in G$ and $p \in M$.

An action can also be viewed as a group homomorphism $\widehat{\Psi}$ from G to the group of bijections of M. For each $g \in G$ we denote by Ψ_g the bijection:

$$\Psi_g : M \to M, \quad p \mapsto g \cdot p$$

When G is a Lie group, M is a smooth manifold and the map $\Psi : G \times M \to M$ is smooth, we say that we have a **smooth action**. In this case each $\Psi_g : M \to M$ is a diffeomorphism of M, so one also says that G is a **transformation group** of M. Note that for a smooth action, for each $p \in M$, the isotropy subgroup

$$G_p \equiv \{g \in G : g \cdot p = p\}.$$

is a closed subgroup, hence it is an (embedded) Lie subgroup of G (see Theorem 13.16).

The results in Lecture 8 concerning smooth structures on orbits spaces of discrete group actions extend to arbitrary smooth actions of Lie groups. A smooth action $\Psi : G \times M \to M$ is called a **proper action** if the map:

$$G \times M \to M \times M, \quad (g, p) \mapsto (p, g \cdot p),$$

is proper. For example, actions of compact Lie groups are always proper. We have:

Theorem 14.1. *Let $\Psi : G \times M \to M$ be a smooth action of a Lie group G on a manifold M. If the action is free and proper, then $G\backslash M$ has a unique smooth structure, compatible with the quotient topology, such that $\pi : M \to G\backslash M$ is a submersion. In particular,*

$$\dim G\backslash M = \dim M - \dim G.$$

Proof. We apply Theorem 8.3 to the orbit equivalence relation defined by the action. This means that we need to verify that its graph:

$$R = \{(p, g \cdot p) : p \in M, g \in G\} \subset M \times M,$$

is a proper submanifold of $M \times M$ and that the restriction of the projection $p_1|_R : R \to M$ is a submersion.

Let us consider the map:

$$\Phi : G \times M \to M \times M, \quad (g, p) \mapsto (p, g \cdot p),$$

whose image is precisely R. Since the action is assumed to be free, we see that the image of this map is injective. The differential of this map $\mathrm{d}_{(g,p)}\Phi : T_gG \times T_pM \to T_pM \times T_{g\cdot p}M$ is given by:

$$(\mathbf{v}, \mathbf{w}) \mapsto (\mathbf{w}, \mathrm{d}\Psi_p \cdot \mathbf{v} + \mathrm{d}\Psi_g \cdot \mathbf{w}).$$

Since this differential is injective we conclude that Φ is an injective immersion with image R. Since, by assumption, Φ is proper, it follows that R is a proper submanifold of $M \times M$.

To verify that $p_1|_R : R \to M$ is a submersion, it is enough to show that the composition $p_1 \circ \Phi : G \times M \to M$ is a submersion. But this composition is just the projection $(g, p) \mapsto p$, which is obviously a submersion. □

It follows from this result that the orbits of a proper and free action are always embedded submanifolds diffeomorphic to G.

Example 14.2.
Consider the action of $\mathbb{S}^1$ on the 3-sphere $\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$, defined by:

$$\theta \cdot (z, w) = (e^{i\theta} z, e^{i\theta} w).$$

This action is free and proper. Hence, the orbits of this action are embedded submanifolds of $\mathbb{S}^3$ diffeomorphic to $\mathbb{S}^1$. The orbit space $\mathbb{S}^1 \backslash \mathbb{S}^3$ is a smooth manifold. We will see later that this manifold is diffeomorphic to $\mathbb{S}^2$.

Let G be a Lie group and consider the action of G on itself by left translations:

$$G \times G \to G, \quad (g, h) \mapsto gh.$$

This action is free and proper. If $H \subset G$ is a closed subgroup, then H is a Lie subgroup and the action of H on G, by left translation is also free and proper. The orbit space for this action consist of the right cosets:

$$H \backslash G = \{Hg : g \in G\}.$$

From Theorem 14.1, we conclude that:

Corollary 14.3. *Let G be a Lie group and let $H \subset G$ be a closed subgroup. Then $H \backslash G$ has a unique smooth structure, compatible with the quotient topology, such that $\pi : G \to H \backslash G$ is a submersion. In particular,*

$$\dim H \backslash G = \dim G - \dim H.$$

Remark 14.4. So far we have discussed **left actions**. We can also discuss **right actions** $M \times G \to M$, $(m, g) \to m \cdot g$, where axioms (a) and (b) are replaced by:

(a) $p \cdot e = p$, for all $p \in M$;
(b) $(p \cdot h) \cdot g = p \cdot (hg)$, for all $g, h \in G$ and $p \in M$.

Given a left action $(g, m) \mapsto g \cdot m$ one obtains a right action by setting $(m, g) \mapsto g^{-1} \cdot m$, and conversely. Hence, every result about left actions yields a result about right actions, and conversely. For example, if G is a Lie group and $H \subset G$ is a closed subgroup, the right action of H on G by right translations is free and proper. Hence, the set of left cosets

$$G/H = \{gH : g \in G\},$$

also has a natural smooth structure.

Given two G-actions, $G \times M \to M$ and $G \times N \to N$, a G**-equivariant map** is a map $\Phi : M \to N$ such that:

$$\Phi(g \cdot p) = g \cdot \Phi(p), \quad \forall g \in G, p \in M.$$

We say that we have **equivalent actions** is there exists a G-equivariant bijection between them.

Given any action $\Psi : G \times M \to M$, for each $p \in M$ the map

$$\Psi_p : G \to M, \quad g \mapsto g \cdot p,$$

induces a bijection $\bar{\Psi}_p$ between G/G_p and the orbit through p. Notice that G acts on the set of right cosets by left translations:

$$G \times G/G_p \to G/G_p, \quad (h, gG_p) \mapsto (hg)G_p.$$

The map $\bar{\Psi}_p$ is a G-equivariant bijection between the set of right cosets G/G_p and the orbit through p.

If we have a smooth action $\Psi : G \times M \to M$ we can use the results above with $H = G_p$ to conclude that G/G_p has a smooth structure and that the map:

$$\bar{\Psi}_p : G/G_p \to M, \quad gG_p \mapsto g \cdot p,$$

is an injective immersion. Since the image of this map is the orbit through p, we conclude that:

Theorem 14.5. *Let $\Psi : G \times M \to M$ be a smooth action of a Lie group G on a manifold M. The orbits of the action are regularly immersed submanifolds of M. Moreover, for every $p \in M$, the map*

$$\bar{\Psi}_p : G/G_p \to M, \quad gG_p \mapsto g \cdot p,$$

is a G-equivariant diffeomorphism between G/G_p and the orbit through p.

Proof. Since G_p is a closed subgroup, by Corollary 14.3, G/G_p has a smooth structure. The map:

$$\bar{\Psi}_p : G/G_p \to M, \quad gG_p \mapsto g \cdot p,$$

is an injective immersion whose image is the orbit through p. This makes the orbit an immersed submanifold and we leave it as an exercise to show that it is regularly immersed.

This smooth structure on the orbit does not depend on the choice of $p \in M$: two points $p, q \in M$ which belong to the same orbit have conjugate isotropy groups:

$$q = g \cdot p \quad \Longrightarrow \quad G_q = g G_p g^{-1}.$$

It follows that $\Phi : G/G_p \to G/G_q$, $hG_p \mapsto ghg^{-1}G_q$, is an equivariant diffeomorphism which makes the following diagram commute:

$$\begin{array}{ccc} G/G_p & \xrightarrow{\bar{\Psi}_p} & M \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \Psi_g} \\ G/G_q & \xrightarrow{\bar{\Psi}_q} & M \end{array}$$

Since $\Psi_g : M \to M$, $m \mapsto g \cdot m$, is a diffeomorphism, it is clear that the two immersions give equivalent smooth structures on the orbit. $\square$

A **transitive action** $\Psi : G \times M \to M$ is an action with only one orbit. This means that for any pair of points $p, q \in M$, there exists $g \in G$ such that $q = g \cdot p$. In this case, fixing any point $p \in M$, we obtain an equivariant bijection $G/G_p \to M$. When the action is smooth, this gives an equivariant diffeomorphism between M and the quotient G/G_p. In this case, one also calls M a **homogeneous space**.

The homogeneous G-spaces are just the manifolds of the form G/H where $H \subset G$ is a closed subgroup. In the homogenous space G/H we have the natural G-action, induced from the action of G on itself by left translations. Homogenous spaces are particularly nice examples of manifolds. The next examples will show that a manifold can be a homogeneous G-space for different choices of Lie groups.

Examples 14.6.

1. *Let $\mathbb{S}^3$ be the unit quarternions. Identifying $\mathbb{R}^3$ with the purely imaginary quaternions, we obtain an action of $\mathbb{S}^3$ on $\mathbb{R}^3$:*

$$q \cdot v = qvq^{-1}.$$

It is easy to see that the orbits of this action are the spheres of radius r and the origin. Let us restrict the action to $\mathbb{S}^2$, the sphere of radius 1. An easy computation shows that the isotropy group of $p = (1,0,0)$ is the subgroup $\mathbb{S}^1 = (\mathbb{S}^3)_p \subset \mathbb{S}^3$ formed by quaternions of the form $q_0 + iq_1 + 0j + 0k$. It follows that the sphere is diffeomorphic to the homogeneous space $\mathbb{S}^3/\mathbb{S}^1$. The surjective submersion $\pi : \mathbb{S}^3 \to \mathbb{S}^2$, $q \mapsto q \cdot (1,0,0)$, whose fibers are diffeomorphic to $\mathbb{S}^1$, is known ***Hopf fibration****.*

2. *Let $O(d+1) \times \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ be the standard action by matrix multiplication:*

$$(A, \vec{v}) \mapsto A\vec{v}.$$

The orbits of this action are the spheres $(x^0)^2+\cdots+(x^d)^2=r^2$ *and the origin. Again, we consider the sphere* $\mathbb{S}^d$ *of radius 1 and we let* $p_N=(0,\ldots,0,1)\in\mathbb{S}^d$*, the north pole. The isotropy group at* p_N *consists of matrices of the form:*

$$\left(\begin{array}{c|c} B & 0 \\ \hline 0 & 1 \end{array}\right)\in O(d+1),$$

so we can identify it with $O(d)$*. It follows that the map*

$$O(d+1)/O(d)\to\mathbb{S}^d,\quad A\ O(d)\mapsto A\cdot p_N,$$

is a diffeomorphism. A similar reasoning shows that $\mathbb{S}^d$ *is also diffeomorphic to the homogeneous space* $SO(d+1)/SO(d)$*.*

3. *Let* $\mathbb{P}^d$ *be the projective space and denote by* $\pi:\mathbb{R}^{d+1}-\{0\}\to\mathbb{P}^d$ *the map*

$$\pi(x^0,\ldots,x^d)=[x^0:\cdots:x^d].$$

The action $SO(d+1)\times\mathbb{R}^{d+1}\to\mathbb{R}^{d+1}$ *by matrix multiplication, induces a smooth transitive action* $SO(d+1)\times\mathbb{P}^d\to\mathbb{P}^d$*. The isotropy subgroup of the point* $[0:\cdots:0:1]$ *consist of matrices of the form:*

$$\left(\begin{array}{c|c} B & 0 \\ \hline 0 & \det B \end{array}\right)\in SO(d+1),$$

so we can identify it with $O(d)$*. We conclude that* $\mathbb{P}^d$ *is diffeomorphic to the homogeneous space* $SO(d+1)/O(d)$*.*

A similar reasoning shows that the complex projective space $\mathbb{CP}^d$ *is diffeomorphic to the homogeneous space* $SU(d+1)/U(d)$*.*

4. *Let* $G_k(\mathbb{R}^d)$ *denote that set of all linear subspaces of* $\mathbb{R}^d$ *of dimension* k*. The usual action of the orthogonal group* $O(d)$ *on* $\mathbb{R}^d$ *by matrix multiplication induces an action* $O(d)\times G_k(\mathbb{R}^d)\to G_k(\mathbb{R}^d)$*: an invertible linear transformation takes linear subspaces of dimension* k *to linear subspaces of dimension* k*. It is easy to check that given any two* k*-dimensional linear subspaces* $S_1,S_2\subset\mathbb{R}^d$ *there exists* $A\in O(d)$ *mapping* S_1 *onto* S_2*. This means that the action* $O(d)\times G_k(\mathbb{R}^d)\to G_k(\mathbb{R}^d)$ *is transitive.*

We fix the point $S_0\in G_k(\mathbb{R}^d$ *to be the subspace* $\mathbb{R}^k\times\{0\}\subset\mathbb{R}^d$*. The isotropy group of this point is:*

$$H=\left\{\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)\in O(d):A\in O(k),B\in O(d-k)\right\}.$$

so we have a bijection

$$O(d)/O(k)\times O(d-k)\to G_k(V).$$

On $G_k(\mathbb{R}^d)$ *we can consider the unique smooth structure for which this bijection becomes a diffeomorphism. This gives* $G_k(\mathbb{R}^d)$ *the structure of a manifold of dimension de* $k(d-k)=\dim O(d)-(\dim O(k)+\dim O(d-k))$*. One can show that this smooth structure is independent of the choice of base point* S_0*. The manifold* $G_k(\mathbb{R}^d)$ *is called the* ***Grassmannian manifold*** *of* k*-planes in* $\mathbb{R}^d$*.*

Since Lie groups have infinitesimal counterparts, it should come as no surprise that Lie group actions also have an infinitesimal counterpart. Let $\Psi : G \times M \to M$ be a smooth action, which we can view as "Lie group" homomorphism:

$$\widehat{\Psi} : G \to \mathrm{Diff}(M).$$

We think of $\mathrm{Diff}(M)$ as a Lie group with Lie algebra $\mathfrak{X}(M)$, then there must exist a homomorphism of Lie algebras

$$\psi = (\widehat{\Psi})_* : \mathfrak{g} \to \mathfrak{X}(M).$$

In fact, if $X \in \mathfrak{g}$ and $p \in M$, the curve

$$t \mapsto \exp(tX) \cdot p,$$

goes through p at $t = 0$, and it is defined and smooth in a small interval $]-\varepsilon, \varepsilon[$. We define the vector field $\psi(X)$ in M, by:

$$\psi(X)_p \equiv \frac{\mathrm{d}}{\mathrm{d}t} \exp(tX) \cdot p \bigg|_{t=0}.$$

The proof of the following lemma is left as an exercise:

Lemma 14.7. *For each $X \in \mathfrak{g}$, $\psi(X)$ is a smooth vector field and the map $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ is linear and satisfies:*

$$\psi([X, Y]_{\mathfrak{g}}) = -[\psi(X), \psi(Y)], \quad \forall X, Y \in \mathfrak{g}.$$

Remark 14.8. An *anti-homomorphism* of Lie algebras is a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ which satisfies:

$$\phi([X, Y]) = -[\phi(X), \phi(Y)], \quad \forall X, Y \in \mathfrak{g}.$$

The appearance of a minus sign in the lemma is easy to explain: with our conventions, where the Lie algebra of a Lie group is formed by the left invariant vector fields, the Lie algebra of the group of diffeomorphisms $\mathrm{Diff}(M)$ is formed by the vector fields $\mathfrak{X}(M)$ with a Lie bracket which is the *simmetric* of the usual Lie bracket of vector fields. One can see this, for example, by determining the 1-parameter subgroups of the group of diffeomorphims. [3]

The lemma above suggests the following definition:

Definition 14.9. *An **infinitesimal action** of a Lie algebra $\mathfrak{g}$ on a manifold M is an anti-homomorphism of Lie algebras $\psi : \mathfrak{g} \to \mathfrak{X}(M)$.*

Example 14.10.

The Lie algebra $\mathfrak{so}(3)$ has a basis consisting of the skew-symmetric matrices:

$$X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

[3] We could have defined the Lie bracket of vector fields with the opposite sign, but this would lead to the presence of negative signs in other formulas.

In this basis, we have the following Lie bracket relations:

$$[X,Y] = -Z, \quad [Y,Z] = -X, \quad [Z,X] = -Y.$$

For the usual action of $SO(3)$ on $\mathbb{R}^3$ by rotations, we can compute the infinitesimal action as follows. First, we compute the exponential

$$\exp(tX) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & \operatorname{sen} t \\ 0 & -\operatorname{sen} t & \cos t \end{bmatrix}.$$

Then:

$$\psi(X)_{(x,y,z)} = \frac{\mathrm{d}}{\mathrm{d}t} \exp(tX) \cdot (x,y,z)\Big|_{t=0} = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}.$$

Similarly, we compute:

$$\psi(Y) = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad \psi(Z) = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

The vector fields $\{\psi(X), \psi(Y), \psi(Z)\}$ are called the ***infinitesimal generators*** *of the action. Using that ψ is an anti-homomorphism of Lie algebras, one recovers the Lie brackets of Example 10.2.*

A smooth action $\Psi : G \times M \to M$ determines an infinitesimal action $\psi : \mathfrak{g} \to \mathfrak{X}(M)$. The converse does not necessarily hold, as our next example shows.

Examples 14.11.

1. *Consider the infinitesimal Lie algebra action of $\mathfrak{so}(3)$ on $\mathbb{R}^3$ given in Example 14.10. We can restrict this action to $M = \mathbb{R}^3 - \{p_0\}$ by taking for each $X \in \mathfrak{g}$, the restriction of $\psi(X)$ to M. This defines an infinitesimal action of $\mathfrak{so}(3)$ on M which, if $p_0 \neq 0$, is not induced from a Lie group action of $SO(3)$ in M.*
2. *Any non-zero vector field X on a manifold M determines an infinitesimal action of the Lie algebra $\mathfrak{g} = \mathbb{R}$ on M by setting $\psi(\lambda) := \lambda X$. This infinitesimal action integrates to a Lie group action of $G = (\mathbb{R}, +)$ on M if and only if the vector field X is complete. The Lie group $\mathbb{S}^1$ also has Lie algebra $\mathbb{R}$, but even if the vector field is complete, there will be no action $\Psi : \mathbb{S}^1 \times M \to M$ with $\Psi_* = \psi$, since the orbits of X may not be periodic.*

Obviously, for any infinitesimal Lie algebra action $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ which is induced from a Lie group action $G \times M \to M$ the infinitesimal generators $\rho(X) \in \mathfrak{X}(M)$ are all complete vector fields. Conversely, one can show that:

Theorem 14.12. *Let $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ be an infinitesimal Lie algebra action such $\psi(X)$ is complete, for all $X \in \mathfrak{g}$. Then there exists a smooth action $\Psi : G \to \mathrm{Diff}(M)$ with $\Psi_* = \phi$, where G is the 1-connected Lie group with Lie algebra $\mathfrak{g}$.*

For example, if M is a compact manifold then every infinitesimal Lie algebra action $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ integrates to a smooth Lie group action $\Psi : G \times M \to M$, where G is the 1-connected Lie group with Lie algebra $\mathfrak{g}$.

Examples 14.13.

1. *A* ***representation of a Lie group*** *G in a vector space V is a Lie group homomorphism $\widehat{\Psi} : G \to GL(V)$. Since $GL(V) \subset \mathrm{Diff}(V)$, this is the same as a smooth linear action $\Psi : G \times V \to V$. A*

   At the Lie algebra level, a ***representation of a Lie algebra*** *$\mathfrak{g}$ is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. We also have a natural inclusion $\mathfrak{gl}(V) \hookrightarrow \mathfrak{X}(V)$, which to a linear map $T : V \to V$ associates a unique (linear) vector field $X_T \in \mathfrak{X}(V)$ which acts on linear functions $l : V \to \mathbb{R}$ by:*

$$X_T(l) = l \circ T.$$

   One checks that $[X_{T_1}, X_{T_2}] = -X_{[T_1,T_2]}$, so that the Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ determines a anti-Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathfrak{X}(V)$.

   It follows that a representation $\widehat{\Psi} : G \to GL(V)$ is the same as a linear action. It yields by differentiation a Lie algebra representation $\widehat{\Psi}_* : \mathfrak{g} \to \mathfrak{gl}(V)$ which is the same as an infinitesimal Lie algebra action $\psi : \mathfrak{g} \to \mathfrak{X}(V)$.

   Conversely, since a linear vector field on a vector space is complete, any Lie algebra representation $\mathfrak{g} \to \mathfrak{gl}(V)$ integrates to a Lie group representation $G \to GL(V)$ of the 1-connected Lie group G with Lie algebra G.

Homework.

1. Let $\Psi : G \times M \to M$ be a proper and free smooth action and denote by $B = G\backslash M$ its orbit space. Show that the projection $\pi : M \to B$ is locally trivial, i.e., for any $b \in B$ there exists a neighborhood $b \in U \subset B$ and diffeomorphism
$$\sigma : \pi^{-1}(U) \to G \times U, \quad q \mapsto (\chi(q), \pi(q)),$$
such that:
$$\sigma(g \cdot q) = (g\chi(q), \pi(q)), \quad \forall q \in \pi^{-1}(U), g \in G.$$

2. Show that the orbits of a smooth action are regularly immersed submanifolds.

3. Let G be a Lie group and $H \subset G$ a closed connected subgroup. Show that:
   (a) H is a normal subgroup of G if and only if its Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ is an **ideal**, i.e.,
$$\forall X \in \mathfrak{g}, Y \in \mathfrak{h}, \quad [X, Y] \in \mathfrak{h}.$$
   (b) If H is normal in G, then G/H is a Lie group and $\pi : G \to G/H$ is a Lie group homomorphism.

4. Let G be a Lie group and let $H \subset G$ be a closed subgroup. Show that if G/H and H are both connected then G is connected. Conclude from this that the groups $SO(d)$, $SU(d)$ and $U(d)$ are all connected. Show that $O(d)$ and $GL(d)$ have two connected components.

5. Let $\Psi : G \times M \to M$ be a smooth transitive action with M connected. Show that:
(a) The connected component of the identity G^0 also acts transitively on M;
(b) For all $p \in M$, G/G^0 is diffeomorphic to $G_p/(G_p \cap G^0)$;
(c) If G_p is connected for some $p \in M$, then G is connected.

6. For any Lie group G, recall that its **adjoint representation** $\mathrm{Ad} : G \to GL(\mathfrak{g})$, $g \mapsto \mathrm{Ad}\, g$, is defined by $\mathrm{Ad}(g) := \mathrm{d}_e i_g$, where $i_g : G \to G$ is given by $i_g(h) = ghg^{-1}$. Show that the induced Lie algebra representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is given by:

$$\mathrm{ad}(X)(Y) = [X, Y], \quad \forall X, Y \in \mathfrak{g}.$$

7. Find the orbits and the isotropy groups for the adjoint representations of the following Lie groups:
(a) $SL(2)$.
(b) $SO(3)$.
(c) $SU(2)$.

8. For a vector space V of dimension d denote by $S_k(V)$ the set of all k-frames of V:

$$S_k(V) = \{(\mathbf{v}_1, \dots, \mathbf{v}_k) \in V \times \cdots \times V : \mathbf{v}_1, \dots, \mathbf{v}_k \text{ are linearly independent}\}.$$

Show that $S_k(V)$ is a homogenous space of dimension dk. $S_k(V)$ is called the **Stiefel manifold** of k-frames of V.
(Hint: Fix a base of V and consider the action $GL(d)$ in V by matrix multiplication.)

9. Give a proof of Lema 14.7.
(Hint: If G is a Lie group with Lie algebra $\mathfrak{g}$, for each $X \in \mathfrak{g}$ denoted by $\overline{X} \in \mathfrak{X}(G)$ the *right invariant* vector field in G which takes the value X_e at the identity. Show that:

$$[\overline{X}, \overline{Y}] = -\overline{[X, Y]}, \quad \forall X, Y \in \mathfrak{g},$$

and express the infinitesimal action $\phi : \mathfrak{g} \to \mathfrak{X}(M)$ in terms of right invariant vector fields.)

10. Let $\Psi : G \times M \to M$ be a smooth action with associated infinitesimal action $\psi : \mathfrak{g} \to \mathfrak{X}(M)$. If G_p is the isotropy group at p, show that its Lie algebra is the **isotropy subalgebra**:

$$\mathfrak{g}_p = \{X \in \mathfrak{g} : \psi(X)_p = 0\}.$$

11. Let $\Psi : G \times M \to M$ be a smooth action with associated infinitesimal action $\psi : \mathfrak{g} \to \mathfrak{X}(M)$. We call $p_0 \in M$ a **fixed point of the action** if:

$$g \cdot p_0 = p_0, \forall g \in G.$$

Show that if p_0 is a fixed point of the action then:
(a) Ψ induces a representation $\Xi : G \to GL(T_{p_0}M)$;
(b) ψ induces a representation $\xi : \mathfrak{g} \to \mathfrak{gl}(T_{p_0}M)$;
(c) The representation Ξ of G integrates the representation ξ of $\mathfrak{g}$: $(\Xi)_* = \xi$.

Part 3. Differential Forms

Differential forms are the objects that can be integrated over a manifold. For this reason, they play a crucial role when passing from *local* to *global* aspects of manifolds. In this third part of the lectures, we will introduce differential forms and we will see how effective they are in the study of global properties of manifolds.

The main concept and ideas that we will introduce in this round of lectures are the following:

- In Lecture 15: the notion of *differential form* and, more generally, of tensor fields. The elementary operations with differential forms: *exterior product*, *inner product* and *pull-back*.
- In Lecture 16: the *differential* and the *Lie derivative* of differential forms, which give rise to the *Cartan calculus* on differential forms.
- In Lecture 17: the *integration* of differential forms on manifolds and *Stokes Theorem*.
- In Lecture 18: the *de Rham complex* formed by the differential forms and its *cohomology*, an important invariant of a differentiable manifold.
- In Lecture 19: the relationship between de Rham cohomology and *singular cohomology*, which shows that de Rham cohomology is a topological invariant.
- In Lecture 20: the basic properties of de Rham cohomology: *homotopy invariance* and the *Mayer-Vietoris sequence*.
- In Lecture 22: applications of the Mayer-Vietoris sequence to deduce further properties of cohomology like finite dimensionality and *Poincaré duality*. How to define and compute the *Euler characteristic* of a manifold.
- In Lecture 21: applications of cohomology: the *degree* of a map and the *index of a zero* of a vector field.

For a finite dimensional vector space V, we denote the dual vector space by V^*:

$$V^* = \{\alpha : \alpha : V \to \mathbb{R} \text{ is a linear map}\}.$$

Its **tensor algebra** is:

$$\bigotimes V^* = \bigoplus_{k=0}^{+\infty} \otimes^k V^*,$$

and is furnished with the **tensor product** $\otimes : \otimes^k V^* \times \otimes^l V^* \to \otimes^{k+l} V^*$. Its **exterior algebra** is:

$$\bigwedge V^* = \bigoplus_{k=0}^{d} \wedge^k V^*$$

and is furnished with the **exterior product** $\wedge : \wedge^k V^* \times \wedge^l V^* \to \wedge^{k+l} V^*$. If $\alpha_1, \dots, \alpha_k \in V^*$ and $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$, our convention is that:

$$\alpha_1 \wedge \cdots \wedge \alpha_k(\mathbf{v}_1, \dots, \mathbf{v}_k) = \det(\alpha_i(\mathbf{v}_j))_{i,j=1}^k.$$

It maybe worth to recall that one can identify $\otimes^k V^*$ (respectively, $\wedge^k V^*$) with the space of k-multilinear (respectively, k-multilinear and alternating) maps $V \times \cdots \times V \to \mathbb{R}$.

If $T : V \to W$ is a linear transformation between two finite dimensional vector spaces, its transpose is the linear transformation $T^* : W^* \to V^*$ defined by:

$$T^*\alpha(\mathbf{v}) = \alpha(T\mathbf{v}).$$

Similarly, there exists an induced application $T^* : \wedge^k W^* \to \wedge^k V^*$ defined by:

$$T^*\omega(\mathbf{v}_1, \dots, \mathbf{v}_k) = \omega(T\mathbf{v}_1, \dots, T\mathbf{v}_k).$$

This is the restriction of a similarly defined map $T^* : \otimes^k W^* \to \otimes^k V^*$.

Let now M be a smooth manifold. If $(x^1, \dots, x^d)$ are local coordinates around $p \in M$, we know that the tangent vectors

$$\left.\frac{\partial}{\partial x^i}\right|_p \quad (i = 1, \dots, d),$$

form a base for T_pM. Similarly, the forms

$$\mathrm{d}_p x^i \quad (i = 1, \dots, d),$$

form a base for T_p^*M. These basis are dual to each other. If we take tensor products and exterior products of elements of these basis, we obtain basis for $\otimes^k T_pM$, $\wedge^k T_pM$, $\otimes^k T_p^*M$, $\wedge^k T_p^*M$, etc. For example, the space $\wedge^k T_p^*M$ has the base

$$\mathrm{d}_p x^{i_1} \wedge \cdots \wedge \mathrm{d}_p x^{i_k} \quad (i_1 < \cdots < i_k).$$

As in the case of the tangent and cotangent spaces, we are interested in the spaces $\otimes^k T_pM$, $\wedge^k T_pM$, $\otimes^k T^*_pM$, $\wedge^k T^*_pM$, etc., when p varies. For example, we define

$$\wedge^k T^*M := \bigcup_{p\in M} \wedge^k T^*_pM.$$

and we have a projection $\pi : \wedge^k T^*M \to M$. We call $\wedge^k T^*M$ the **k-exterior bundle** of M. Just like the case of the tangent bundle, one has a smooth structure on this bundle.

Proposition 15.1. *There exists a canonical smooth structure on $\wedge^k T^*M$ such that the canonical projection in M is a submersion.*

The proof is similar to the case of the tangent bundle and is left as an exercise. One has also similar smooth structures on the bundles $\wedge^k TM$, $\otimes^k T^*M$, $\otimes^k TM$, $\otimes^k T^*M \otimes^s T^*M$, etc. For any such bundle $\pi : E \to M$ a **section** is a map $s : M \to E$ such that $\pi \circ s(p) = p$, for all $p \in M$.

Definition 15.2. *Let M be a manifold.*

*(i) A **differential form of degree** k is a section of $\wedge^k T^*M$.*
*(ii) A **multivector field of degree** k is a section of $\wedge^k TM$.*
*(iii) A **tensor field of degree** (k, s) is a section of $\otimes^k TM \otimes^s T^*M$.*

We will consider only smooth differential forms, smooth multivector fields and smooth tensor fields, meaning that the corresponding sections are smooth maps.

If $(U, \phi) = (U, x^1, \dots, x^d)$ is a chart then a differential form ω of degree k can be written in the form:

$$\begin{aligned}\omega|_U &= \sum_{i_1<\cdots<i_k} \omega_{i_1\cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \\ &= \sum_{i_1\cdots i_k} \frac{1}{k!} \omega_{i_1\cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k},\end{aligned}$$

where the components $\omega_{i_1\cdots i_k}$ are alternating: for every permutation $\sigma \in S_k$ one has

$$\omega_{\sigma(i_1)\cdots\sigma(i_k)} = (-1)^{\operatorname{sgn}\sigma} \omega_{i_1\cdots i_k}.$$

It should be clear that ω is smooth if and only if for any open cover by charts the components $\omega_{i_1\cdots i_k} \in C^\infty(U)$ are smooth. If $(V, \psi) = (V, y^1, \dots, y^d)$ is another chart, so that

$$\omega|_V = \sum_{j_1<\cdots<j_k} \overline{\omega}_{j_1\cdots j_k} \mathrm{d}y^{j_1} \wedge \cdots \wedge \mathrm{d}y^{j_k},$$

where $\overline{\omega}_{j_1\cdots j_k} \in C^\infty(V)$. If $U \cap V \neq \emptyset$ the components in the overlap of the two charts are related by:

$$\overline{\omega}_{j_1\cdots j_k}(y) = \sum_{i_1<\cdots<i_k} \omega_{i_1\cdots i_k}(\phi \circ \psi^{-1}(y)) \frac{\partial(x^{i_1} \cdots x^{i_k})}{\partial(y^{j_1} \cdots y^{j_k})}.$$

The symbol in the right side of this expression is an abbreviation for the minor consisting of the rows $i_1, \ldots, i_k$ and the columns $j_1, \ldots, j_k$ of the Jacobian matrix of the change of coordinates $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$.

If Π is a multivector field of degree k, we have similar expressions in a local chart:

$$\Pi|_U = \sum_{i_1 < \cdots < i_k} \Pi^{i_1 \cdots i_k} \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}},$$

and similarly for a tensor field T of degree (k, s), which can be written in a local chart:

$$T|_U = \sum_{i_1, \ldots, i_k, j_1, \ldots, j_s} T^{i_1, \ldots, i_k}_{j_1, \ldots, j_s} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_k}.$$

We leave it as an exercise to determine the formulas of transformation of variables for multivector fields and tensor fields.

Remark 15.3. One maybe intrigued with the relative positions of the indices, as subscripts and superscripts, in the different objects. The convention that we follow is such that an index is only summed if it appears in a formula repeated both as a subscript and as a superscript. With this convention, one can even omit the summation sign from the formula, with the agreement that one sums over an index whenever that index is repeated. This convention is called the **Einstein convention sum**.

From now on we will concentrate on the study of differential forms. Although other objects, such as multivector fields and tensor fields, are also interesting, differential forms play a more fundamental role because they are the objects one can integrate over a manifold.

We will denote the vector space of smooth differential forms of degree k on a manifold M by $\Omega^k(M)$. Given a differential form $\omega \in \Omega^k(M)$ its value at a point $\omega_p \in \wedge^k T_p^* M$ can be seen as an alternating, multilinear, map

$$\omega_p : T_pM \times \cdots \times T_pM \to \mathbb{R}.$$

Hence, if $X_1, \ldots, X_k \in \mathfrak{X}(M)$ are smooth vector fields M we obtain a smooth function $\omega(X_1, \ldots, X_k) \in C^\infty(M)$:

$$p \mapsto \omega_p(X_1|_p, \ldots, X_k|_p).$$

Therefore every differential form $\omega \in \Omega^k(M)$ can be seen as a map

$$\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M).$$

This map is $C^\infty(M)$-multilinear and alternating. Conversely, every $C^\infty(M)$-multilinear, alternating, map $\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$ defines a smooth differential form.

We consider now several basic operations with differential forms.

Exterior product of differential forms. The exterior (or wedge) product $\wedge$ in the exterior algebra $\wedge T_p^*M$ induces an exterior (or wedge) product of differential forms

$$\wedge : \Omega^k(M) \times \Omega^s(M) \to \Omega^{k+s}(M),\ (\omega \wedge \eta)_p \equiv \omega_p \wedge \eta_p.$$

If we consider the space of all differential forms:

$$\Omega(M) = \bigoplus_{k=0}^{d} \Omega^k(M).$$

where we convention that $\Omega^0(M) = C^\infty(M)$ and $f\omega = f \wedge \omega$, the exterior product turns $\Omega(M)$ into a **Grassmannn algebra** over the ring $C^\infty(M)$, i.e., the following properties hold:

(a) $(f\omega + g\eta) \wedge \theta = f\omega \wedge \theta + g\eta \wedge \theta$.
(b) $\omega \wedge \eta = (-1)^{\deg \omega \deg \eta} \eta \wedge \omega$.
(c) $(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta)$.

If $\alpha_1, \ldots, \alpha_k \in \Omega^1(M)$ and $X_1, \ldots, X_k \in \mathfrak{X}(M)$, according to our conventions we have:

$$\alpha_1 \wedge \cdots \wedge \alpha_k(X_1, \ldots, X_k) = \det \left[\alpha_i(X_j)\right]_{i,j=1}^k.$$

This properties is all that we need to know to compute exterior products in local coordinates, as we illustrate in the next example:

Example 15.4.
In $\mathbb{R}^4$, with coordinates (x, y, z, w), consider the differential forms of degree 2:

$$\begin{aligned}\omega &= (x + w^2)\mathrm{d}x \wedge \mathrm{d}y + e^z \mathrm{d}x \wedge \mathrm{d}w + \cos x \mathrm{d}y \wedge \mathrm{d}z,\\ \eta &= x\mathrm{d}y \wedge \mathrm{d}z - e^z \mathrm{d}z \wedge \mathrm{d}w.\end{aligned}$$

Then:

$$\begin{aligned}\omega \wedge \eta &= -(x + w^2)e^z \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}w + xe^z \mathrm{d}x \wedge \mathrm{d}w \wedge \mathrm{d}y \wedge \mathrm{d}z\\ &= -w^2 e^z \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}w.\end{aligned}$$

Also, if we would like to compute, e.g., η on the vector fields $X = y\frac{\partial}{\partial z} - \frac{\partial}{\partial y}$ and $Y = e^z \frac{\partial}{\partial w}$ we proceed as follows:

$$\begin{aligned}\eta(X, Y) &= x\mathrm{d}y \wedge \mathrm{d}z(X, Y) - e^z \mathrm{d}z \wedge \mathrm{d}w(X, Y)\\ &= x \begin{vmatrix} \mathrm{d}y(X) & \mathrm{d}y(Y) \\ \mathrm{d}z(X) & \mathrm{d}z(Y) \end{vmatrix} - e^z \begin{vmatrix} \mathrm{d}z(X) & \mathrm{d}z(Y) \\ \mathrm{d}w(X) & \mathrm{d}w(Y) \end{vmatrix}\\ &= x \begin{vmatrix} -1 & 0 \\ y & 0 \end{vmatrix} - e^z \begin{vmatrix} y & 0 \\ 0 & e^z \end{vmatrix} = -ye^{2z}\end{aligned}$$

Pull-back of differential forms. Let $\Phi : M \to N$ be a smooth map. For each $p \in M$, the transpose of the differential

$$(\mathrm{d}_p\Phi)^* : T^*_{\Phi(p)}N \to T^*_pM.$$

induces a linear map

$$(\mathrm{d}_p\Phi)^* : \wedge^k T^*_{\Phi(p)}N \to \wedge^k T^*_p M.$$

The **pull-back of differential forms** $\Phi^* : \Omega^k(N) \to \Omega^k(M)$ is defined as:

$$\begin{aligned}(\Phi^*\omega)(X_1, \dots, X_k)_p &= ((\mathrm{d}_p\Phi)^*\omega)(X_1|_p, \dots, X_k|_p)\\ &= \omega_{\Phi(p)}(\mathrm{d}_p\Phi \cdot X_1|_p, \dots, \mathrm{d}_p\Phi \cdot X_k|_p).\end{aligned}$$

This defines a $C^\infty(M)$-multilinear, alternating, map $\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$, hence $\Phi^*\omega$ is a smooth differential form of degree k in M.

It is easy to check that for any smooth map $\Phi : M \to N$, the pull-back $\Phi^* : \Omega(N) \to \Omega(M)$ is a homomorphism of Grassmann algebras, i.e., the following properties hold:

(a) $\Phi^*(a\omega + b\eta) = a\Phi^*\omega + b\Phi^*\eta$, $a, b \in \mathbb{R}$;
(b) $\Phi^*(\omega \wedge \eta) = \Phi^*\omega \wedge \Phi^*\eta$;
(c) $\Phi^*(f\omega) = (f \circ \Phi)\Phi^*\omega$, $f \in C^\infty(M)$;

Note that if $f : N \to \mathbb{R}$ is a smooth function then the differential $\mathrm{d}f$ can be viewed as a differential form of degree 1. We have also that:

(d) $\Phi^*(\mathrm{d}f) = \mathrm{d}(f \circ \Phi)$.

These properties is all that it is needed to compute pull-backs in local coordinates, as we illustrate in the next example:

Example 15.5.
Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^4$ be the smooth map:

$$\Phi(u, v) = (u + v, u - v, v^2, \frac{1}{1 + u^2}).$$

In order to compute the pull-back under Φ of the form:

$$\eta = x\mathrm{d}y \wedge \mathrm{d}z - e^z \mathrm{d}z \wedge \mathrm{d}w \in \Omega^2(\mathbb{R}^4),$$

we proceed as follows:

$$\begin{aligned}\Phi^*\eta &= (x \circ \Phi)\mathrm{d}(y \circ \Phi) \wedge \mathrm{d}(z \circ \Phi) - e^{(z\circ\Phi)}\mathrm{d}(z \circ \Phi) \wedge \mathrm{d}(w \circ \Phi)\\ &= (u + v)\mathrm{d}(u - v) \wedge \mathrm{d}(v^2) - e^{v^2}\mathrm{d}(v^2) \wedge \mathrm{d}(\frac{1}{1 + u^2})\\ &= (u + v)\mathrm{d}u \wedge 2v\mathrm{d}v - 2ve^{v^2}\mathrm{d}v \wedge \frac{-2u\mathrm{d}u}{(1 + u^2)^2}\\ &= \left(2v(u + v) - \frac{4uve^{v^2}}{(1 + u^2)^2}\right)\mathrm{d}u \wedge \mathrm{d}v.\end{aligned}$$

In other words, to compute the pull-back $\Phi^*\eta$, one replaces in η, the coordinates (x, y, z, w) by its expressions in terms of the coordinates (u, v).

Remark 15.6. When (N, i) is a submanifold of M the pull-back of a differential form $\omega \in \Omega^k(M)$ by the inclusion map $i : N \hookrightarrow M$ is called the **restriction of the differential form** ω to N. Often one denotes the restriction $\omega|_N$ instead of $i^*\omega$.

For example, for the sphere

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\},$$

we can write

$$\omega = (x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y)|_{\mathbb{S}^3},$$

meaning that ω is the pull-back by the inclusion $i : \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ of the differential form $x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y \in \Omega^2(\mathbb{R}^3)$. Sometimes, one even drops the restriction sign.

One should also notice that if $\Phi : M \to N$ and $\Psi : N \to Q$ are smooth maps, then $\Psi \circ \Phi : M \to Q$ is a smooth map and we have:

$$(\Psi \circ \Phi)^* \omega = \Phi^*(\Psi^* \omega).$$

In categorial language, we have a *contravariant functor* from the category of smooth manifolds to the category of Grassmann algebras, which to a smooth manifold M associates the algebra $\Omega(M)$ and to a smooth map $\Phi : M \to N$ associates a homomorphism $\Phi^* : \Omega(N) \to \Omega(M)$.

Interior Product. Given a vector field $X \in \mathfrak{X}(M)$ and a differential form $\omega \in \Omega^k(M)$, the **interior product** of ω by X, denoted $i_X\omega \in \Omega^{k-1}(M)$, is the the differential form of degree $(k-1)$ defined by:

$$i_X\omega(X_1, \dots, X_{k-1}) = \omega(X, X_1, \dots, X_{k-1}).$$

Since $i_X\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$ is a $C^\infty(M)$-multilinear, alternating, map, it is indeed a smooth differential form of degree $k-1$.

It is easy to check that the following properties hold:

(a) $i_X(f\omega + g\theta) = f i_X\omega + g i_X\theta$.
(b) $i_X(\omega \wedge \theta) = (i_X\omega) \wedge \theta + (-1)^{\deg \omega}\omega \wedge (i_X\theta)$.
(c) $i_{(fX+gY)}\omega = f i_X\omega + g i_Y\omega$.
(d) $i_X(\mathrm{d}f) = X(f)$;

Again, these properties is all that it is needed to compute interior products in local coordinates.

Example 15.7.
Let $\omega = e^x\mathrm{d}x \wedge \mathrm{d}y + e^z\mathrm{d}y \wedge \mathrm{d}z \in \Omega^2(\mathbb{R}^3)$, *and* $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^3)$. *Then:*

$$\begin{aligned}
i_{\frac{\partial}{\partial x}}(\mathrm{d}x \wedge \mathrm{d}y) &= (i_{\frac{\partial}{\partial x}}\mathrm{d}x) \wedge \mathrm{d}y - \mathrm{d}x \wedge (i_{\frac{\partial}{\partial y}}\mathrm{d}y) = \mathrm{d}y,\\
i_{\frac{\partial}{\partial y}}(\mathrm{d}x \wedge \mathrm{d}y) &= (i_{\frac{\partial}{\partial y}}\mathrm{d}x) \wedge \mathrm{d}y - \mathrm{d}x \wedge (i_{\frac{\partial}{\partial y}}\mathrm{d}y) = -\mathrm{d}x,\\
i_{\frac{\partial}{\partial x}}(\mathrm{d}y \wedge \mathrm{d}z) &= (i_{\frac{\partial}{\partial x}}\mathrm{d}y) \wedge \mathrm{d}z - \mathrm{d}y \wedge (i_{\frac{\partial}{\partial x}}\mathrm{d}z) = 0,\\
i_{\frac{\partial}{\partial y}}(\mathrm{d}y \wedge \mathrm{d}z) &= (i_{\frac{\partial}{\partial y}}\mathrm{d}y) \wedge \mathrm{d}z - \mathrm{d}y \wedge (i_{\frac{\partial}{\partial y}}\mathrm{d}z) = \mathrm{d}z.
\end{aligned}$$

Hence, we conclude that:

$$i_X\omega = -xe^x\mathrm{d}x - ye^x\mathrm{d}y + xe^z\mathrm{d}z.$$

Remark 15.8. One can extend the interior product in a more or less obvious way to other objects (multivector fields, tensor fields, etc.). For these objects it is frequent to use the designation **contraction**, instead of interior product. For example, one can define the contraction of a differential form ω of degree k by a multivector field Π of degree $l < k$, to be a differential form $i_\Pi \omega$ of degree $k - l$. In a local chart $(U, x^1, \ldots, x^d)$, if

$$\omega|_U = \sum_{i_1 \cdots i_k} \omega_{i_1 \cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}, \quad \Pi|_U = \sum_{j_1 \cdots j_l} \Pi^{j_1 \cdots j_l} \frac{\partial}{\partial x^{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{j_l}},$$

then:

$$(i_\Pi \omega)|_U = \sum_{i_1 \cdots i_k} \omega_{i_1 \cdots i_k} \Pi^{i_1 \cdots i_l} \mathrm{d}x^{i_{l+1}} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

As a first application of differential forms, we are going to formalize the notion of *orientation* of a manifold.

Recall that if V is a linear vector space of dimension d and $\mu \in \wedge^d(V^*)$ is a non-zero element, then for any base $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ of V we have

$$\mu(\mathbf{v}_1, \ldots, \mathbf{v}_d) \neq 0.$$

This implies that μ splits the ordered basis of V into two classes: a base $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ has positive (respectively, negative) μ-orientation if this number is positive (respectively, negative). Hence, μ determines a orientation for V.

Definition 15.9. *For a smooth manifold M of dimension d, we call a differential form $\mu \in \Omega^d(M)$ a* ***volume form*** *if $\mu_p \neq 0$, for all $p \in M$. A manifold M is said to be* ***orientable*** *if it admits a volume form.*

Notice that if $\mu \in \Omega^d(M)$ is a volume form then any other differential form of degree d in M is of the form $f\mu$ for a smooth function $f \in C^\infty(M)$. In particular, if $\mu_1, \mu_2 \in \Omega^d(M)$ are two volume forms then there exists a unique smooth *non-vanishing* function $f \in C^\infty(M)$ such that $\mu_2 = f\mu_1$.

Let M be an orientable manifold of dimension d. If $\mu_1, \mu_2 \in \Omega^d(M)$ are volumes forms we say that μ_1 e μ_2 define the same orientation if for all $p \in M$ and any ordered base $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ of T_pM, one has:

$$\mu_1(\mathbf{v}_1, \ldots, \mathbf{v}_d)\mu_2(\mathbf{v}_1, \ldots, \mathbf{v}_d) > 0.$$

Note that if μ_1 and μ_2 define the same orientation, then a base is μ_1-positive if and only if it is μ_2-positive. We leave the proof of the following lemma as an exercise:

Lemma 15.10. *Let M be manifold of dimension d. Two volume forms $\mu_1, \mu_2 \in \Omega^d(M)$ define the same orientation if and only if $\mu_2 = f\omega_1$ for a smooth everywhere positive function $f \in C^\infty(M)$.*

The property "define the same orientation" is an equivalence relation on the set of volume forms in an orientable manifold M.

Definition 15.11. *An **orientation** for an orientable manifold M is a choice of an equivalence class $[\mu]$. A pair $(M,[\mu])$ is called an **oriented manifold**.*

A connected orientable manifold has two orientations. More generally, an orientable manifold with k connected components has 2^k orientations.

Examples 15.12.

1. *The euclidean space $\mathbb{R}^d$ is orientable. The canonical orientation of $\mathbb{R}^d$ is the orientation defined by the volume form $\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^d$. For this canonical orientation, the canonical base of $T_p\mathbb{R}^d\simeq\mathbb{R}^d$ has positive orientation.*
2. *A Lie group G is always orientable. If $\{\alpha_1,\ldots,\alpha_d\}$ is a base of left invariant 1-forms then $\mu=\alpha_1\wedge\cdots\wedge\alpha_d$ a left invariant volume form.*
3. *The sphere $\mathbb{S}^d$ is an orientable manifold. A volume form is given by:*
$$\omega=\sum_{i=1}^{d+1}(-1)^i x^i\mathrm{d}x^1\wedge\cdots\wedge\widehat{\mathrm{d}x^i}\wedge\cdots\wedge\mathrm{d}x^{d+1}\Big|_{\mathbb{S}^d}.$$
We leave it as an exercise to check that this form never vanishes.
4. *The projective space $\mathbb{P}^2$ is not orientable. To see this let $\mu\in\Omega^2(\mathbb{P}^2)$ be any differential 2-form. If $\pi:\mathbb{S}^2\to\mathbb{P}^2$ is the quotient map, then the pull-back $\pi^*\mu$ is a differential 2-form in $\mathbb{S}^2$. It follows from the previous example that*
$$\pi^*\mu=f\omega,$$
for some smooth function $f\in C^\infty(\mathbb{S}^2)$.
Let $\Phi:\mathbb{S}^2\to\mathbb{S}^2$ be the anti-podal map: $p\mapsto -p$. Since $\pi\circ\Phi=\pi$, we have:
$$\Phi^*(\pi^*\mu)=(\pi\circ\Phi)^*\mu=\pi^*\mu.$$
On the other, it is easy to check that $\Phi^*\omega=-\omega$. Hence:
$$\begin{aligned}f\omega=\pi^*\mu&=\Phi^*(\pi^*\mu)\\&=\Phi^*(f\omega)=(f\circ\Phi)\Phi^*(\omega)=-(f\circ\Phi)\omega.\end{aligned}$$
We conclude that $f(-p)=-f(p)$, for all $p\in\mathbb{S}^2$. But then we must have $f(p_0)=0$, at some $p_0\in\mathbb{S}^2$. Hence, $\pi^*\mu$ vanishes at some point. Since π is a local diffeomorphism, we conclude that every differential form $\mu\in\Omega^2(\mathbb{P}^2)$ vanishes at some point, so $\mathbb{P}^2$ has no volume forms, and it is non-orientable.

Let $(M,[\mu_M])$ and $(N,[\mu_N])$ be oriented manifolds. We say that a diffeomorphism $\Phi:M\to N$ **preserves orientations** or that it is **positive**, if $[\Phi^*\mu_N]=[\mu_M]$.

Example 15.13.
Let $[\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^d]$ be the standard orientation for $\mathbb{R}^d$. Given a diffeomorphism $\phi:U\to V$, where U,V are open sets in $\mathbb{R}^d$, we have:
$$\phi^*(\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^d)=\det[\phi'(x)]\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^d.$$
Hence ϕ preserves the standard orientation if and only if $\det[\phi'(x)]>0$, for all $x\in\mathbb{R}^d$.

One can also express the possibility of orienting a manifold in terms of an atlas, as shown by the following proposition. The proof is left as an exercise.

Proposition 15.14. *Let M be a manifold of dimension d. The following statements are equivalents:*

(i) M is orientable, i.e., M has a volume form.
(ii) There exists an atlas $\{(U_i, \phi_i)\}_{i\in I}$ for M such that for all $i, j \in I$ the transition functions preserve the standard orientation of $\mathbb{R}^d$.

In particular, if $[\mu_M]$ is an orientation for M, then there exists an atlas $\{(U_i, \phi_i)\}_{i\in I}$ for M such that each chart $\phi_i : U_i \to \mathbb{R}^d$ is positive, where in $\mathbb{R}^d$ we consider the canonical orientation.

Homework.

1. Construct the natural differentiable structure on $\wedge^k T^*M$, for which the canonical projection $\pi : \wedge^k T^*M \to M$ is a submersion.

2. Determine the formulas of transformation of variables for multivector fields and tensor fields.

3. Show that a Riemannian structure on a manifold M (see Exercise 8 in Lecture 9) defines a symmetric tensor field of degree (0,2).
Note: In a chart (U, x^i), a symmetric tensor field of degree (0,2) is written as

$$g|_U = \sum_{i,j} g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j,$$

where the components $g_{ij} \in C^\infty(U)$ satisfy $g_{ij} = g_{ji}$.

4. Proof the basic properties of the pull-back and interior product of differential forms.

5. Let $\Phi : M \to N$ be a smooth map and let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ be Φ-related smooth vector fields. Show that

$$\Phi^*(i_Y \omega) = i_X \Phi^* \omega,$$

for any differential form $\omega \in \Omega(N)$.

6. Proof Proposition 15.14.

7. Show that for any orientable manifolds M and N the product $M \times N$ is orientable. Conclude that the torus $\mathbb{T}^d$ is orientable. Give an example of a volume form in $\mathbb{T}^d$.

8. Show that the projective space $\mathbb{P}^d$ is orientable if and only if d is odd.

9. Verify that the Klein bottle (see Example 6.8.4) is a non-orientable manifold.

10. Show that every oriented manifold $(M, [\mu])$ has an atlas consisting of positive charts.

11. Let M be a Riemannian manifold of dimension d. Show that:
(a) Each inner product on the tangent space T_pM induces an inner product on the cotangent space T_p^*M.
(b) For each $p \in M$, there exists a neighborhood U of p and orthonormal smooth vector fields $X_1, \dots, X_d \in \mathfrak{X}(U)$:

$$\langle X_i, X_j \rangle = \delta_{ij} \text{ (Kronecker symbol).}$$

The set $\{X_1, \dots, X_d\}$ is called a **(local) orthonormal frame**.
(c) For each $p \in M$, there exists a neighborhood U of p and orthonormal differential forms $\alpha_1, \dots, \alpha_d \in \Omega^1(U)$:

$$\langle \alpha_i, \alpha_j \rangle = \delta_{ij} \text{ (Kronecker symbol).}$$

The set $\{\alpha_1, \dots, \alpha_d\}$ is called a **(local) orthonormal coframe**.

12. Let $(M, [\mu])$ be an oriented Riemannian manifold of dimension d. Show that there exists a unique linear map $* : \Omega^k(M) \to \Omega^{d-k}(M)$ such that for every local orthonormal coframe $\alpha_1, \dots, \alpha_d$ which is positive (i.e., $\alpha_1 \wedge \cdots \wedge \alpha_d$ is positive) the following properties hold:
(a) $*1 = \alpha_1 \wedge \cdots \wedge \alpha_d$ and $*(\alpha_1 \wedge \cdots \wedge \alpha_d) = 1$;
(b) $*(\alpha_1 \wedge \cdots \wedge \alpha_k) = \alpha_{k+1} \wedge \cdots \wedge \alpha_d$.
Show also that:

$$* * \omega = (-1)^{k(d-k)}\omega, \text{ where } k = \deg \omega.$$

$*$ is called the **Hodge star operator**.

Lecture 16. Differential and Cartan Calculus

We will introduce now two important differentiation operations on differential forms: the *differential* of forms, which is an intrinsic derivative, and the *Lie derivative* of differential forms, which is a derivative along vector fields. These differential operations together with the algebraic operations on differential forms that we studied in the previous lecture, are the basis of a calculus on differential forms on which is usually called *Cartan Calculus*.

Let $\omega \in \Omega^k(M)$. The **differential** of ω is the differential form of degree $k+1$, denoted $\mathrm{d}\omega$, defined by:

$$\begin{aligned}(16.1) \quad \mathrm{d}\omega(X_0, \dots, X_k) = &\sum_{i=0}^{k} (-1)^i X_i(\omega(X_0, \dots, \widehat{X}_i, \dots, X_k)) + \\ &+ \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j \dots, X_k),\end{aligned}$$

for any smooth vector fields $X_0, \dots, X_k \in \mathfrak{X}(M)$. Since this formula defines a $C^\infty(M)$-multilinear, alternating, map $\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$, we see that $\mathrm{d}\omega$ is indeed a smooth differential (k+1)-form.

A smooth function $f \in \mathcal{C}^\infty(M)$ is a degree 0 form. In this case, formula (16.1) gives:

$$\mathrm{d}f(X) = X(f).$$

Therefore this definition matches our previous definition of the differential of a smooth function. Our next result shows that the differential is the only operation on the forms which extends the differential of functions in a reasonable way:

Theorem 16.1. *The differential*

$$\mathrm{d} : \Omega^{\bullet}(M) \to \Omega^{\bullet+1}(M)$$

is the only operation on forms satisfying the following properties:

(i) d *is* $\mathbb{R}$*-linear:*

$$\mathrm{d}(a\omega + b\theta) = a\mathrm{d}\omega + b\mathrm{d}\theta.$$

(ii) d *is a derivation:*

$$\mathrm{d}(\omega \wedge \theta) = (\mathrm{d}\omega) \wedge \theta + (-1)^{\deg \omega} \omega \wedge (\mathrm{d}\theta).$$

(iii) d *extends the differential of smooth functions: if* $f \in C^\infty(M)$*, then*

$$\mathrm{d}f(X) = X(f), \forall X \in \mathfrak{X}(M).$$

(iv) $\mathrm{d}^2 = 0$.

Moreover, if $\Phi : M \to N$ *is a smooth map, then for every* $\omega \in \Omega^k(N)$*:*

$$\Phi^* \mathrm{d}\omega = \mathrm{d}\Phi^* \omega.$$

Proof. We leave it for the exercises to check that d, as defined by (16.1), satisfies properties (i) through (iv). To prove uniqueness, we need to check that given $\omega \in \Omega^k(M)$, then $\mathrm{d}\omega$ is determined by properties (i)–(iv).

Since d is a derivation, it is local: if $\omega|_U = 0$ on an open set U then $(\mathrm{d}\omega)|_U = 0$. In fact, let $p \in U$ and $f \in C^\infty(M)$ with $f(p) > 0$ and $\operatorname{supp} f \subset U$. Since $f\omega \equiv 0$, we find that:

$$0 = \mathrm{d}(f\omega) = \mathrm{d}f \wedge \omega + f\mathrm{d}\omega.$$

If we evaluate both sides of this identity at p, we conclude that $f(p)(\mathrm{d}\omega)_p = 0$. Hence $\mathrm{d}\omega|_U = 0$, as claimed.

Therefore, to prove uniqueness, it is enough to consider the case where $\omega \in \Omega^k(U)$, where U is the domain of some local chart $(x^1, \dots, x^d)$. In this case we have:

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

Using only properties (i)–(iv) we find:

$$\begin{aligned}
\mathrm{d}\omega &= \sum_{i_1 < \cdots < i_k} \mathrm{d}(\omega_{i_1 \cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}) && \text{(by (i))} \\
&= \sum_{i_1 < \cdots < i_k} \mathrm{d}(\omega_{i_1 \cdots i_k}) \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} && \text{(by (ii) and (iv))} \\
&= \sum_{i_1 < \cdots < i_k} \sum_i \frac{\partial \omega_{i_1 \cdots i_k}}{\partial x^i} \, \mathrm{d}x^i \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} && \text{(by (iii)).}
\end{aligned}$$

The last expression defines a differential form of degree $k+1$ in U. Hence, $\mathrm{d}\omega$ is completely determine by properties (i)–(iv), as claimed.

The proof that the differential commutes with pull-backs can also be reduced to a computation in local charts, and we leave it to the exercises. $\square$

As this proof shows, one can compute the differential of a form using only properties (i)–(iv). This is often much more efficient than applying directly the formula (16.1), as we illustrate in the next example.

EXAMPLE 16.2.
Let $\omega = e^y \mathrm{d}x \wedge \mathrm{d}z + e^z \mathrm{d}y \wedge \mathrm{d}z \in \Omega^2(\mathbb{R}^3)$. Then using properties (i)–(iv), we find:

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}(e^y \mathrm{d}x \wedge \mathrm{d}z + e^z \mathrm{d}y \wedge \mathrm{d}z) \\
&= (\mathrm{d}e^y) \wedge \mathrm{d}x \wedge \mathrm{d}z + \mathrm{d}(e^z) \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= e^y \mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}z + e^z \mathrm{d}z \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= -e^y \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.
\end{aligned}$$

The operation $\mathrm{d} : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ is also referred to as *exterior differentiation*, since it increases the degree of a form. There is another type of differentiation of a form which preserves the degree:

Definition 16.3. *The **Lie derivative** of a differential form $\omega \in \Omega^k(M)$ along a vector $X \in \mathfrak{X}(M)$ is the differential form $\mathcal{L}_X\omega \in \Omega^k(M)$ defined by:*

$$\mathcal{L}_X\omega = \left.\frac{\mathrm{d}}{\mathrm{d}t}(\phi_X^t)^*\omega\right|_{t=0} = \lim_{t\to 0} \frac{1}{t}\left((\phi_X^t)^*\omega - \omega\right).$$

EXAMPLE 16.4.
Let $\omega = e^x \mathrm{d}x \wedge \mathrm{d}y + e^y \mathrm{d}y \wedge \mathrm{d}z \in \Omega^2(\mathbb{R}^3)$ and $X = x\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^3)$. The flow of X is given by $\phi_X^t(x,y,z) = (x, y+tx, z)$. Hence, we find that:

$$\begin{aligned}
(\phi_X^t)^*\omega &= e^x \mathrm{d}x \wedge \mathrm{d}(y+tx) + e^{y+tx}\mathrm{d}(y+tx) \wedge \mathrm{d}z \\
&= e^x \mathrm{d}x \wedge \mathrm{d}y + e^{y+tx}\mathrm{d}y \wedge \mathrm{d}z + te^{y+tx}\mathrm{d}x \wedge \mathrm{d}z
\end{aligned}$$

Then:

$$\begin{aligned}
\mathcal{L}_X\omega &= \left.\frac{\mathrm{d}}{\mathrm{d}t}(\phi_X^t)^*\omega\right|_{t=0} \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left((e^{y+tx} - e^y)\mathrm{d}y \wedge \mathrm{d}z + e^{y+tx}\mathrm{d}x \wedge \mathrm{d}z\right) \\
&= xe^y \mathrm{d}y \wedge \mathrm{d}z + e^y \mathrm{d}x \wedge \mathrm{d}z.
\end{aligned}$$

In most examples, it is impossible to find explicitly the flow of a vector field. Still the basic properties of the Lie derivative listed in the next proposition allow one to find the Lie derivative without knowledge of the flow. The proof is left as an exercise:

Proposition 16.5. *Let $X \in \mathfrak{X}(M)$ and $\omega, \eta \in \Omega^\bullet(M)$. Then:*

(i) $\mathcal{L}_X(a\omega + b\eta) = a\mathcal{L}_X\omega + b\mathcal{L}_X\eta$ *for all* $a, b \in \mathbb{R}$.
(ii) $\mathcal{L}_X(\omega \wedge \eta) = \mathcal{L}_X\omega \wedge \eta + \omega \wedge \mathcal{L}_X\eta$.
(iii) $\mathcal{L}_X(f) = X(f)$, *if* $f \in \Omega^0(M) = \mathcal{C}^\infty(M)$.
(iv) $\mathcal{L}_X \mathrm{d}\omega = \mathrm{d}\mathcal{L}_X\omega$.

Example 16.6.
Let us redo Example 16.4 using only properties (i)-(iv) in the previous proposition:

$$\begin{aligned}
\mathcal{L}_X\omega &= \mathcal{L}_X(e^x \mathrm{d}x \wedge \mathrm{d}y + e^y \mathrm{d}y \wedge \mathrm{d}z) \\
&= \mathcal{L}_X(e^x)\mathrm{d}x \wedge \mathrm{d}y + e^x\mathcal{L}_X(\mathrm{d}x) \wedge \mathrm{d}y + e^x \mathrm{d}x \wedge \mathcal{L}_X(\mathrm{d}y) + \\
&\quad + \mathcal{L}_X(e^y)\mathrm{d}y \wedge \mathrm{d}z + e^y\mathcal{L}_X(\mathrm{d}y) \wedge \mathrm{d}z + e^y \mathrm{d}y \wedge \mathcal{L}_X(\mathrm{d}z) \\
&= e^x \mathrm{d}x \wedge \mathrm{d}X(y) + X(e^y)\mathrm{d}y \wedge \mathrm{d}z + e^y \mathrm{d}X(y) \wedge \mathrm{d}z \\
&= xe^y \mathrm{d}y \wedge \mathrm{d}z + e^y \mathrm{d}x \wedge \mathrm{d}z.
\end{aligned}$$

There is still another efficient way to compute the Lie derivative. In fact, there is an important "magical" formula, often playing an unexpected role, which relates all three operations: Lie derivative, exterior differential and interior product:

Theorem 16.7 (Cartan's Magic Formula)**.** *Let $X \in \mathfrak{X}(M)$ and $\omega \in \Omega(M)$. Then:*

$$\mathcal{L}_X\omega = i_X \mathrm{d}\omega + \mathrm{d}i_X\omega. \tag{16.2}$$

Proof. By Proposition 16.5 (iii), $\mathcal{L}_X : \Omega(M) \to \Omega(M)$ is a derivation. The properties of d and i_X give that $i_X\mathrm{d} + \mathrm{d}i_X : \Omega(M) \to \Omega(M)$ is also a derivation. Hence, it is enough to check that both derivations take the same values on differential forms of the type $\omega = f$ and $\omega = \mathrm{d}g$, where $f, g \in C^\infty(M)$.

On the one hand, the properties in Proposition 16.5, give:

$$\mathcal{L}_X(f) = X(f), \quad \mathcal{L}_X(\mathrm{d}g) = \mathrm{d}\mathcal{L}_X g = \mathrm{d}(X(g)).$$

On the other hand, the properties of d and i_X yield:

$$\begin{aligned}
(i_X\mathrm{d} + \mathrm{d}i_X)f &= i_X\mathrm{d}f = X(f), \\
(i_X\mathrm{d} + \mathrm{d}i_X)\mathrm{d}g &= \mathrm{d}(i_X\mathrm{d}g) = \mathrm{d}(X(g)).
\end{aligned}$$

∎

Example 16.8.
Let us redo Example 16.4 using Cartan's Magic Formula:

$$\begin{aligned}
\mathcal{L}_X\omega &= i_X\mathrm{d}\omega + \mathrm{d}i_X\omega \\
&= i_X(0) + \mathrm{d}(-xe^x\mathrm{d}x + xe^y\mathrm{d}z) \\
&= e^y\mathrm{d}x \wedge \mathrm{d}z + xe^y\mathrm{d}y \wedge \mathrm{d}z.
\end{aligned}$$

As an pplication of the notion of differential, we show how one can restate the Frobenius Theorem in the language of differential forms. If D is a smooth distribution in a manifold M we will say that $\omega \in \Omega^k(M)$ **annihilates** D if:

$$\omega(X_1, \dots, X_k) = 0 \quad \text{whenever } X_1, \dots, X_k \in \mathfrak{X}(D).$$

We denote by $I(D)$ the collection of all such forms:

$$I(D) \equiv \{\omega \in \Omega(M) : \omega \text{ annihilates } D\}.$$

Also, given a collection of differential 1-forms $\alpha_1, \dots, \alpha_k \in \Omega^1(M)$ we will say that the collection is **linearly independent** if for each $p \in M$ they yield a linearly independent set in T_p^*M.

One can describe any distribution using differential forms:

Proposition 16.9. *Let D be a smooth k-dimensional distribution in a manifold M of dimension d. Then:*

(i) $I(D)$ is an ideal in the Grassmann algebra $\Omega(M)$.
(ii) $I(D)$ is locally generated by $d-k$ linearly independent 1-forms.

Conversely, if $I \subset \Omega(M)$ is an ideal which is locally generated by $d-k$ differential forms of degree 1, then there exists a unique smooth k-dimensional distribution D such that $I = I(D)$.

Proof. Assume that D be a smooth k-dimensional distribution in a manifold M of dimension d. Item (i) follows immediately from the definition of $I(D)$ and the properties of the wedge product. To show that (ii) holds, for each $p \in M$ let U be an open neighborhood of p and let $X_{d-k+1}, \dots, X_d \in \mathfrak{X}(U)$ be linearly independent vector fields generating $D|_U$. We can complete this collection with vector fields so that $X_1, \dots, X_d \in \mathfrak{X}(U)$, is a base for T_pM, at each $p \in U$. Let $\alpha_1, \dots, \alpha_d \in \Omega^1(U)$ be the dual base of 1-forms:

$$\alpha_i(X_j) = \delta_{ij}.$$

We claim that $\alpha_1, \dots, \alpha_k$ are the desired 1-differential forms:

- The collection $\alpha_1, \dots, \alpha_k$ is linearly independent: this is obvious, since $\{\alpha_1, \dots, \alpha_d\}$ yield a base for form T_p^*M, for each $p \in U$.
- The collection $\alpha_1, \dots, \alpha_k$ is a generating set for $I(D)$: if $\omega \in \Omega^r(M)$, there exist $a_{i_1 \cdots i_r} \in C^\infty(U)$ such that

  $$\omega|_U = \sum_{1 \le i_1 < \cdots < i_r \le d} a_{i_1 \cdots i_r} \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_r}.$$

  When $\omega \in I(D)$ we see that $a_{i_1 \cdots i_r} = 0$ for all $i_j \ge k$, by evaluating both sides on the vector fields $X_{d-k+1}, \dots, X_d$. Hence,

  $$\omega|_U = \sum_{1 \le i_1 < \cdots < i_r \le k} a_{i_1 \cdots i_r} \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_r},$$

  so the collection $\alpha_1, \dots, \alpha_k$ is a generating set for $I(D)$.

In order to show the converse, let $p \in M$ and let $\alpha_1, \dots, \alpha_k$ be independent 1-forms which generate the ideal I in some neighborhood U of p. Define $D_p \subset T_pM$ to be the subspace:

$$D_p = \{v \in T_pM : \alpha_1(v) = \cdots = \alpha_k(v) = 0\}.$$

It is easy to check that D is a smooth k-dimensional distribution in M such that $I = I(D)$. Uniqueness follows from the fact that if $D_1 \neq D_2$, then $I(D_1) \neq I(D_2)$. $\square$

Define a **differential ideal** to be an ideal $I \subset \Omega(M)$ which is closed under the differential:

$$\omega \in I \implies \mathrm{d}\omega \in I.$$

The Frobenius Theorem can be restated in the language of differential forms:

Theorem 16.10 (Frobenius)**.** *A distribution D is integrable if and only if $I(D)$ is a differential ideal.*

Proof. It is enough to show that D is involutive if and only if $I(D)$ is a differential ideal.

On the one hand, expression (16.1) for the differential shows that if D is involutive then $I(D)$ is a differential ideal. On the other hand, if $I(D)$ is a differential ideal and $X, Y \in \mathfrak{X}(D)$, then for any degree 1 form $\omega \in I(D)$ we find, using (16.1), that:

$$\omega([X,Y]) = -\mathrm{d}\omega(X,Y) + X(\omega(Y)) - Y(\omega(X)) = 0.$$

Hence, $[X,Y] \in \mathfrak{X}(D)$, so we conclude that D is involutive. $\square$

Example 16.11.

Let $\omega \in \Omega^1(M)$ be a differential 1-form which is nowhere vanishing. Then ω define a smooth distribution of codimension 1. By the Theorem, this distribution is integrable if and only if

$$\mathrm{d}\omega = \eta \wedge \omega,$$

for some 1-form $\eta \in \Omega^1(M)$.

Homework.

1. Show that d defined by formula (16.1), satisfies properties (i)–(iv) in Theorem 16.1.

2. Let $\Phi : M \to N$ be a smooth map. Show that for any form $\omega \in \Omega^k(M)$:

$$\Phi^*\mathrm{d}\omega = \mathrm{d}\Phi^*\omega.$$

3. Let $I \subset \Omega(M)$ be an ideal generated by k independent differential forms $\alpha_1, \dots, \alpha_k$ of degree 1. Show that the following statements are equivalent:
   (a) I is a differential ideal;
   (b) $\mathrm{d}\alpha_i = \sum_j \omega_{ij} \wedge \alpha_j$, for some 1-forms $\omega_{ij} \in \Omega^1(M)$;
   (c) If $\omega = \alpha_1 \wedge \cdots \wedge \alpha_k$, then $\mathrm{d}\omega = \alpha \wedge \omega$, for some 1-form $\alpha \in \Omega^1(M)$.

4. Prove the properties of the Lie derivative given in Proposition 16.5.

5. Let $X, Y \in \mathfrak{X}(M)$ be vector fields and $\omega \in \Omega(M)$ a differential form. Show that:

$$\mathcal{L}_{[X,Y]}\omega = \mathcal{L}_X(\mathcal{L}_Y\omega) - \mathcal{L}_Y(\mathcal{L}_X\omega).$$

6. Let $\Phi : M \to N$ be smooth. Show that if $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are Φ-related vector fields, then

$$\Phi^*(\mathcal{L}_Y\omega) = \mathcal{L}_X(\Phi^*\omega),$$

for every differential form $\omega \in \Omega(N)$.

7. Let $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$. Show that:

$$\mathcal{L}_X(\omega(X_1, \ldots, X_k)) = \mathcal{L}_X\omega(X_1, \ldots, X_k) + \sum_{i=1}^{k} \omega(X_1, \ldots, \mathcal{L}_X X_i, \ldots, X_k). \tag{16.3}$$

Use this formula to compute $\mathcal{L}_X\omega$ when $X = x\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^3)$ and $\omega = e^x \mathrm{d}x \wedge \mathrm{d}y + e^y \mathrm{d}y \wedge \mathrm{d}z \in \Omega^2(\mathbb{R}^3)$.

(Hint: Take X_1 and X_2 to be the vector fields $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$.)

8. Let M be an oriented Riemannian manifold. If $\mathbf{v} \in T_pM$ denoted by $\mathbf{v}^\flat \in T^*M$ the unique convector defined by $\mathbf{v}^\flat(\mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle$. The map $\mathbf{v} \mapsto \mathbf{v}^\flat$ is an isomorphism and we denote its inverse by $\alpha \mapsto \alpha^\sharp$. The **gradient** of a function $f : M \to \mathbb{R}$ is the vector field $\operatorname{grad} f \in \mathfrak{X}(M)$ defined by:

$$\operatorname{grad} f := (\mathrm{d}f)^\sharp.$$

The **divergence** of a vector field $X \in \mathfrak{X}(M)$ is the function $\operatorname{div} X : M \to \mathbb{R}$ defined by

$$\operatorname{div} X := *\mathrm{d} * X.$$

The **laplacian** of $f : M \to \mathbb{R}$ is the function $\Delta f : M \to \mathbb{R}$ defined by:

$$\Delta f := -\operatorname{div}(\operatorname{grad} f).$$

When $M = \mathbb{R}^3$ with its canonical Riemannian structure, find the gradient, the divergence and the laplacian in cylindrical and in spherical coordinates.

9. In a smooth manifold M denote by $\mathfrak{X}^k(M)$ the vector space of multivector fields of degree k. Show that there exists a unique $\mathbb{R}$-bilinear operation $[\ ,\ ] : \mathfrak{X}^k(M) \times \mathfrak{X}^s(M) \to \mathfrak{X}^{k+s}(M)$ which coincides with the usual Lie bracket of vector fields when $k = s = 1$ and satisfies:
(a) $[P, Q] = (-1)^{pq}[Q, P]$;
(b) $[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{q(p+1)} Q \wedge [P, R]$;
Verify that this bracket satisfies the following Jacobi type identity:

$$(-1)^{p(r-1)}[P, [Q, R]] + (-1)^{q(p-1)}[Q, [R, P]] + (-1)^{r(q-1)}[R, [P, Q]] = 0.$$

In all these identities, $p = \deg P$, $q = \deg Q$ and $r = \deg R$.
Note: This operation is known as the **Schouten bracket** and is the counterpart for multivector fields of the exterior differential for forms. It is an example of a graded or super Lie bracket.

Ultimately, our interest on differential forms of degree d lies in the fact that they can be integrated over oriented d-manifolds, as we now explain.

Let us start with the case where $M = \mathbb{R}^d$, with the usual orientation. If $U \subset \mathbb{R}^d$ is open, then every differential form $\omega \in \Omega^d(U)$ can be written as:

$$\omega = f \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d, \quad (f \in C^\infty(U)).$$

We say that ω is **integrable in** U and we define its integral by:

$$\int_U \omega = \int_U f(x^1, \dots, x^d)\mathrm{d}x^1 \cdots \mathrm{d}x^d,$$

provided the integral in the right hand side exists and is finite.

The usual change of variable formula for the integral in $\mathbb{R}^d$ yields the following result:

Lemma 17.1. *Let $\Phi : U \to \mathbb{R}^d$ be a diffeomorphism defined in an open connected set $U \subset \mathbb{R}^d$. If ω is a differential form integrable in $\Phi(U)$, then $\Phi^*\omega$ is integrable in U and*

$$\int_{\Phi(U)} \omega = \pm \int_U \Phi^*\omega,$$

where $\pm$ is the sign of the determinant of the Jacobian matrix $\Phi'(p)$.

Therefore, as long as we consider only orientation preserving diffeomorphisms, the integral is invariant under diffeomorphisms. For this reason, we will only consider the integral of differential forms over *oriented* manifolds. It is possible to define the integral over non-oriented manifolds, but this requires introducing **odd differential forms**, which generalize the **even differential forms** that we have been discussing.

We will also assume, in order to avoid convergence issues, that the differential forms $\omega \in \Omega^k(M)$ to be integrated have support

$$\sup \omega = \overline{\{p \in M : \omega_p \neq 0\}},$$

a compact set. We will denote by $\Omega^k_c(M)$ the smooth differential forms of degree k with compact support.

Definition 17.2. *If M is an oriented d-manifold and $\omega \in \Omega^d_c(M)$ has compact support, we define its* ***integral*** *over M as follows:*

- *If $\sup \omega \subset U$, where (U, ϕ) is a positive coordinate chart, then:*

$$\int_M \omega := \int_{\phi(U)} (\phi^{-1})^*\omega.$$

- *More generally, we consider an open cover of M by positive charts (U_α, ϕ_α) and a partition of unit $\{\rho_\alpha\}$ subordinated to this cover, and we define:*

$$\int_M \omega = \sum_\alpha \int_M \rho_\alpha \omega.$$

We remark that the sum in this definition is finite, since we assume that $\operatorname{supp}\omega$ is compact. The definition gives distinct ways to compute the integral of a form with support in a chart. It is easy to check that they yield the same result. One can also show that the definition is independent of the choice of covering and partition of unit. We leave it to the exercises the check of all these details.

It is also easy to check, that the integral satisfies the following basic properties:

(a) *Linearity:* If $\omega, \eta \in \Omega^d_c(M)$ and $a, b \in \mathbb{R}$, then:

$$\int_M (a\omega + b\eta) = a \int_M \omega + b \int_M \eta.$$

(b) *Additivity:* If $M = M_1 \cup M_2$ and $\omega \in \Omega^d_c(M)$, then:

$$\int_M \omega = \int_{M_1} \omega + \int_{M_2} \omega,$$

provided that $M_1 \cap M_2$ has zero measure.

Moreover, we have:

Theorem 17.3 (Change of Variables Formula)**.** *Let M and N be oriented manifolds of dimension d and let $\Phi : M \to N$ be an orientation preserving diffeomorphism. Then, for every differential form $\omega \in \Omega^d_c(N)$, one has:*

$$\int_N \omega = \int_M \Phi^*\omega.$$

Proof. Since Φ is a diffeomorphism and preserves orientations, we can find an open cover of M by positive charts (U_α, ϕ_α) positivos, such that the open sets $\Phi(U_\alpha)$ are domains of positive charts $\psi_\alpha : \Phi(U_\alpha) \to \mathbb{R}^d$ for N. Let $\{\rho_\alpha\}$ be a partition of unity for N subordinated to the cover $\{\Phi(U_\alpha)\}$, so that $\{\rho_\alpha \circ \Phi\}$ is a partition of unity for M subordinated to the cover $\{U_\alpha\}$. By Lemma 17.1, we find:

$$\int_{\Phi(U_\alpha)} \rho_\alpha\omega = \int_{U_\alpha} \Phi^*(\rho_\alpha\omega) = \int_{U_\alpha} (\rho_\alpha \circ \Phi)\Phi^*\omega.$$

Hence, we obtain:

$$\begin{aligned}
\int_N \omega &= \sum_\alpha \int_N \rho_\alpha\omega \\
&= \sum_\alpha \int_{\Phi(U_\alpha)} \rho_\alpha\omega \\
&= \sum_\alpha \int_{U_\alpha} (\rho_\alpha \circ \Phi)\Phi^*\omega \\
&= \sum_\alpha \int_M (\rho_\alpha \circ \Phi)\Phi^*\omega = \int_M \Phi^*\omega.
\end{aligned}$$

$\square$

The computation of the integral of differential forms from the definition is not practical since it uses a partition of unity. The following result can often be applied to avoid the use of partitions of unity:

Proposition 17.4. *Let M be an oriented manifold of dimension d and let $C \subset M$ be a closed subset of zero measure. For any differential form $\omega \in \Omega^d_c(N)$, we have:*

$$\int_M \omega = \int_{M-C} \omega.$$

Proof. Using a partition of unity we can reduce the result to the case where M is an open subset of $\mathbb{R}^d$. For any open set $U \subset \mathbb{R}^d$, the result reduced to the equality:

$$\int_U f \mathrm{d}x^1 \dots \mathrm{d}x^d = \int_{U-C} f \mathrm{d}x^1 \dots \mathrm{d}x^d,$$

where $f : U \to \mathbb{R}$ is smooth and bounded. This result holds, since C has zero measure. $\square$

Example 17.5.

Let $i : \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ be the 2-sphere and consider the standard orientation defined by the volume form $\mu \in \Omega^2(\mathbb{R}^3)$:

$$\mu = i^* x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y.$$

By the proposition, we have that:

$$\int_{\mathbb{S}^2} \mu = \int_{\mathbb{S}^2 - p} \mu,$$

for any $p \in \mathbb{S}^2$. Let us take the north pole $p = p_N$. Then the stereographic projection o pólo $\pi_N : \mathbb{S}^2 - N \to \mathbb{R}^2$ defines a global chart for $\mathcal{S}^2 - \{p_N\}$ whose inverse is the parameterization:

$$\pi_N^{-1}(u, v) = \frac{1}{u^2 + v^2 + 1}(2u, 2v, u^2 + v^2 - 1).$$

We then compute:

$$(\pi_N^{-1})^* i^* \omega = (i \circ \pi_N^{-1})^* \omega = -\frac{4}{(u^2 + v^2 + 1)^2} \mathrm{d}u \wedge \mathrm{d}v.$$

Which shows that π_N is a negative chart. Therefore:

$$\int_{\mathbb{S}^2} \mu = \int_{\mathbb{R}^2} \frac{4}{(u^2 + v^2 + 1)^2} \mathrm{d}u \wedge \mathrm{d}v.$$

The integral on the right can be computed using polar coordinates, and the result is:

$$\int_{\mathbb{S}^2} \mu = \int_0^{+\infty} \int_0^{2\pi} \frac{4r}{(r^2 + 1)^2} \mathrm{d}\theta \mathrm{d}r = 4\pi$$

Our next aim is to generalize Stokes Theorem to differential forms.

Let M be a manifold with boundary and $p \in \partial M$. In local coordinates $(U, x^1, \ldots, x^d)$ centered at p, a tangent vector $\mathbf{v} \in T_pM$ can be written in the form:

$$\mathbf{v} = \sum_{i=1}^{d} v^i \left.\frac{\partial}{\partial x^i}\right|_p.$$

and the tangent vectors in $T_p(\partial M)$ are exactly the tangent vectors whose last component vanishes:

$$T_p(\partial M) = \{\mathbf{v} \in T_pM : v^d = 0\}.$$

We will say that a tangent vector is **exterior** to ∂M if $v^d < 0$. It is easy to see that this condition is independent of the choice of charts centered at p.

We can use this remark to construct the **induced orientation** on ∂M, whenever $(M, [\mu])$ is an oriented manifold with boundary: if $p \in \partial M$, the orientation of $T_p(\partial M)$ is, by definition, $[i_{\mathbf{v}}\mu_p]$ where $\mathbf{v} \in T_pM$ is any exterior tangent vector to ∂M. Is easy to see that this definition is independent of choice of exterior tangent vector so we have a well defined orientation $[\partial\mu]$ for ∂M. Henceforth, whenever M is an oriented manifold with boundary, we will always consider the induced orientation on ∂M.

Theorem 17.6 (Stokes Formula)**.** *Let M be an oriented manifold with boundary of dimension d. If $\omega \in \Omega_c^{d-1}(M)$ then:*

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega.$$

Proof. We consider first two special cases.

The case $M = \mathbb{R}^d$: By linearity of the integral, we can reduce to the case where $\omega = f\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{d-1}$ where f has compact support. Then:

$$\mathrm{d}\omega = (-1)^{d-1}\frac{\partial f}{\partial x^d}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d.$$

By Fubini's Theorem:

$$\int_M \mathrm{d}\omega = (-1)^{d-1}\int_{\mathbb{R}^{d-1}}\left(\int_{-\infty}^{+\infty}\frac{\partial f}{\partial x^d}\mathrm{d}x^d\right)\mathrm{d}x^1\cdots\mathrm{d}x^{d-1} = 0.$$

since f compacto support. Since $\partial\mathbb{R}^d = \emptyset$, Stokes Formula for $\mathbb{R}^d$ follows.

The case $M = H^d$: In this case, we can write:

$$\omega = \sum_{i=1}^{d} f_i \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^d,$$

hence:

$$\mathrm{d}\omega = \sum_{i=1}^{d}(-1)^{i-1}\frac{\partial f_i}{\partial x^i}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d,$$

For $i \neq d$, by a computation entirely similar to the previous case, we obtain:

$$\int_{H^d} \frac{\partial f_i}{\partial x^i} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d = 0.$$

For $i = d$, we compute:

$$\begin{aligned}
(-1)^{d-1} \int_{H^d} \frac{\partial f_d}{\partial x^d} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d = \\
&= (-1)^{d-1} \int_{\mathbb{R}^{d-1}} \left(\int_0^{+\infty} \frac{\partial f_d}{\partial x^d} \mathrm{d}x^d \right) \mathrm{d}x^1 \cdots \mathrm{d}x^{d-1} \\
&= (-1)^d \int_{\mathbb{R}^{d-1}} f_d(x^1, \dots, x^{d-1}, 0) \mathrm{d}x^1 \cdots \mathrm{d}x^{d-1}.
\end{aligned}$$

Assim, obtemos:

$$\int_{H^d} \mathrm{d}\omega = (-1)^d \int_{\mathbb{R}^{d-1}} f_d(x^1, \dots, x^{d-1}, 0) \mathrm{d}x^1 \cdots \mathrm{d}x^{d-1}.$$

Por outro lado, $\partial H^d = \{(x^1, \dots, x^d) : x^d = 0\}$, logo

$$\int_{\partial H^d} \omega = \int_{\partial H^d} f_d(x^1, \dots, x^{d-1}, 0) \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{d-1}.$$

In $\partial H^d = \mathbb{R}^{d-1}$ we must take the induced orientation from the canonical orientation $[\mu] = [\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d]$ in H^d. The induced orientation is given by: $[(-1)^d \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{d-1}]$ so we conclude that:

$$\int_{\partial H^d} \omega = (-1)^d \int_{\partial \mathbb{R}^{d-1}} f_d(x^1, \dots, x^{d-1}, 0) \mathrm{d}x^1 \cdots \mathrm{d}x^{d-1}.$$

Therefore, Stokes Formula also hold for the half space H^d.

We now consider the general case of a manifold of dimension d. We fix an open cover of M by positive charts (U_α, ϕ_α) and we choose a partition of unity $\{\rho_\alpha\}$ subordinated to this cover. The forms $\rho_\alpha \omega$ have compact support contained in U_α:

$$\operatorname{supp} \rho_\alpha \omega \subset \operatorname{supp} \rho_\alpha \cap \operatorname{supp} \omega.$$

Since each U_α is diffeomorphic to either $\mathbb{R}^d$ or to H^d, we already know that:

$$\int_{U_\alpha} \mathrm{d}(\rho_\alpha \omega) = \int_{\partial U_\alpha} \rho_\alpha \omega.$$

By the linearity and the additivity of the integral, we obtain:

$$\begin{aligned}\int_M \mathrm{d}\omega &= \sum_\alpha \int_M \mathrm{d}(\rho_\alpha\omega)\\ &= \sum_\alpha \int_{U_\alpha} \mathrm{d}(\rho_\alpha\omega)\\ &= \sum_\alpha \int_{\partial U_\alpha} \rho_\alpha\omega\\ &= \int_{\partial M} \sum_\alpha \rho_\alpha\omega = \int_{\partial M} \omega.\end{aligned}$$

□

Corollary 17.7. *Let M be a compact, oriented, manifold of dimension d (without boundary). For any $\omega \in \Omega^{d-1}(M)$:*

$$\int_M \mathrm{d}\omega = 0.$$

HOMEWORK.

1. Show that the integral of differential forms is linear and additive relative to the region of integration.

2. Show that if in H^d one considers the standard orientation $[\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d]$, then the induced orientation in $\partial H^d = \mathbb{R}^{d-1}$ is given by $[(-1)^d \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{d-1}]$

3. Consider the 2-torus T^2 as an embedded submanifold of $\mathbb{R}^4$:
$$T^2 = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 = 1, z^2 + y^2 = 1\}.$$
Let ω be the restriction of the form $\mathrm{d}x \wedge \mathrm{d}z \in \Omega^2(\mathbb{R}^4)$ to T^2. Compute the integral $\int_{T^2} \omega$ for an orientation of your choice.

4. Given a volume form μ on a compact manifold M one defines the **volume of** M to be the integral $\int_M \mu$, where the integral is taken relative to the orientation $[\mu]$. Find the volume of $\mathbb{S}^d$ for the standard volume form on the sphere:
$$\mu = \sum_{i=1}^{d+1} (-1)^i x^i \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^{d+1}\Big|_{\mathbb{S}^d}.$$

5. Let M be an oriented Riemannian manifold with boundary. If $f : M \to \mathbb{R}$ is a smooth, compactly supported function, define the integral of f over M by:
$$\int_M f \equiv \int_M *\omega.$$
Also, if X is any vector field proof the classical **Divergence Theorem**:
$$\int_M \operatorname{div} X = \int_{\partial M} X \cdot n,$$
where $n : \partial M \to T_{\partial M}M$ is the unit exterior normal vector along ∂M.

6. Let M be an oriented Riemannian manifold with boundary. For any smooth function $f : M \to \mathbb{R}$ denote by $\frac{\partial f}{\partial n}$ the function $n(f) : \partial M \to \mathbb{R}$, where n is the unit exterior normal vector along ∂M. Verify the following Green identities:

$$\int_{\partial M} f\frac{\partial g}{\partial n} = \int_M \langle \operatorname{grad} f, \operatorname{grad} g\rangle - \int_M f\Delta g,$$
$$\int_{\partial M}\left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) = \int_M (g\Delta f - f\Delta g),$$

where $f, g \in C^\infty(M)$.

7. Let G be a Lie group of dimension d.
   (a) Show that if $\omega, \omega' \in \Omega^d(M)$ are left invariant and $[\omega] = [\omega']$, then

$$\int_G f\omega = a\int_G f\omega', \ \forall f \in C^\infty(M),$$

   for some real number $a > 0$.

Fix an orientation μ for G and choose a left invariant form $\omega \in \Omega^d(M)$ such that $\mu = [\omega]$. Define the integral of $f : G \to \mathbb{R}$ by:

$$\int_G f \equiv \int_G f\omega.$$

   (b) Show that the integral is left invariant, i.e., for every $g \in G$ is valid the identity:

$$\int_G f \circ L_g = \int_G f.$$

   (c) Give an example of a Lie group where the integral is not right invariant.

For each $g \in G$, the differential form $R_g^*\omega$ is left invariant, hence

$$R_g^*\omega = \tilde{\lambda}(g)\omega,$$

for some smooth function $\tilde{\lambda} : G \to \mathbb{R}$. The **modular function** $\lambda : G \to \mathbb{R}_+$ is defined to be $\lambda(g) = |\tilde{\lambda}(g)|$.

   (d) Show that the integral is right invariant if and only if G is **unimodular**, i.e., $\lambda \equiv 1$.
   (e) Show that a compact Lie group is unimodular.

8. Let G be a compact Lie group and let $\Phi : G \to GL(V)$ be a representation of G. Show that there exists an inner product $\langle\ ,\ \rangle$ in V such that this representation is by orthogonal transformations:

$$\langle \Phi(g)\cdot \mathbf{v}, \Phi(g)\cdot \mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{w}\rangle, \quad \forall g \in G.$$

(Hint Use the fact that a compact Lie group is unimodular.)

9. Let G be a compact Lie group. Show that G has a bi-invariant Riemannian metric, i.e., a Riemannian metric which is both right and left invariant.
(Hint: A left invariant Riemannian metric in G is also right invariant if and only if the inner product $\langle\ ,\ \rangle$ induced in $\mathfrak{g} \simeq T_eG$ satisfies:

$$\langle \operatorname{Ad}(g)\cdot X, \operatorname{Ad}(g)\cdot Y\rangle = \langle X, Y\rangle, \quad \forall g \in G, X, Y \in \mathfrak{g}.$$

The equation $\mathrm{d}^2 = 0$, which so far we have made little use, has in fact some deep consequences, as we shall see in the next few lectures.

Definition 18.1. *Let $\omega \in \Omega^k(M)$.*

*(i) ω is called a **closed form** if $\mathrm{d}\omega = 0$.*
*(ii) ω is called an **exact form** if $\omega = \mathrm{d}\eta$, for some $\eta \in \Omega^{k-1}(M)$.*

We will denote by $Z^k(M)$, respectively $B^k(M)$, the subspaces of closed, respectively exact, differential forms of degree k.

In other words, the closed forms form the kernel of d, while the exact forms form the image of d. The pair $(\Omega(M), \mathrm{d})$ is called the *de Rham complex* of M and we will often represent it as:

$$\cdots \longrightarrow \Omega^{k-1}(M) \xrightarrow{\mathrm{d}} \Omega^k(M) \xrightarrow{\mathrm{d}} \Omega^{k+1}(M) \longrightarrow \cdots$$

The fact that $\mathrm{d}^2 = 0$ means that every exact form is closed:

$$B^k(M) \subset Z^k(M).$$

One should think of $(\Omega(M), \mathrm{d})$ as a set of differential equations associated with the manifold M. Finding the closed forms, means to solve the differential equation:

$$\mathrm{d}\omega = 0.$$

On the other hand, the exact forms can be thought of as the trivial solutions of this equation. We are interested in the space of all solutions modulus the trivial solutions, and this is called the de Rham cohomology of M:

Definition 18.2. *The **de Rham cohomology space of order** k is the vector space:*

$$H^k(M) \equiv Z^k(M)/B^k(M).$$

One can also consider differential forms whose support is compact, which we denote by $\Omega^k_c(M)$. Clearly, the differential d takes a compactly supported form to a compactly supported form, so we have another complex $(\Omega_c(M), \mathrm{d})$, and we let:

Definition 18.3. *The **compactly supported de Rham cohomology space of order** k is the vector space:*

$$H^k_c(M) \equiv Z^k_c(M)/B^k_c(M),$$

where $Z^k_c(M) \subset \Omega^k_c(M)$, respectively $B^k_c(M) \subset \Omega^k_c(M)$, denotes the subspaces of closed, respectively exact, compactly supported forms of degree k.

Obviously, $H^k(M) = H^k_c(M)$ if M is compact. In general these two groups are different, as we can see already in degree 0, where the de Rham cohomology spaces have the following meaning:

Theorem 18.4. *Let M be a smooth manifold. Then:*

$$H^0(M) = \mathbb{R}^l,$$

where l is the number of connected components of M, and

$$H^0_c(M) = \mathbb{R}^{l'},$$

where l' is the number of compact connected components of M.

Proof. We have $\Omega^0(M) = C^\infty(M)$ and if $f \in C^\infty(M)$ satisfies $\mathrm{d}f = 0$, then f is locally constant. Hence:

$$Z^0(M) = \mathbb{R}^l,$$

where l is the number of connected components of M. Since

$$B^0(M) = \{0\},$$

we have that $H^0(M) = \mathbb{R}^l$.

On the other hand, if we consider compactly supported forms, we note that if $f \in C^\infty_c(M)$ satisfies $\mathrm{d}f = 0$, then f is constant in the compact connected components of M and is zero in the non-compact connected components. Since $B^0_c(M) = \{0\}$, we conclude that

$$H^0_c(M) = \mathbb{R}^{l'},$$

where l' is the number of compact connected components of M. $\square$

In geral, the computation of the cohomology groups $H^k(M)$ and $H^k_c(M)$, for $k \geq 1$, directly from the definition is very hard. Note however, that we have

$$H^k(M) = H^k_c(M) = 0, \text{ if } k > \dim M,$$

since $\Omega^k(M) = 0$ if $k > \dim M$.

In the next lectures we will study several properties of the de Rham cohomology spaces which can be used to compute them. For now, we consider a simple example where one can still use the definition to compute them.

Example 18.5.
Let $M = \mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Since $\mathbb{S}^1$ is compact, we have that $H^\bullet(\mathbb{S}^1) = H^\bullet_c(\mathbb{S}^1)$. Since $\mathbb{S}^1$ is connected, it follows that:

$$H^0(\mathbb{S}^1) = \mathbb{R}.$$

Now to compute $H^1(\mathbb{S}^1)$, we consider the 1-form $-y\mathrm{d}x + x\mathrm{d}y \in \Omega^1(\mathbb{R}^2)$. This form restricts to a 1-form in $\mathbb{S}^1$ which we will denote by ω. Since $\dim \mathbb{S}^1 = 1$, ω is closed. On the other hand, consider the parameterization $\sigma :]0, 2\pi[\to \mathbb{S}^1 - \{(1, 0)\}$, given by $\sigma(t) = (\cos t, \operatorname{sen} t)$. Then:

$$\begin{aligned}
\int_{\mathbb{S}^1} \omega &= \int_{]0,2\pi[} \sigma^*\omega \\
&= \int_{]0,2\pi[} (-\sin t)\mathrm{d}\cos t + \cos t \mathrm{d}\sin t = \int_0^{2\pi} \mathrm{d}t = 2\pi.
\end{aligned}$$

By the corollary to Stokes Formula, we see that ω is not exact, so it represents a non-trivial cohomology class $[\omega] \in H^1(\mathbb{S}^1)$.

The form ω has a simple geometric meaning: since $\sigma^*\omega = \mathrm{d}t$, we have that $\omega = \mathrm{d}\theta$ in $\mathbb{S}^1 - \{(1,0)\}$, where $\theta : \mathbb{S}^1 - \{(1,0)\} \to \mathbb{R}$ is the angle coordinate (the inverse of the parameterization σ). Sometimes one denotes ω by $\mathrm{d}\theta$, in spite of the fact that this is not an exact form.

We claim that $[\omega]$ is a basis for $H^1(\mathbb{S}^1)$. Given a form $\alpha \in \Omega^1(\mathbb{S}^1)$ we have that $\alpha = f\omega$, for some function $f : \mathbb{S}^1 \to \mathbb{R}$. Let

$$c = \frac{1}{2\pi}\int_{\mathbb{S}^1} \alpha = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\mathrm{d}\theta,$$

and define $g : \mathbb{R} \to \mathbb{R}$ by:

$$g(t) = \int_0^t (\alpha - c\omega) = \int_0^t (f(\theta) - c)\mathrm{d}\theta.$$

Since:

$$\begin{aligned} g(t+2\pi) &= \int_0^{t+2\pi} (f(\theta) - c)\mathrm{d}\theta \\ &= \int_0^t (f(\theta) - c)\mathrm{d}\theta + \int_t^{t+2\pi} (f(\theta) - c)\mathrm{d}\theta \\ &= g(t) + \int_0^{2\pi} (f(\theta) - c)\mathrm{d}\theta = g(t), \end{aligned}$$

we obtain a C^∞ function $g : \mathbb{S}^1 \to \mathbb{R}$. In $\mathbb{S}^1 - \{(1,0)\}$, we have that

$$\mathrm{d}g = f(\theta)\mathrm{d}\theta - c\mathrm{d}\theta = \alpha - c\omega.$$

Hence, we must have $\mathrm{d}g = \alpha - c\omega$ in $\mathbb{S}^1$ so that $[\alpha] = c[\omega]$. This shows that $[\omega]$ generates $H^1(\mathbb{S}^1)$ so we conclude that:

$$H^1(\mathbb{S}^1) \simeq \mathbb{R}.$$

The wedge product $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$ induces a product in the de Rham cohomology of M by setting:

$$[\alpha] \cup [\beta] := [\alpha \wedge \beta].$$

We leave it as an exercise to check that this definition is independent of the choice of representatives of the cohomology classes. With this product $H^\bullet(M) = \oplus_k H^k(M)$ and $H_c^\bullet(M) = \oplus_k H_c^k(M)$ become rings.

If $\Phi : M \to N$ is a smooth map, then pull-back map gives a linear map $\Phi^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$ which commutes with the differentials:

$$\Phi^*\mathrm{d}\omega = \mathrm{d}(\Phi^*\omega).$$

Therefore, Φ^* takes closed (respectively, exact) forms to closed (respectively, exact) forms, and we have an induced map in cohomology:

$$\Phi^* : H^\bullet(N) \to H^\bullet(M),\ [\omega] \longmapsto [\Phi^*\omega].$$

This correspondence has the following properties:

(i) $\Phi^*(\alpha \wedge \beta) = (\Phi^*\alpha) \wedge (\Phi^*\beta)$, so $\Phi^* : H^\bullet(N) \to H^\bullet(M)$ is a ring homomorphism.
(ii) If $\Phi : M \to N$ and $\Psi : N \to Q$ are smooth maps, then the composition $(\Psi \circ \Phi)^* : H^\bullet(Q) \to H^\bullet(M)$ satisfies $(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*$;
(iii) The identity map $M \to M$ induces the identity linear transformation $H^\bullet(M) \to H^\bullet(M)$.

In particular, when $\Phi : M \to N$ is a diffeomorphism, the induced linear map $\Phi^* : H^\bullet(N) \to H^\bullet(M)$ is an isomorphism in cohomology. Hence, we have:

Corollary 18.6. *The de Rham cohomology ring is an* invariant *of differentiable manifolds: if M and N are diffeomorphic, then $H^\bullet(M)$ and $H^\bullet(N)$ are isomorphic rings.*

Note that if $\Phi : M \to N$ is a smooth map, in general, the pull-back $\Phi^*\omega$ of a compactly supported form $\omega \in \Omega_c(N)$ is not compactly supported. This will be the case, however, if $\Phi : M \to N$ is a *smooth proper map*. Hence, we can still conclude that:

Corollary 18.7. *The compactly supported de Rham cohomology ring is an* invariant *of differentiable manifolds: if M and N are diffeomorphic, then $H^\bullet_c(M)$ and $H^\bullet_c(N)$ are isomorphic rings.*

Remark 18.8 (*A Crash Course in Homological Algebra - part I*)**.** The de Rham complex $(\Omega^\bullet(M), \mathrm{d})$ and the compactly supported de Rham complex $(\Omega^\bullet_c(M), \mathrm{d})$ are examples of *cochain complexes*. In general, one calls a **cochain complex** a pair (C, d) where:

(a) C is a $\mathbb{Z}$-graded vector space, i.e., $C = \oplus_{k\in\mathbb{Z}} C^k$ is the direct sum of vector spaces [4];
(b) $\mathrm{d} : C \to C$ is a linear transformation of degree 1, i.e., $\mathrm{d}(C^k) \subset C^{k+1}$, such that $\mathrm{d}^2 = 0$.

One represents a complex by the diagram:

$$\cdots \longrightarrow C^{k-1} \xrightarrow{\mathrm{d}} C^k \xrightarrow{\mathrm{d}} C^{k+1} \longrightarrow \cdots$$

The transformation d is called the **differential** of the complex.

For any cochain complex, (C, d) one defines the subspace of all **cocycles**:

$$Z^k(C) \equiv \{z \in C^k : \mathrm{d}z = 0\},$$

and the subspace of all **coboundaries**

$$B^k(C) \equiv \{\mathrm{d}z : z \in C^{k-1}\}.$$

Since $\mathrm{d}^2 = 0$, we have that $B^k(C) \subset Z^k(C)$. The **cohomology of** (C, d) is the direct sum $H(C) = \oplus_{k\in\mathbb{Z}} H^k(C)$ of all the cohomology spaces of order

[4]More generally, one can consider complexes formed by $\mathbb{Z}$-graded modules over commutative rings with unit (e.g., abelian groups). Most of the following statements are valid for the category of modules over commutative rings with unit, with obvious modifications.

k, which are defined by:

$$H^k(C) = \frac{Z^k(C)}{B^k(C)}.$$

Given two cochain complexes (A, d_A) and (B, d_B), a **cochain map** is a linear map $f : A \to B$ such that:

(a) f preserves the grading, i.e., $f(A^k) \subset B^k$;

(b) f commutes with the differentials, i.e., $f\mathrm{d}_A = \mathrm{d}_B f$.

One represents a cochain map by a commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{k-1} \arrow[r, "\mathrm{d}_A"] \arrow[d, "f"'] & A^k \arrow[r, "\mathrm{d}_A"] \arrow[d, "f"'] & A^{k+1} \arrow[r] \arrow[d, "f"'] & \cdots \\
\cdots \arrow[r] & B^{k-1} \arrow[r, "\mathrm{d}_B"'] & B^k \arrow[r, "\mathrm{d}_B"'] & B^{k+1} \arrow[r] & \cdots
\end{tikzcd}$$

It should be clear that a cochain map $f : A \to B$ takes cocycles to cocycles and coboundaries to coboundaries. Hence, f induces a linear map in cohomology, which we denote by the same letter: $f : H^\bullet(A) \to H^\bullet(B)$.

The cochain complexes and cochain maps form a category, and their study is one of the central themes of *Homological Algebra*.

One can summarize the remarks above as follows: the assignment which associates to a differential manifold M its de Rham complex $(\Omega(M), \mathrm{d})$ and to each smooth map $\Phi : M \to N$ the pull-back $\Phi^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$ is a *contravariant functor* from the category of differential manifolds to the category of cochain complexes. Similarly, the same assignment gives a functor from the category of differential manifolds and smooth proper maps to the to the category of cochain complexes which assigns to a differentiable manifold M its compactly supported de Rham complex.

HOMEWORK.

1. Consider the 2-sphere $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.
   (a) Show that every closed 1-form $\omega \in \Omega^1(\mathbb{S}^2)$ is exact.
   (b) Show that the 2-form in $\mathbb{R}^3 - 0$ given by
   $$\omega = x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y.$$
   induces by restriction to $\mathbb{S}^2$ a non-trivial cohomology class $[\omega] \in H^2(\mathbb{S}^2)$.

2. Show that $H^1(\mathbb{T}^d) = \mathbb{R}^d$, using the definition of the de Rham cohomology.

   HINT: Show that a basis for $H^1(\mathbb{T}^d)$ is given by $\{[\mathrm{d}\theta_1], \ldots, [\mathrm{d}\theta_d]\}$, where $(\theta_1, \ldots, \theta_d)$ are the angles on each $\mathbb{S}^1$ factor.

3. Show that if M is a compact, orientable, d-manifold, then $H^d(M) \neq 0$.

   HINT: Consider the cohomology class defined by some volume form $\mu \in \Omega^d(M)$.

4. Use de Rham cohomology to prove that $\mathbb{T}^2$ and $\mathbb{S}^2$ are not diffeomorphic manifolds.

HINT: Show that every closed 1-form on $\mathbb{S}^2$ is exact as follows: if $\omega \in \Omega^1(\mathbb{S}^2)$ is closed, then $\omega = \mathrm{d}f$, where $f \in C^\infty(\mathbb{S}^2)$ is defined by $f(x) := \int_\gamma \omega$, where γ is any curve in $\mathbb{S}^2$ with $\gamma(0) = p_N$ and $\gamma(1) = x$.

5. Show that the wedge product $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$ induces a product $\cup$ in the de Rham cohomology of M for which $H(M) = \oplus_k H^k(M)$ becomes a ring.

6. A symplectic form on a manifold M of dimension $2n$ is a 2-form $\omega \in \Omega^2(M)$ such that $\mathrm{d}\omega = 0$ and $\wedge^n\omega$ is a volume form. Show that if M is compact and admits some symplectic form, then $H^{2k}(M) \neq 0$ for $k = 0, \ldots, n$.

HINT: Use the ring structure of $H^\bullet(M)$.

LECTURE 19. THE DE RHAM THEOREM

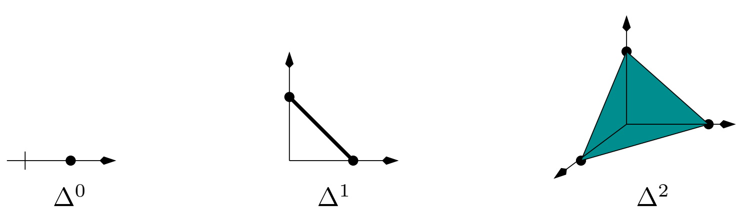
We saw in the previous lecture that de Rham cohomology is an invariant of differential manifolds. Actually, de Rham cohomology is a topological invariant: if M and N are smooth manifolds which are homeomorphic as topological spaces, then their de Rham cohomologies are isomorphic. This is a consequence of the famous de Rham Theorem, which shows that for any smooth manifold its singular cohomology with real coefficients is isomorphic with its de Rham cohomology.

Singular Homology. We recall the definition of the singular homology of a topological space M. Although we will continue to use the letter M, the following discussion only uses the topology of M.

We denote by $\Delta^k \subset \mathbb{R}^{k+1}$ the **standard k-simplex**:

$$\Delta^k = \{(t_0, \ldots, t_k) \in \mathbb{R}^{k+1} : \sum_{i=0}^{k} t_i = 1, t_i \geq 0\}.$$

Note that $\Delta^0 = \{1\}$ has only one element.



Definition 19.1. *A **singular k-simplex** in M is a continuous map $\sigma : \Delta^k \to M$. A **singular k-chain** is a formal linear combination*

$$c = \sum_{i=1}^{p} a_i \sigma_i,$$

where $a_i \in \mathbb{R}$ and the σ_i are singular k-simplices.

We will denote by $S_k(M;\mathbb{R})$ the set of all singular k-chains. Note that $S_k(M;\mathbb{R})$ is a real vector space. In fact, formally, $S_k(M;\mathbb{R})$ is the free vector space generated by the set of all singular k-simplices. One can also consider other coefficients besides $\mathbb{R}$, but here we will consider only real coefficients.

We define the ***i*-face map** of the standard k-simplex, where $0 \leq i \leq k$, to be the map $\varepsilon^i : \Delta^{k-1} \to \Delta^k$ defined by:

$$\varepsilon^i(t_0, \dots, t_{k-1}) = (t_0, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_{k-1}).$$

These face maps of the standard k-simplex induce face maps ε_i of any singular k-simplex $\sigma : \Delta^k \to M$ by setting:

$$\varepsilon_i(\sigma) = \sigma \circ \varepsilon^i.$$

These clearly extend by linearity to any k-chain, yielding linear maps

$$\varepsilon_i : S_k(M;\mathbb{R}) \to S_{k-1}(M;\mathbb{R}),$$

and these lead to the following definition:

Definition 19.2. *The **boundary of a k-chain** c is the $(k-1)$-chain ∂c defined by*

$$\partial c = \sum_{i=0}^{k} (-1)^i \varepsilon_i(c).$$

The geometric meaning of this definition is that we consider the faces of each simplex with a certain choice of signs, which one should view as some kind of orientations of the faces. We illustrate this choice in the next example.

Example 19.3.
The boundary of the standard 2-simplex $\sigma =$id: $\Delta^2 \to \mathbb{R}^3$ is the chain:

$$\partial\sigma = \varepsilon_0(\sigma) - \varepsilon_1(\sigma) + \varepsilon_2(\sigma),$$

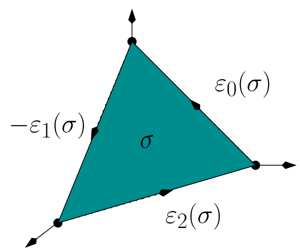
where ε_0, ε_1 and ε_2 are the 1-simplices (faces) given by:

$$\begin{aligned}
\varepsilon_0(\sigma)(t_0, t_1) &= (0, t_0, t_1),\\
\varepsilon_1(\sigma)(t_0, t_1) &= (t_0, 0, t_1),\\
\varepsilon_2(\sigma)(t_0, t_1) &= (t_0, t_1, 0).
\end{aligned}$$

Also, the simplices ε_0, ε_1 and ε_2 have boundaries:

$$\begin{aligned}
\partial\varepsilon_0(\sigma)(0,1) &= (0,0,1) - (0,1,0),\\
\partial\varepsilon_1(\sigma)(0,1) &= (1,0,0) - (0,0,1),\\
\partial\varepsilon_2(\sigma)(0,1) &= (1,0,0) - (0,1,0).
\end{aligned}$$

We can represent this choice of signs by including orientations on the faces of the simplex, as shown schematically by the following figure:



Note that:

$$\partial^2\sigma = \partial(\partial\sigma) = \partial\varepsilon_0(\sigma) - \partial\varepsilon_1(\sigma) + \partial\varepsilon_2(\sigma) = 0.$$

We noticed in this example that $\partial^2\sigma = 0$. This is actually a general fact which is a consequence of the judicious choice of signs and parameterizations of the faces. We leave its proof as an exercise:

Lemma 19.4. *For every singular chain c:*

$$\partial(\partial c) = 0.$$

In this way we obtain a complex $S(M;\mathbb{R}) = \oplus_{k\in\mathbb{Z}} S_k(M;\mathbb{R})$:

$$\cdots \longleftarrow S_{k-1}(M;\mathbb{R}) \xleftarrow{\partial} S_k(M;\mathbb{R}) \xleftarrow{\partial} S_{k+1}(M;\mathbb{R}) \longleftarrow \cdots$$

One calls $(S(M;\mathbb{R}), \partial)$ the **complex of singular chains** in M.

Remark 19.5 (*A Crash Course in Homological Algebra - part II*)**.** In the cochain complexes that we studied related to de Rham cohomology the differentials *increase* the degree, while for the singular chains the differential *decreases* the degree.

We call a complex $C = \oplus_{k\in\mathbb{Z}} C_k$ where the differential decreases the degree

$$\cdots \longleftarrow C_{k-1} \xleftarrow{\partial} C_k \xleftarrow{\partial} C_{k+1} \longleftarrow \cdots$$

a **chain complex**. We say that $z \in C_k$ is a **cycle** if $\partial z = 0$ and we say that z is a **boundary** if $z = \partial b$ ([5]). In this case, once defines the **homology** of the complex C is the direct sum $H(C) = \oplus_{k\in\mathbb{Z}} H_k(C)$ of the vector spaces:

$$H_k(C) = \frac{Z_k(C)}{B_k(C)},$$

where $Z_k(C)$ is the subspace of all cycles and $B_k(C)$ is the subspace of all boundaries. Note also the position of the indices.

[5]Notice that given a complex (C, ∂) where the differential decreases one can define a new complex $(\bar{C}, \mathrm{d})$ setting $\bar{C}^k \equiv C_{-k}$ and $\mathrm{d} = \partial$, obtaining a complex where the differential increases the degree. Therefore, these conventions are somewhat arbitrary.

The homology of the complex $(S(M;\mathbb{R}),\partial)$ is called the **singular homology** of M with real coefficients, and is denoted

$$H_k(M;\mathbb{R}) = \frac{Z_k(M;\mathbb{R})}{B_k(M;\mathbb{R})},$$

If $\Phi : M \to N$ continuous map, then for any singular simplex $\sigma : \Delta^k \to M$, we have that $\Phi_*(\sigma) \equiv \Phi \circ \sigma : \Delta^k \to N$ is a singular simplex in N. We extend this map to any chain $c = \sum_j a_j\sigma_j$ requiring linearity to hold:

$$\Phi_*(c) \equiv \sum_j a_j(\Phi \circ \sigma_j).$$

It follows that $\Phi_* : S(M;\mathbb{R}) \to S(N;\mathbb{R})$ is a chain map:

$$\begin{tikzcd}
\cdots & S_{k-1}(M;\mathbb{R}) \arrow[l] \arrow[d, "\Phi_*"'] & S_k(M;\mathbb{R}) \arrow[l, "\partial"'] \arrow[d, "\Phi_*"'] & S_{k+1}(M;\mathbb{R}) \arrow[l, "\partial"'] \arrow[d, "\Phi_*"'] & \cdots \arrow[l] \\
\cdots & S_{k-1}(N;\mathbb{R}) \arrow[l] & S_k(N;\mathbb{R}) \arrow[l, "\partial"'] & S_{k+1}(N;\mathbb{R}) \arrow[l, "\partial"'] & \cdots \arrow[l]
\end{tikzcd}$$

Therefore, Φ_* induces a linear map in singular homology:

$$\Phi_* : H_\bullet(M;\mathbb{R}) \to H_\bullet(N;\mathbb{R}).$$

One checks easily that this assignment has the following properties:

(i) If $\Phi : M \to N$ e $\Psi : N \to Q$ are continuous maps, then:

$$(\Psi \circ \Phi)_* = \Psi_* \circ \Phi_*;$$

(ii) The identity map id: $M \to M$ induces the identity map in homology:

$$\mathrm{id}_* = \mathrm{id} : H_\bullet(M;\mathbb{R}) \to H_\bullet(M;\mathbb{R}).$$

It follows that singular homology is a *topological invariant*:

Theorem 19.6. *If M and N are are homeomorphic spaces then $H_\bullet(M,\mathbb{R}) \simeq H_\bullet(N,\mathbb{R})$.*

Smooth Singular Homology. Assume now that M is a manifold. The chain complex $(S_\bullet(M;\mathbb{R}),\partial)$ has a subcomplex $(S^\infty_\bullet(M;\mathbb{R}),\partial)$ formed by the **smooth singular k-chains**:

$$S^\infty_k(M;\mathbb{R}) = \{\sum_{i=1}^p a_i\sigma_i : \sigma_i : \Delta^k \to M \text{ is smooth}\}$$

This is a sub complex because if $c \in S^\infty_k(M;\mathbb{R})$ is a smooth k-chain, then so is $\partial c \in S^\infty_k(M;\mathbb{R})$.

Remark 19.7. Even when c is smooth, the use of the term "singular" is justified by the absence of any assumption on the differentials of the maps σ_i: in general, a smooth k-simplex *does not* parameterize any submanifold and its image may be contained in a submanifold of dimension less than k.

One has the following important fact, which we will not prove here:

Proposition 19.8. *The inclusion $S^{\infty}_{\bullet}(M,\mathbb{R}) \hookrightarrow S_{\bullet}(M,\mathbb{R})$ induces an isomorphism in homology:*

$$H(S^{\infty}_{\bullet}(M,\mathbb{R})) \simeq H(S_{\bullet}(M,\mathbb{R})).$$

This proposition says that:

(i) every homology class in $H_{\bullet}(M;\mathbb{R})$ has a representative c which is a C^{∞} cycle, and

(ii) if two C^{∞} cycles c and c' differ by a continuous boundary ($c-c'=\partial b$), then they also differ by a C^{∞} boundary b' ($c-c'=\partial b'$).

Hence, smooth singular homology and singular homology coincide.

Singular Cohomology. Dually, one defines the singular cohomology of M as follows. First, one defines the space of **singular k-cochains** with real coefficients to be the vector space dual to $S_k(M.\mathbb{R})$

$$S^k(M;\mathbb{R}) := \mathrm{Hom}(S_k(M;\mathbb{R}),\mathbb{R}).$$

We have a differential $\mathrm{d}: S^k(M;\mathbb{R}) \to S^{k+1}(M;\mathbb{R})$ obtained by transposing the singular boundary operator: if $c: S_k(M,\mathbb{R}) \to \mathbb{R}$ is a linear function we can think of it as collection of numbers $c=(c_\sigma)$, indexed by singular simplices, and then we define:

$$(\mathrm{d}c)_\sigma = \sum_{i=0}^{k} (-1)^i c_{\varepsilon_i(\sigma)}.$$

It follows that $\mathrm{d}^2=0$, so we do have a cochain complex $(S^{\bullet}(M;\mathbb{R}),\mathrm{d})$. The corresponding cohomology is called the **singular cohomology** with real coefficients and is denoted by $H^{\bullet}(M;\mathbb{R})$.

If $\Phi: M \to N$ we can transpose the map $\Phi_*: S_k(M;\mathbb{R}) \to S_k(N;\mathbb{R})$, obtaining a cochain map $\Phi^*: S^k(N;\mathbb{R}) \to S^k(M;\mathbb{R})$:

$$\Phi^*\mathrm{d} = \mathrm{d}\Phi^*,$$

Therefore, we have an induced linear map in singular cohomology $\Phi^*: H^{\bullet}(N;\mathbb{R}) \to H^{\bullet}(M;\mathbb{R})$, which satisfies the obvious functorial properties, and hence we also have:

Theorem 19.9. *If M and N are homeomorphic spaces then $H^{\bullet}(M,\mathbb{R}) \simeq H^{\bullet}(N,\mathbb{R})$.*

Of course, one can also consider smooth singular k-cochains:

$$S^k_{\infty}(M;\mathbb{R}) := \mathrm{Hom}(S^{\infty}_k(M;\mathbb{R}),\mathbb{R}).$$

which form a a complex $(S^{\bullet}_{\infty}(M;\mathbb{R}),\mathrm{d})$. There is an obvious restriction map $S^k(M;\mathbb{R}) \to S^k_{\infty}(M;\mathbb{R})$, which is a cochain map, and this yields an isomorphism in cohomology:

$$H(S^{\bullet}(M;\mathbb{R}),\mathrm{d}) \simeq H(S^{\bullet}_{\infty}(M;\mathbb{R}),\mathrm{d}).$$

For this reason, in the sequel we will not distinguish between these cohomologies.

Singular Cohomology vs. de Rham Cohomology. We now take advantage of the fact that singular cohomology and differentiable singular cohomology coincide to relate it with the de Rham cohomology. For that, we start by explaining that one can integrate differential forms over singular chains.

First, we observe that we can parameterize the standard k-simplex Δ^k by the map $\phi : \Delta_0^k \to \Delta^k$, where:

$$\Delta_0^k := \{(x^1, \dots, x^k) : x^i \geq 0, \sum_{i=1}^{k} x^i \leq 1\}$$

$$\phi(x^1, \dots, x^k) = (1 - \sum_{i=1}^{k} x^i, x^1, \dots, x^k),$$

Hence, if $\omega \in \Omega^k(U)$ is a k-form which is defined in some open set $U \subset \mathbb{R}^{k+1}$ containing the standard k-simplex Δ^k, we can write:

$$\phi^*\omega = f(x^1, \dots, x^k)\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^k,$$

and define:

$$\int_{\Delta^k} \omega := \int_{\Delta_0^k} f \mathrm{d}x^1 \cdots \mathrm{d}x^k.$$

Next, given any differential form $\omega \in \Omega^k(M)$ in a smooth manifold M, we define the **integral of ω over a smooth simplex** $\sigma : \Delta^k \to M$ to be the real number:

$$\int_\sigma \omega := \int_{\Delta^k} \sigma^*\omega.$$

We extend this definition to any smooth singular k-chain $c = \sum_{j=1}^p a_j\sigma_j$ by linearity:

$$\int_c \omega := \sum_{j=1}^{p} a_j \int_{\sigma_j} \omega.$$

We leave it to the exercises the proof of the following version of Stokes formula:

Theorem 19.10 (Stokes II)**.** *Let M be a smooth manifold, $\omega \in \Omega^{k-1}(M)$ a $(k-1)$-differential form, and c a smooth singular k-chain. Then:*

$$\int_c \mathrm{d}\omega = \int_{\partial c} \omega.$$

Now we can define an *integration map* $I : \Omega^\bullet(M) \to S^\bullet_\infty(M; \mathbb{R})$:

$$I(\omega)_\sigma = \int_\sigma \omega, \quad \omega \in \Omega^k(M), \sigma \in S_k^\infty(M; \mathbb{R}).$$

and we have:

Proposition 19.11. *The integration map $I : (\Omega^\bullet(M), \mathrm{d}) \to (S^\bullet_\infty(M; \mathbb{R}), \mathrm{d})$ is a chain map:*

$$I(\mathrm{d}\omega) = \mathrm{d}I(\omega).$$

Proof. This follows from the following computation, using Stokes formula for chains:

$$\begin{aligned}
I(\mathrm{d}\omega)_\sigma &= \int_\sigma \mathrm{d}\omega \\
&= \int_{\partial\sigma} \omega \\
&= \sum_{i=0}^{k+1} (-1)^i \int_{\varepsilon_i(\sigma)} \omega \\
&= \sum_{i=0}^{k+1} (-1)^i (I(\omega))_{\varepsilon_i(\sigma)} = \mathrm{d}I(\omega).
\end{aligned}$$

□

It follows that we have an induced linear map in cohomology:

$$I : H^k_{dR}(M) \to H^k(M;\mathbb{R}).$$

Theorem 19.12 (de Rham)**.** *For any smooth manifold the integration map* $I : H^\bullet_{dR}(M) \to H^\bullet(M;\mathbb{R})$ *is an isomorphism.*

We will not prove this result in these lectures. We note however that it has the following very important consequence: the de Rham cohomology is actually a topological invariant of smooth manifolds, i.e., if M and N are homeomorphic smooth manifolds then their de Rham cohomologies are isomorphic. For example, the different exotic smooth structures on the spheres all have the same de Rham cohomology!

Homework.

1. Show that for every singular chain c one has $\partial(\partial c) = 0$.

2. Give a proof of Stokes Formula for singular chains, by showing the following:
   (a) It is enough to prove the formula for chains consisting of a singular simplex.
   (b) It is enough to prove the formula for the standard k-simplex.
   (c) It is enough to prove the formula for $(k-1)$-differential forms in $\mathbb{R}^k$ of the type:
   $$\omega = f\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^k.$$
   (d) Show that
   $$\int_{\Delta^k} \mathrm{d}\omega = \int_{\partial\Delta^k} \omega,$$
   where ω is a differential form of the type (c).

3. In the torus $\mathbb{T}^d = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ consider the 1-chains $c_1, \ldots, c_d : [0,1] \to \mathbb{T}^d$ defined by:
$$c_j(t) \equiv (1, \ldots, e^{2\pi i t}, \ldots, 1) \quad (j = 1, \ldots, d).$$
Show that:

(a) The c_j's are 1-cycles: $\partial c_j = 0$;
(b) The c_j's are not 1-boundaries;
(c) The classes $\{[c_1], \dots, [c_d]\} \subset H_1(\mathbb{T}^d, \mathbb{R})$ form a linearly independent set.

Hint: Use Stokes formula.

4. The de Rham Theorem, shows that the exterior product induces a product

$$\cup : H^k(M;\mathbb{R}) \times H^l(M:\mathbb{R}) \to H^{k+l}(M;\mathbb{R}),$$

so that $H^\bullet(M;\mathbb{R})$ becomes a ring. This product is called the **cup product**. Here is one way of constructing it directly:

(a) Show that for $l < k$ and $0 \leq i_0 < \cdots < i_l \leq k$ one has maps $\varepsilon_{i_0,\dots,i_l} : \Delta^l \to \Delta^k$, defined by:

$$\varepsilon_{i_0,\dots,i_l}(t_0,\dots,t_l) = (s_0,\dots,s_k), \quad \text{where } \begin{cases} s_l = 0, \text{ if } l \notin \{i_0,\dots,i_l\} \\ s_{i_j} = t_j, \text{ otherwise.} \end{cases}$$

(b) Show that if $c_1 \in S^k(M;\mathbb{R})$ and $c_2 \in S^l(M;\mathbb{R})$ the formula:

$$(c_1 \cup c_2)(\sigma) := c_1(\sigma \circ \varepsilon_{1,\dots,k}) c_2(\sigma \circ \varepsilon_{k+1,\dots,k+l}),$$

defines an element $c_1 \cup c_2 \in S^{k+l}(M;\mathbb{R})$.

(c) Show that for any chains $c_1 \in S^k(M;\mathbb{R})$ and $c_2 \in S^l(M;\mathbb{R})$ one has:

$$\mathrm{d}(c_1 \cup c_2) = (\mathrm{d}c_1) \cup c_2 + (-1)^k c_1 \cup (\mathrm{d}c_2).$$

It follows that one can define $\cup : H^k(M;\mathbb{R}) \times H^l(M:\mathbb{R}) \to H^{k+l}(M;\mathbb{R})$ by

$$[c_1] \cup [c_2] := [c_1 \cup c_2].$$

Note that for the integration map $I : \Omega^k(M) \to S^k_\infty(M)$, in general, $I(\omega \wedge \eta) \neq I(\omega) \cup I(\eta)$. However, this equality holds in cohomology:

$$I([\omega] \wedge [\eta]) = I([\omega]) \cup I([\eta]), \quad [\omega] \in H^k_{dR}(M), [\eta] \in H^l_{dR}(M).$$

Lecture 20. Homotopy Invariance and Mayer-Vietoris Sequence

We shall now study some properties of de Rham cohomology which are very useful in the computation of these rings in specific examples.

The Poincaré Lemma. We start with the simplest example of manifold, namely $M = \mathbb{R}^d$. In order to compute its cohomology we will proceed by induction in the dimension d. Since $\mathbb{R}^{d+1} = \mathbb{R}^d \times \mathbb{R}$, we consider the projection $\pi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ and the inclusion $i : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}$ given by:

$$\begin{tikzcd} \mathbb{R}^d \times \mathbb{R} \arrow[d, shift left, "\pi"] \\ \mathbb{R}^d \arrow[u, shift left, "i"] \end{tikzcd} \qquad \begin{aligned} \pi(x,t) &= x, \\ i(x) &= (x,0). \end{aligned}$$

The associated pull-back maps give linear maps

$$\begin{tikzcd} \Omega^\bullet(\mathbb{R}^d \times \mathbb{R}) \arrow[d, shift right, "i^*"'] \\ \Omega^\bullet(\mathbb{R}^d) \arrow[u, shift right, "\pi^*"'] \end{tikzcd}$$

and we will see that these induce isomorphisms in cohomology:

Proposition 20.1. *The induced maps* $i^* : H^\bullet(\mathbb{R}^d \times \mathbb{R}) \to H^\bullet(\mathbb{R}^d)$ *and* $\pi^* : H^\bullet(\mathbb{R}^d) \to H^\bullet(\mathbb{R}^d \times \mathbb{R})$ *are inverse to each other.*

Remark 20.2 (*A Crash Course in Homological Algebra - part III*)**.** In order to prove this proposition we will use the notion of homotopy operator. Given two cochain complexes (A, d) and (B, d) and cochain maps $f, g : A \to B$ a **homotopy operator** is a linear map $h : A \to B$ of degree -1, such that

$$f - g = \pm(\mathrm{d}h \pm h\mathrm{d})$$

(the choice of signs is irrelevant). In this case, we say that f and g are **homotopic cochain maps** and we express it by the diagram:

$$\begin{tikzcd} \cdots \arrow[r] & A^{k-1} \arrow[r, "\mathrm{d}"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dl] & A^k \arrow[r, "\mathrm{d}"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dl, "h"'] & A^{k+1} \arrow[r] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dl, "h"'] & \cdots \arrow[dl] \\ \cdots \arrow[r] & B^{k-1} \arrow[r, "\mathrm{d}"'] & B^k \arrow[r, "\mathrm{d}"'] & B^{k+1} \arrow[r] & \cdots \end{tikzcd}$$

Since $\pm(\mathrm{d}h \pm h\mathrm{d})$ maps closed forms to exact forms, it is induces the zero map in cohomology. Hence. if f and g are homotopic chain maps, they induce the same map in cohomology:

$$f_* = g_* : H^\bullet(A) \to H^\bullet(B).$$

Proof of Proposition 20.1. Note that $\pi \circ i =$id, hence $i^* \circ \pi^* =$id. To complete the proof we need to check that $\pi^* \circ i^* =$id. For this we construct a homotopy operator $h : \Omega^\bullet(\mathbb{R}^d \times \mathbb{R}) \to \Omega^{\bullet-1}(\mathbb{R}^d \times \mathbb{R})$ such that:

$$\mathrm{id} - \pi^* \circ i^* = \mathrm{d}h + h\mathrm{d}.$$

To construct h, note that a differential form in $\mathbb{R}^d \times \mathbb{R}$ is a linear combination of differential forms of two kinds:

$$f(x,t)\pi^*\omega,$$
$$f(x,t)\mathrm{d}t \wedge \pi^*\omega,$$

where ω is a differential form in $\mathbb{R}^d$. So we define the homotopy operator in each of these kinds of forms by:

$$h : \begin{cases} f(x,t)\pi^*\omega \longmapsto 0, \\ f(x,t)\mathrm{d}t \wedge \pi^*\omega \longmapsto \int_0^t f(x,s)\mathrm{d}s\ \pi^*\omega, \end{cases}$$

and then we extend it by linearity to all forms. We now check that h is indeed a homotopy operator:

Let $\theta = f(x,t)\pi^*\omega \in \Omega^k(\mathbb{R}^d \times \mathbb{R})$ be a form of the first kind. Then:

$$(\mathrm{id} - \pi^* \circ i^*)\theta = \theta - \pi^*(f(x,0)\omega) = (f(x,t) - f(x,0))\pi^*\omega.$$

On the other hand,

$$\begin{aligned}
(\mathrm{d}h + h\mathrm{d})\theta &= h\mathrm{d}\theta \\
&= h\left((\sum_i \frac{\partial f}{\partial x^i}\mathrm{d}x^i + \frac{\partial f}{\partial t}\mathrm{d}t) \wedge \pi^*\omega + f\pi^*\mathrm{d}\omega \right) \\
&= h\left(\frac{\partial f}{\partial t}\mathrm{d}t \wedge \pi^*\omega \right) \\
&= \int_0^t \frac{\partial f}{\partial t}(x,s)\mathrm{d}s \ \pi^*\omega = (f(x,t) - f(x,0))\pi^*\omega.
\end{aligned}$$

Hence, for any form θ of the first kind:

$$(\mathrm{id} - \pi^* \circ i^*)\theta = (\mathrm{d}h + \mathrm{d}h)\theta.$$

Let now $\theta = f(x,t)\mathrm{d}t \wedge \pi^*\omega$ be a differential form of the second kind. On the one hand,

$$(\mathrm{id} - \pi^* \circ i^*)\theta = \theta.$$

On the other hand,

$$\begin{aligned}
(\mathrm{d}h + h\mathrm{d})\theta = \mathrm{d}(\int_0^t f(x,s)\mathrm{d}s \ \pi^*\omega) + h\left(\sum_i \frac{\partial f}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}t \wedge \pi^*\omega - f\mathrm{d}t \wedge \pi^*\mathrm{d}\omega \right) \\
= f(x,t)\mathrm{d}t \wedge \pi^*\omega + \sum_i \int_0^t \frac{\partial f}{\partial x^i}\mathrm{d}s \ \mathrm{d}x^i \wedge \pi^*\omega + \int_0^t f(x,s)\mathrm{d}s \ \mathrm{d}\pi^*\omega \\
- \sum_i \int_0^t \frac{\partial f}{\partial x^i}\mathrm{d}s \ \mathrm{d}x^i \wedge \pi^*\omega - \int_0^t f(x,s)\mathrm{d}s \ \pi^*\mathrm{d}\omega \\
= f(x,t)\mathrm{d}t \wedge \pi^*\omega = \theta.
\end{aligned}$$

Therefore, for any form θ of the second kind:

$$(\mathrm{id} - \pi^* \circ i^*)\theta = (\mathrm{d}h + \mathrm{d}h)\theta.$$

□

It is should be clear that $H^0(\mathbb{R}^0) = \mathbb{R}$, since a set with one point is connected. On the other hand, $H^k(\mathbb{R}^0) = 0$ if $k \neq 0$. By induction we conclude that the cohomology of euclidean space is:

Corollary 20.3 (Poincaré Lemma)**.**

$$H^k(\mathbb{R}^d) = H^k(\mathbb{R}^0) = \begin{cases} \mathbb{R} & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}$$

Note that the Poincaré Lemma states that in $\mathbb{R}^d$ every closed form is exact.

Homotopy Invariance. The argument used above to show that $H^\bullet(\mathbb{R}^d \times \mathbb{R}) \simeq H^\bullet(\mathbb{R}^d)$ can be extended easily from $\mathbb{R}^d$ to any smooth manifold M yielding:

Proposition 20.4. *If M is a smooth manifold, consider the maps $\pi : M \times \mathbb{R} \to M$ and $i : M \to M \times \mathbb{R}$:*

$$\begin{tikzcd} M \times \mathbb{R} \arrow[d, shift left, "\pi"] \\ M \arrow[u, shift left, "i"] \end{tikzcd} \qquad \begin{aligned} \pi(p,t) &= p, \\ i(p) &= (p,0). \end{aligned}$$

The induced maps $i^* : H^\bullet(M \times \mathbb{R}) \to H^\bullet(M)$ and $\pi^* : H^\bullet(M) \to H^\bullet(M \times \mathbb{R})$ are inverse to each other.

We leave the proof for the homework at the end of the lecture.

Actually, this proposition is a very special case of a general property of cohomology: if a manifold can be continuously deformed into another manifold then their cohomologies are isomorphic. In order to turn this into a precise statement, we make the following definition.

Definition 20.5. *Let $\Phi, \Psi : M \to N$ be smooth maps. A* ***smooth homotopy*** *between Φ and Ψ is a smooth map $H : M \times \mathbb{R} \to N$ such that:*[6]

$$H(p,t) = \begin{cases} \Phi(p) & \text{if } t \leq 0, \\ \Psi(p) & \text{if } t \geq 1. \end{cases}$$

The general property we alluded too is the following:

Theorem 20.6 (Homotopy Invariance)**.** *If $\Phi, \Psi : M \to N$ are homotopic maps then $\Phi^* = \Psi^* : H^\bullet(N) \to H^\bullet(M)$.*

Proof. Denote by $\pi : M \times \mathbb{R} \to M$ the projection and $i_0, i_1 : M \to M \times \mathbb{R}$ the sections:

$$i_0(p) = (p,0) \text{ and } i_1(p) = (p,1).$$

By Proposition 20.4, i_0^* and i_1^* are linear maps which both invert π^*, so they must coincide: $i_0^* = i_1^*$.

Now let $H : M \times \mathbb{R} \to N$ be a homotopy between Φ and Ψ. Then $\Phi = H \circ i_0$ and $\Psi = H \circ i_1$. At the level of cohomology we find:

$$\Phi^* = (H \circ i_0)^* = i_0^* \, H^*,$$
$$\Psi^* = (H \circ i_1)^* = i_1^* \, H^*.$$

Since $i_0^* = i_1^*$, we conclude that $\Phi^* = \Psi^*$. □

[6]One can show that two smooth maps are C^∞-homotopic iff they are C^0-homotopic and also that any continuous map between two smooth manifolds is C^0-homotopic to a smooth map. For this reason, one often defines the homotopy in the interval $[0,1]$.

We say that two manifolds M and N have the **same homotopy type** if there exist smooth map $\Phi : M \to N$ and $\Psi : N \to M$ such that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are homotopic to id_M and id_N, respectively. A manifold is said to be **contractible** if is has the same homotopy type as $\mathbb{R}^0$.

Corollary 20.7. *If M and N have the same homotopy type then $H^\bullet(M) \simeq H^\bullet(N)$. In particular, if M is a contractible manifold then:*

$$H^k(M) = \begin{cases} \mathbb{R} & \text{if } k = 0, \\ \\ 0 & \text{if } k \neq 0. \end{cases}$$

Examples 20.8.

1. *An open set $U \subset \mathbb{R}^d$ is called **star shaped** if there exists some $x_0 \in U$ such that for any $x \in U$, the segment $tx + (1-t)x_0$ lies in U. We leave it as exercise to show that a star shaped open set U is contractible, so that*

$$H^k(U) = \begin{cases} \mathbb{R} & \text{if } k = 0, \\ \\ 0 & \text{if } k \neq 0. \end{cases}$$

2. *The manifold $M = \mathbb{R}^{d+1} - 0$ has the same homotopy type as $\mathbb{S}^d$: the inclusion $i : \mathbb{S}^d \hookrightarrow \mathbb{R}^{d+1} - 0$ and the projection $\pi : \mathbb{R}^{d+1} - 0 \to \mathbb{S}^d$, $x \mapsto x/||x||$, are homotopic inverses to each other. Hence:*

$$H^\bullet(\mathbb{S}^d) = H^\bullet(\mathbb{R}^{d+1} - 0).$$

Notice that we don't know yet how to compute $H^\bullet(\mathbb{R}^{d+1} - 0)$!

Mayer-Vietoris Sequence. Let us discuss now another important property of cohomology, which allows to compute the cohomology of a manifold M from a decomposition of M into more elementary pieces of which we already know the cohomology.

Theorem 20.9 (Mayer-Vietoris Sequence)**.** *Let M be a smooth manifold and let $U, V \subset M$ be open subsets such that $M = U \cup V$. There exists a long exact sequence:*

$$\longrightarrow H^k(M) \longrightarrow H^k(U) \oplus H^k(V) \longrightarrow H^k(U \cap V) \xrightarrow{\delta^*} H^{k+1}(M) \longrightarrow$$

Remark 20.10 (*A Crash Course in Homological Algebra - part IV*)**.** A sequence of vector spaces and linear maps

$$\cdots \longrightarrow C^{k-1} \xrightarrow{f_{k-1}} C^k \xrightarrow{f_k} C^{k+1} \longrightarrow \cdots$$

is called **exact** if $\operatorname{Im} f_{k-1} = \operatorname{Ker} f_k$. An exact sequence of the form:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called **a short exact sequence**. This means that:
(a) f is injective,

(b) $\operatorname{Im} f = \operatorname{Ker} g$, and
(c) g is surjective.

A basic property of exact sequences is the following: given any exact sequence ending in trivial vector spaces

$$\begin{tikzcd} 0 \arrow[r] & C^0 \arrow[r] & \cdots \arrow[r] & C^k \arrow[r] & \cdots \arrow[r] & C^d \arrow[r] & 0 \end{tikzcd}$$

the alternating sum of the dimensions is zero:

$$\sum_{i=0}^{d} (-1)^i \dim C^i = 0.$$

We leave the (easy) proof for the exercises.

Note that a short exact sequences of complexes:

$$\begin{tikzcd} 0 \arrow[r] & (A^\bullet, \mathrm{d}) \arrow[r, "f"] & (B^\bullet, \mathrm{d}) \arrow[r, "g"] & (C^\bullet, \mathrm{d}) \arrow[r] & 0 \end{tikzcd}$$

can be represented by a large commutative diagram where all rows are exact:

$$\begin{tikzcd}
 & {} & {} & {} & \\
0 \arrow[r] & A^{k+1} \arrow[u] \arrow[r, "f"] & B^{k+1} \arrow[u] \arrow[r, "g"] & C^{k+1} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & A^k \arrow[u, "\mathrm{d}"] \arrow[r, "f"] & B^k \arrow[u, "\mathrm{d}"] \arrow[r, "g"] & C^k \arrow[u, "\mathrm{d}"] \arrow[r] & 0 \\
0 \arrow[r] & A^{k-1} \arrow[u, "\mathrm{d}"] \arrow[r, "f"] & B^{k-1} \arrow[u, "\mathrm{d}"] \arrow[r, "g"] & C^{k-1} \arrow[u, "\mathrm{d}"] \arrow[r] & 0 \\
 & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}$$

We have the following basic fact: given a short exact sequence of complexes as above there exists an *associated long exact sequence* in cohomology

$$\begin{tikzcd} \cdots \arrow[r] & H^k(A) \arrow[r, "f"] & H^k(B) \arrow[r, "g"] & H^k(C) \arrow[r, "\delta^*"] & H^{k+1}(A) \arrow[r] & \cdots \end{tikzcd}$$

where $\delta^* : H^k(C) \to H^{k+1}(A)$ is called the *connecting homomorphism*. The fact that $\operatorname{Im} f = \operatorname{Ker} g$ follows immediately from the definition of short exact sequence. On the other hand, the identities $\operatorname{Im} g = \operatorname{Ker} \delta^*$ and $\operatorname{Im} \delta^* = \operatorname{Ker} f$ follow from the way δ^* is constructed, and which we now describe.

For the construction of d^* one should keep in mind the large commutative diagram above. Given a cocycle $c \in C^k$ so that $\mathrm{d}c = 0$, it follows from the fact that the rows are exact that there exists $b \in B^k$ such that $g(b) = c$. Since the diagram commutes, we have

$$g(\mathrm{d}b) = \mathrm{d}g(b) = \mathrm{d}c = 0.$$

Using again that the rows are exact, we conclude that there exists a unique $a \in A^{k+1}$ such that $f(a) = \mathrm{d}b$. Note that:

$$f(\mathrm{d}a) = \mathrm{d}f(a) = \mathrm{d}^2 b = 0,$$

and since f is injective, we have $\mathrm{d}a = 0$, i.e., a is cocycle. In this way, we have associated to a cocycle $c \in C^k$ a cocycle $a \in A^{k+1}$.

This association depends on a choice of an intermediate element $b \in C^k$. If we choose a different $b' \in C^k$ such $g(b') = c$, we obtain a different element $a' \in A^{k+1}$. However, note that

$$g(b - b') = g(b') - g(b) = c - c = 0,$$

so there exist $\bar{a} \in A^k$ such that $f(\bar{a}) = b - b'$. Hence, we find

$$f(a - a') = f(a) - f(a') = \mathrm{d}b - \mathrm{d}b' = \mathrm{d}f(\bar{a}) = f(\mathrm{d}\bar{a}).$$

Since f is injective, we conclude that $a - a' = \mathrm{d}\bar{a}$. This shows that different intermediate choices lead to elements in the same cohomology class.

Finally, note that this assignment associates a coboundary to a coboundary. In fact, if $c \in C^k$ is a coboundary, i.e., $c = \mathrm{d}c'$, then there exists $b' \in C^{k-1}$ such that $g(b') = c'$. Moreover,

$$g(b - \mathrm{d}b') = g(b) - \mathrm{d}g(b') = c - \mathrm{d}c' = 0.$$

Therefore, there exists $a' \in A^k$ such that $f(a') = b - \mathrm{d}b'$, and:

$$f(a - \mathrm{d}a') = f(a) - \mathrm{d}f(a') = \mathrm{d}b - \mathrm{d}b + \mathrm{d}^2 b' = 0.$$

Since f is injective, we conclude that $a = \mathrm{d}a'$ is a coboundary, as claimed.

This discussion shows that we have a well-defined linear map

$$\delta^* : H^k(C) \to H^{k+1}(A), [c] \mapsto [a].$$

We leave it as an exercise to check that this definition leads to $\operatorname{Im} g = \operatorname{Ker} \delta^*$ and $\operatorname{Im} \delta^* = \ker f$.

Proof of Theorem 20.9. We claim that we have a short exact sequence:

$$0 \longrightarrow \Omega^\bullet(M) \longrightarrow \Omega^\bullet(U) \oplus \Omega^\bullet(V) \longrightarrow \Omega^\bullet(U \cap V) \longrightarrow 0$$

where the first map is given by:

$$\omega \mapsto (\omega|_U, \omega|_V),$$

while the second map is defined by:

$$(\theta, \eta) \mapsto \theta|_{U\cap V} - \eta|_{U\cap V}.$$

Since $M = U \cup V$, the first map is injective. Also, it is clear that the image of the first map is contained in the kernel of the second map. On the other hand, if $(\theta, \eta) \in \Omega^\bullet(U) \oplus \Omega^\bullet(V)$ belongs to the kernel of the second map, then

$$\theta|_{U\cap V} = \eta|_{U\cap V}.$$

Hence, we can define a smooth differential form in M by:

$$\omega_p = \begin{cases} \theta_p & \text{if } p \in U, \\ \\ \eta_p & \text{if } p \in V. \end{cases}$$

Therefore the image of the first map coincides with the kernel of the second map. Finally, let $\alpha \in \Omega^\bullet(U \cap V)$ and choose a partition of unit $\{\rho_U, \rho_V\}$ subordinated to the cover $\{U, V\}$. Then $\rho_V \alpha \in \Omega^\bullet(U)$ and $\rho_U \alpha \in \Omega^\bullet(V)$ and this pair of forms is transformed by the second map to

$$(\rho_V \alpha, -\rho_U \alpha) \mapsto \rho_V \alpha + \rho_U \alpha = \alpha.$$

Therefore, the second map is surjective and we have a short exact sequence as claimed. The corresponding long exact sequence in cohomology yields the statement of the theorem. □

Example 20.11.
Let us use the Mayer-Vietoris sequence to compute the cohomology of $\mathbb{S}^d$ *for* $d \geq 2$ *(we already know the cohomology* $H^\bullet(\mathbb{S}^1)$*; see in Example 18.5).*

Let $p_N \in \mathbb{S}^d$ *be the north pole and let* $U = \mathbb{S}^d - p_N$*. The stereographic projection* $\pi_N : U \to \mathbb{R}^{d-1}$ *is a diffeomorphism, so* U *is contractible. Similarly if* $p_S \in \mathbb{S}^d$ *is the south pole, the open set* $V = \mathbb{S}^d - p_S$ *is contractible. On the other hand, we have that* $M = U \cap V$ *and the intersection* $U \cap V$ *is diffeomorphic to* $\mathbb{R}^{d-1} - 0$ *(via any of the stereographic projections). We saw in Example 20.8 that* $\mathbb{R}^{d-1} - 0$ *as the same homotopy type as* $\mathbb{S}^{d-1}$*.*

We have all the ingredients to compute the Mayer-Vietoris sequence:

- *if* $k \geq 1$*, the sequence gives:*

$$\cdots \longrightarrow 0 \oplus 0 \longrightarrow H^k(\mathbb{S}^{d-1}) \xrightarrow{\mathrm{d}^*} H^{k+1}(\mathbb{S}^d) \longrightarrow 0 \oplus 0 \longrightarrow \cdots$$

  Hence, $H^{k+1}(\mathbb{S}^d) \simeq H^k(\mathbb{S}^{d-1})$*. By induction, we conclude that:*

$$H^k(\mathbb{S}^d) \simeq H^{k-1}(\mathbb{S}^{d-1}) \simeq \cdots \simeq H^1(\mathbb{S}^{d-k+1}).$$

- *On the other hand, since* U*,* V *and* $U \cap V$ *are connected, the first terms of the sequence are*

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \xrightarrow{\delta^*} H^1(\mathbb{S}^d) \longrightarrow 0 \longrightarrow \cdots$$

  It follows that $\dim H^1(\mathbb{S}^d) = 0$ *if* $d \geq 2$*, since the alternating sum of the dimensions must be zero.*

Since $H^1(\mathbb{S}^1) = \mathbb{R}$*, we conclude that:*

$$H^k(\mathbb{S}^d) = \begin{cases} \mathbb{R} & \text{if } k = 0, d, \\ \\ 0 & \text{otherwise.} \end{cases}$$

Compactly supported cohomology. As we saw in the previous lecture, compactly supported cohomology does not behave functorialy under smooth maps. Still this cohomology behaves functorialy under proper maps and, because of this, compactly supported cohomology still satisfies properties analogous, but distinct, to the properties we have studied for de Rham cohomology.

Proposition 20.12. *Let M be a smooth manifold. Then:*

$$H_c^\bullet(M\times\mathbb{R})\simeq H_c^{\bullet-1}(M).$$

Proof. It is enough to consider the case of $M=\mathbb{R}^d$. Note that if $\pi:\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}^d$ is the projection then $\pi^*\omega$ does not have compact support, if $\omega\neq 0$. Instead, one has "push-forward" maps

$$\begin{aligned}\pi_*&:\Omega_c^{\bullet+1}(\mathbb{R}^d\times\mathbb{R})\to\Omega_c^\bullet(\mathbb{R}^d),\\ e_*&:\Omega_c^\bullet(\mathbb{R}^d)\to\Omega_c^{\bullet+1}(\mathbb{R}^d\times\mathbb{R}).\end{aligned}$$

which are cochains maps homotopic inverse to each other.

We start by constructing π_*. Note that every compactly supported form in $\mathbb{R}^d\times\mathbb{R}$ is a linear combination of forms of two kinds:

$$\begin{aligned}&f(x,t)(\pi^*\omega),\\ &f(x,t)\mathrm{d}t\wedge\pi^*\omega,\end{aligned}$$

where ω is a differential form in $\mathbb{R}^d$ with compact support and f is a compactly supported smooth function. The map π_* is given by:

$$\begin{aligned}f(x,t)(\pi^*\omega)&\longmapsto 0,\\ f(x,t)\mathrm{d}t\wedge\pi^*\omega&\longmapsto\int_{-\infty}^{+\infty}f(x,t)\mathrm{d}t\;\omega.\end{aligned}$$

and it is known as *integration along the fibers.*

On the other hand, in order to construct e_* one chooses some 1-form $\theta=g(t)\mathrm{d}t\in\Omega_c^1(\mathbb{R})$ with $\int_{\mathbb{R}}\theta=1$ and sets:

$$e_*:\ \omega\to\pi^*\omega\wedge\theta.$$

It follows from these definitions that:

$$\pi_*\circ e_*=\mathrm{id},\quad \mathrm{d}\pi_*=\pi_*\mathrm{d},\quad e_*\mathrm{d}=\mathrm{d}e_*.$$

To finish the proof, it is enough to check that $e_*\circ\pi_*$ is homotopic to the the identity. We leave it as an exercise to check that the map $h:\Omega_c^\bullet(\mathbb{R}^d\times\mathbb{R})\to\Omega_c^{\bullet-1}(\mathbb{R}^d\times\mathbb{R})$ defined by:

$$\begin{aligned}f(x,t)(\pi^*\omega)&\longmapsto 0,\\ f(x,t)\mathrm{d}t\wedge\pi^*\omega&\longmapsto\left(\int_{-\infty}^{t}f(x,s)\mathrm{d}s-\int_{-\infty}^{+\infty}f(x,s)\mathrm{d}s\int_{-\infty}^{t}g(s)\mathrm{d}s\right)\pi^*\omega,\end{aligned}$$

is indeed a homotopy from $e_*\circ\pi_*$ to the identity. □

The proposition shows that compactly supported cohomology *is not* homotopy invariant. On the other hand, the proposition shows that the Poincaré Lemma must be modified as follows:

Corollary 20.13 (Poincaré Lemma for compactly supported cohomology)**.**

$$H_c^k(\mathbb{R}^d) = \begin{cases} \mathbb{R} & \text{if } k = d, \\ \\ 0 & \text{if } k \neq d. \end{cases}$$

Next we construct the Mayer-Vietoris sequence for compactly supported cohomology. Notice that if $U, V \subset M$ are open sets with $U \cup V = M$, the inclusions $U, V \hookrightarrow M$, $U \cap V \hookrightarrow U$ and $U \cap V \hookrightarrow V$ give a short exact sequence

$$0 \longleftarrow \Omega_c^\bullet(M) \longleftarrow \Omega_c^\bullet(U) \oplus \Omega_c^\bullet(V) \longleftarrow \Omega_c^\bullet(U \cap V) \longleftarrow 0$$

where the first map is:

$$(\theta, \eta) \mapsto \theta + \eta,$$

while the second map is:

$$\omega \mapsto (-\omega, \omega).$$

Hence, it follows that

Theorem 20.14 (Mayer-Vietoris sequence for compactly supported cohomology)**.** *Let M be a smooth manifold and $U, V \hookrightarrow M$ open subsets such that $M = U \cup V$. There exists a long exact sequence*

$$\longleftarrow H_c^k(M) \longleftarrow H_c^k(U) \oplus H_c^k(V) \longleftarrow H_c^k(U \cap V) \xleftarrow{\delta_*} H_c^{k-1}(M) \longleftarrow$$

We leave the details of the argument for the exercises. For now you should observe that in the Mayer-Vietoris sequence for compact supported cohomology the inclusions $U, V \hookrightarrow M$, $U \cap V \hookrightarrow U$ and $U \cap V \hookrightarrow V$ induce maps in the same direction, while for the ordinary de Rham cohomology the inclusions are reversed in the sequence. In the next lecture we will relate these two cohomology theories, and this will explain all the differences of behavior that we have just discussed.

Homework.

1. Give a proof of Proposition 20.4.

2. Show that a star shaped open set is contractible.

3. Let $i : N \hookrightarrow M$ be a submanifold. We say that a map $r : M \to N$ is a **retraction of** M **in** N if $r \circ i = \mathrm{id}_N$ and that N is a **deformation retract** of M if there exists a retraction $r : M \to N$ such that $i \circ r$ is homotopic to id_M. Show that:
   (a) If N is a deformation retract of M, then $H^\bullet(N) \simeq H^\bullet(M)$.
   (b) Show that $\mathbb{S}^2$ is a deformation retract of $\mathbb{R}^3 - 0$.
   (c) Show that $\mathbb{T}^2$, viewed as a submanifold of $\mathbb{R}^3$ as in Example 6.8.2, is a deformation retract of $\mathbb{R}^3 - \{L \cup S\}$ where L is the z-axis and S is the circle in the xy-plane of radius R and center the origin.

4. In Remark 20.10, show that the connecting homomorphism in the long exact sequence satisfies $\operatorname{Im} g = \operatorname{Ker} \mathrm{d}^*$ and $\operatorname{Im} \mathrm{d}^* = \ker f$.

5. Given a long exact sequence of vector spaces

$$0 \longrightarrow C^0 \longrightarrow \cdots \longrightarrow C^{k-1} \longrightarrow C^k \longrightarrow \cdots \longrightarrow C^d \longrightarrow 0$$

show that:

$$\sum_{i=0}^{d} (-1)^i \dim C^i = 0.$$

6. Show that a generator of $H^d(\mathbb{S}^d)$ is given by the restriction to $\mathbb{S}^d$ of the form $\omega \in \Omega^d(\mathbb{R}^{d+1})$ defined by:

$$\omega = \sum_{i=1}^{d+1} (-1)^i x^i \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^{d+1}.$$

7. Compute the cohomology of $\mathbb{T}^2$ and $\mathbb{P}^2$.

8. Complete the construction of the Mayer-Vietoris sequence for compactly supported cohomology, by showing that:

$$0 \longleftarrow \Omega_c^\bullet(M) \longleftarrow \Omega_c^\bullet(U) \oplus \Omega_c^\bullet(V) \longleftarrow \Omega_c^\bullet(U \cap V) \longleftarrow 0$$

is a short exact sequence of complexes.

9. Compute the compactly supported cohomology of $\mathbb{R}^d - 0$.

Lecture 21. Computations in Cohomology

In the previous lecture we constructed the Mayer-Vietoris sequence relating the cohomology of the union of open sets with the cohomology of its factors. This sequence leads to a very useful technique to compute cohomology by induction, which also allows to extract many properties of cohomology. In order to apply it, we need to cover M by open sets whose intersections have trivial cohomology.

Definition 21.1. *An open cover $\{U_\alpha\}$ of a smooth manifold M is called a* ***good cover*** *if all finite intersections $U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}$ are diffeomorphic to $\mathbb{R}^d$. We say that M is a* ***manifold of finite type*** *if it admits a finite good cover.*

Proposition 21.2. *Every smooth manifold M admits a good cover. If M is compact then it admits a finite good cover.*

Proof. [7] Let g be a Riemannian metric for M. In Riemannian geometry one shows that each point $p \in M$ has a geodesically convex neighborhood U_p

[7] This proof requires some knowledge of Riemannian geometry. If you are not familiar with the notion of geodesics, you may wish to skip the proof and admit the result as valid.

(i.e., for any $q, q' \in U_p$ there exists a unique geodesic in U_p which connects q and q'), such that:

(i) each U_p is diffeomorphic to $\mathbb{R}^d$, and
(ii) the intersection of geodesically convex neighborhoods is geodesically convex.

It follows that $\{U_p\}_{p\in M}$ is a good cover of M.

If M is compact, then a finite number of geodesically convex neighborhoods cover M. $\square$

Finite dimensional cohomology. We can use good covers and the Mayer-Vietoris sequence to show that the cohomology is often finite dimensional:

Theorem 21.3. *If M is a manifold of finite type then the cohomology spaces $H^k(M)$ and $H^k_c(M)$ have finite dimension.*

Proof. For any two open sets U e V, the Mayer-Vietoris sequence:

$$\cdots \longrightarrow H^{k-1}(U\cap V) \xrightarrow{\delta^*} H^k(U\cup V) \xrightarrow{r} H^k(U)\oplus H^k(V) \longrightarrow \cdots$$

shows that:

$$H^k(U\cup V) \simeq \operatorname{Im}\delta^* \oplus \operatorname{Im} r.$$

Hence, if the cohomologies of U, V and $U\cap V$ are finite dimensional, then so is the cohomology of $U\cup V$.

Now we can use induction on the number of open sets in a cover, to show that manifolds which admit a finite good cover have finite dimensional cohomology:

- If M is diffeomorphic to $\mathbb{R}^d$ the Poincaré Lemma shows that M has finite dimensional cohomology.
- Now assume that all manifolds admitting a good cover with at most p open sets have finite dimensional cohomology. Let M be manifold which admits a good cover with $p+1$ open sets $\{U_1,\dots,U_{p+1}\}$. We observe that the open sets:

  $$U_{p+1},$$
  $$U_1\cup\cdots\cup U_p, \text{ and}$$
  $$(U_1\cup\cdots\cup U_p)\cap U_{p+1} = (U_1\cap U_{p+1})\cup\cdots\cup(U_p\cap U_{p+1}),$$

  all have finite dimensional cohomology, since they all admit a good cover with at most p open sets. Hence, the cohomology of $M = U_1\cup\cdots\cup U_{p+1}$ is also finite dimensional.

The proof for compactly supported cohomology is similar. $\square$

Triangulations and Euler's formula. As another application of the Mayer-Vietoris sequence, we show how the familiar Euler's formula for regular polygons can be extended to any compact manifold M admitting a *triangulation*, i.e., a nice decomposition of M into regular simplices as we now explain([8]).

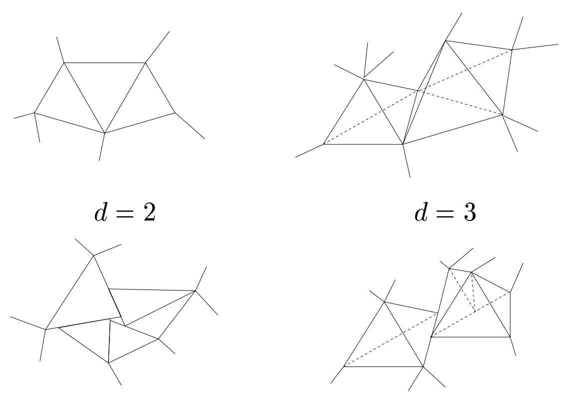
A **regular simplex** is a simplex $\sigma : \Delta^d \to M$ which can be extended to a diffeomorphism $\tilde{\sigma} : U \to \tilde{\sigma}(U) \subset M$, where U is some open neighborhood of Δ^d. We have defined before the $(d-1)$-dimensional faces of a simplex $\sigma : \Delta^d \to M$. For a regular simplex, these are regular $(d-1)$-simplices $\varepsilon_i(sigma) : \Delta^{d-1} \to M$ of dimension $(d-1)$. By iterating this construction we obtain the $d-k$**-dimensional faces** of a simplex, which are regular $(d-k)$-simplices $\varepsilon_{i_1,i_2,\dots,i_{d-k}}(\sigma) : \Delta^{d-k} \to M$.

Definition 21.4. *A **triangulation** of a compact manifold M of dimension d is a finite collection $\{\sigma_i\}$ of regular d-simplices such that:*

(i) the collection $\{\sigma_i\}$ covers M, and

(ii) if two simplices in $\{\sigma_i\}$ have non-empty intersection, then there intersection $\sigma_i \cap \sigma_j$ is a face of both simplices σ_i and σ_j.

The next figure illustrates condition (ii) in this definition for dimensions 2 and 3. Notice that on the top the condition is satisfied while on the bottom



the condition fails.

If M is a manifold with finite dimensional cohomology (e.g., if M is compact) one defines the **Euler characteristic** of M to be the integer $\chi(M)$ given by:

$$\chi(M) = \dim H^0(M) - \dim H^1(M) + \cdots + (-1)^d \dim H^d(M).$$

The generalization we alluded to before is the following:

Theorem 21.5 (Euler's Formula)**.** *If M is a compact manifold of dimension d, for any triangulation we have:*

$$(-1)^d \chi(M) = r_0 - r_1 + \cdots + (-1)^d r_d,$$

where r_i denotes the number of faces of dimension i of the triangulation.

[8]Actually, one can show that every compact manifold can be triangulated. This result is very technical and we will not discuss it in these notes.

Proof. Let us fix a triangulation $\{\sigma_1, \sigma_2, \ldots, \sigma_{r_d}\}$ of M and define open sets:

$$V_k := M - \{k\text{-faces of the triangulation}\}.$$

We claim that for $0 \le k \le d-1$ we have:

$$\chi(M) = \chi(V_k) + (-1)^d (r_0 - r_1 + \cdots + (-1)^k r_k). \tag{21.1}$$

Assuming this claim, since

$$V_{d-1} = \bigcup_{j=1}^{r_d} \operatorname{int}(\sigma_j),$$

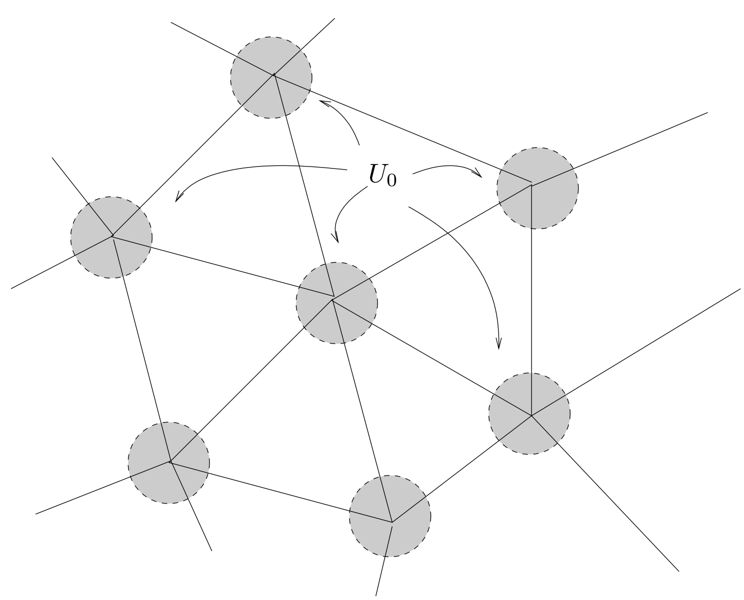
and each open set $\operatorname{int}(\sigma_j)$ is contractible, we have $H^k(V_{d-1}) = 0$, for $k > 0$. Hence:

$$\chi(V_{d-1}) = \dim H^0(V_{d-1}) = r_d.$$

Relation (21.1) for $k = d-1$ and this last identity, together show that Euler's formula holds.

Lets us start by verifying (21.1) for $k = 0$. For each 0-dimensional face we can choose disjoint open neighborhoods $U_{0,1}, \ldots, U_{0,r_0}$, each diffeomorphic to the open ball $B_1^d = \{x \in \mathbb{R}^d : ||x|| < 1\}$, and we let

$$U_0 = \bigcup_{i=0}^{r_0} U_{0,i}.$$



Notice that $V_0 \cup U_0 = M$. Since each $U_{0,i}$ is contractible, we have:

$$\dim H^k(U_0) = \begin{cases} r_0, & \text{if } k = 0, \\ 0, & \text{if } k \neq 0. \end{cases}$$

On the other hand, the intersection $V_0 \cap U_{0,i}$ deformation retracts in $\mathbb{S}^{d-1}$, hence

$$\dim H^k(V_0 \cap U_0) = \begin{cases} r_0, \text{ if } k = 0, d-1, \\ 0, \text{ if } k \neq 0, d-1. \end{cases}$$

We can apply the Mayer-Vietoris argument to the pair (U_0, V_0): if $d > 2$, this sequence gives the following information:

(i) The lowest degree terms in the sequence are:

$$0 \longrightarrow H^0(M) \longrightarrow H^0(U_0) \oplus H^0(V_0) \longrightarrow H^0(U_0 \cap V_0) \longrightarrow$$
$$\longrightarrow H^1(M) \longrightarrow 0 \oplus H^1(V_0) \longrightarrow 0$$

so it follows that:

$$\dim H^0(M) - \dim H^0(U_0) - \dim H^0(V_0) +$$
$$+ \dim H^0(U_0 \cap V_0) - \dim H^1(M) + \dim H^1(V_0) = 0.$$

Since M and V_0 have the same number of connected components we find

$$\dim H^0(M) = \dim H^0(V_0).$$

On the other hand, the number of connected components of U_0 and $V_0 \cap U_0$ are also the same, hence we conclude that:

$$\dim H^1(M) = \dim H^1(V_0).$$

(ii) For $1 < k < d-1$, the Mayer-Vietoris sequence gives:

$$0 \longrightarrow H^k(M) \longrightarrow 0 \oplus H^k(V_0) \longrightarrow 0$$

Hence:

$$\dim H^k(M) = \dim H^k(V_0).$$

(iii) Finally, the last terms in the sequence give:

$$0 \longrightarrow H^{d-1}(M) \longrightarrow 0 \oplus H^{d-1}(V_0) \longrightarrow H^{d-1}(U_0 \cap V_0) \longrightarrow$$
$$\longrightarrow H^d(M) \longrightarrow 0 \oplus H^d(V_0) \longrightarrow 0$$

Since $\dim H^{d-1}(U_0 \cap V_0) = r_d$, we conclude that:

$$\dim H^{d-1}(M) - \dim H^{d-1}(V_0) + \dim H^{d-1}(V_0) - \dim H^d(M) = -r_d.$$

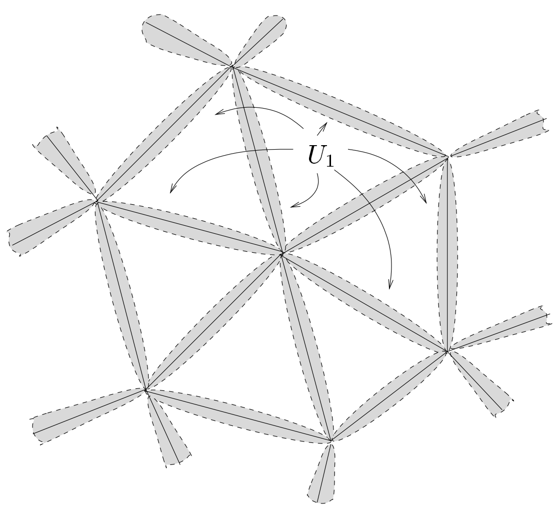
When $d = 2$, we obtain exactly the same results except that we can consider the whole sequence at once. In an any case, we conclude that:

$$\begin{aligned}\chi(M) &= \sum_{i=0}^{d}(-1)^i \dim H^i(M)\\ &= \sum_{i=0}^{d}(-1)^i \dim H^i(V_0) + (-1)^d r_d = \chi(V_0) + (-1)^d r_d.\end{aligned}$$

which yields (21.1) if $k = 0$.

In order to prove (21.1) when $k = 1$, we can proceed as follows: for each 1-face we choose open disjoint neighborhoods $U_{1,1}, \ldots, U_{1,r_1}$ of the (1-faces)-(0-faces), diffeomorphic to $(\mathrm{int}\ \Delta^1) \times B_1^{d-1}$, and we define the open set:

$$U_1 = \bigcup_{i=0}^{r_1} U_{1,i}.$$



We have that $V_0 = U_1 \cup V_1$. Moreover, U_1 is a disjoint union of r_1 contractible open sets, while $U_1 \cap V_1$ as the same homotopy type as the disjoint union of $(d-2)$-spheres This allows one to show, exactly like in the case $k = 0$, that the Mayer-Vietoris sequence yields:

$$\chi(V_0) = \chi(V_1) + (-1)^{d-1} r_1.$$

In general, for each k, we choose open disjoint neighborhoods $U_{k,1}, \ldots, U_{k,r_k}$ of $\{k\text{-faces}\} - \{(k-1)\text{-faces}\}$, diffeomorphic to $(\mathrm{int}\ \Delta^k) \times B_1^{d-k}$, and we define the open set:

$$U_k = \bigcup_{i=0}^{r_k} U_{k,i}.$$

We have that $V_k = U_k \cup V_k$, where U_k is a union of r_k contractible open sets, while $U_k \cap V_k$ as the same homotopy type as the disjoint union of $(d-k-1)$-spheres. The Mayer-Vietoris sequence then shows that:

$$\chi(V_{k-1}) = \chi(V_k) + (-1)^{d-k} r_k.$$

This proves (21.1) and finishes the proof of Euler's formula. □

Poincaré duality. Recall (see the exercises in Lecture 18) that the exterior product induces a ring structure in cohomology:

$$\cup : H^k(M) \times H^l(M) \to H^{k+l}(M),\ [\omega] \cup [\eta] \equiv [\omega \wedge \eta].$$

Obviously, if η has compact support then $\omega \wedge \eta$ also has compact support, hence we obtain also a "product":

$$\cup : H^k(M) \times H^l_c(M) \to H^{k+l}_c(M).$$

Stokes formula shows that the integral of differential forms descends to the level of cohomology. Hence, if M is an oriented manifold of dimension d we obtain a bilinear form

$$H^k(M) \times H^{d-k}_c(M) \to \mathbb{R},\ ([\omega],[\eta]) \mapsto \int_M \omega \wedge \eta. \tag{21.2}$$

Theorem 21.6 (Poincaré duality)**.** *If M is an oriented manifold of finite type the bilinear form (21.2) is non-degenerate. In particular:*

$$H^k(M) \simeq H^{d-k}_c(M)^*.$$

Remark 21.7 (*A Crash Course in Homological Algebra - part V*)**.** For the proof of Poincaré duality we turn once more to Homological Algebra.

Lemma 21.8 (Five Lemma)**.** *Consider a commutative diagram of homomorphisms of vector spaces:*

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "\alpha"'] & B \arrow[r, "f_2"] \arrow[d, "\beta"'] & C \arrow[r, "f_3"] \arrow[d, "\gamma"'] & D \arrow[r, "f_4"] \arrow[d, "\delta"'] & E \arrow[d, "\varepsilon"'] \\
A' \arrow[r, "f_1'"] & B' \arrow[r, "f_2'"] & C' \arrow[r, "f_3'"] & D' \arrow[r, "f_4'"] & E'
\end{tikzcd}$$

where the rows are exact. If α, β, δ and ε are isomorphisms, then γ is also an isomorphism.

The proof of this lemma is by diagram chasing and is left as an easy exercise.

Proof of Theorem 21.6. Let us start by observing that the bilinear form (21.2) gives always a linear map $H^k(M) \to H^{d-k}_c(M)^*$. If U and V are open sets, one checks easily that the Mayer-Vietoris sequence for $\Omega^\bullet$ and $\Omega_c^\bullet$, give a diagram of exact sequences:

$$\begin{tikzcd}
\arrow[r] & H^k(U\cup V) \arrow[r] \arrow[d] & H^k(U)\oplus H^k(V) \arrow[r] \arrow[d] & H^k(U\cap V) \arrow[r] \arrow[d] & {} \\
\arrow[r] & H_c^{d-k}(U\cup V)^* \arrow[r] & H_c^{d-k}(U)^*\oplus H_c^{d-k}(V)^* \arrow[r] & H_c^{d-k}(U\cap V)^* \arrow[r] & {}
\end{tikzcd}$$

which commutes up to signs: for example, we have

$$\int_{U\cap V}\omega\wedge\delta^*\theta=\pm\int_{U\cup V}\delta^*\omega\wedge\tau.$$

If we apply the Five Lemma to this diagram, we conclude that if Poincaré duality holds for U, V and $U\cap V$, then it also holds for $U\cup V$.

Now let M be a manifold with a finite good cover. We show that Poincaré duality holds using induction on the cardinality of the cover:

- If $M\simeq\mathbb{R}^d$, the Poincaré Lemmas give:

  $$H^k(\mathbb{R}^d)=\begin{cases}\mathbb{R} & \text{if } k=0,\\ 0 & \text{if } k\neq 0.\end{cases}\qquad H_c^k(\mathbb{R}^d)=\begin{cases}\mathbb{R} & \text{if } k=d,\\ 0 & \text{if } k\neq d.\end{cases}$$

  Therefore, the bilinear form is non-degenerate in this case.
- Now assume that Poincaré duality holds for any manifold admitting a good cover with at most p open sets. If M is a manifold which admits an open cover $\{U_1,\dots,U_{p+1}\}$ with $p+1$ open sets, we note that the open sets:

  $U_{p+1},\ U_1\cup\cdots\cup U_p,$ and
  $(U_1\cup\cdots\cup U_p)\cap U_{p+1}=(U_1\cap U_{p+1})\cup\cdots\cup(U_p\cap U_{p+1}),$

  all satisfy Poincaré duality, since they all admit a good cover with at most p open sets. It follows that $M=U_1\cup\cdots\cup U_{p+1}$ also satisfies Poincaré duality.

□

If M is a compact manifold, we have $H_c^\bullet(M)=H^\bullet(M)$. Hence:

Corollary 21.9. *Se M is a compact oriented manifold then:*

$$H^k(M)\simeq H^{d-k}(M).$$

In particular, if additionally $\dim M$ *is odd, then:*

$$\chi(M)=0.$$

Remark 21.10. One can show that Poincaré duality still holds for manifolds which do not admit a finite good cover. However, when the cohomology of M is not finite dimensional one must be careful in stating it. The correct statement is that for any oriented manifold M one has an isomorphism:

$$H^k(M)\simeq(H_c^{d-k}(M))^*.$$

In general, *one does not have* a dual isomorphism $H_c^{d-k}(M) \simeq H^k(M)^*$. The reason is that while the dual of direct product is a direct sum, the dual of an infinite direct sum is not a direct product. We discuss an example in the exercises.

Because of the previous remark, in the next corollary we omit the assumption that M has a finite good cover.

Corollary 21.11. *Let M be a connected manifold of dimension d. Then:*

$$H_c^d(M) \simeq \begin{cases} \mathbb{R} & \text{if } M \text{ is orientable,} \\ \\ 0 & \text{if } M \text{ is not orientable.} \end{cases}$$

In particular, if M is compact and connected of dimension d, then M is orientable if and only if $H^d(M) \simeq \mathbb{R}$.

Proof. By Poincaré duality, if M is a connected orientable manifold of dimension d, then $H_c^d(M) \simeq H^0(M)^* \simeq \mathbb{R}$. We leave the converse to the exercises. □

Homework.

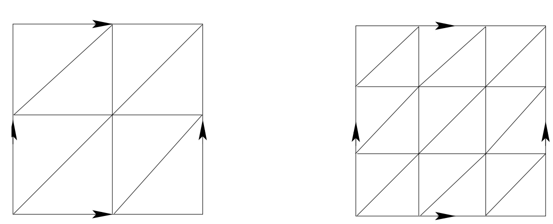
1. Given an example of a connected manifold which is not of finite type.

2. Prove the Five Lemma and find weaker conditions on the maps α, β, ε and δ, so that the conclusion still holds.

3. Check the commutativity, up to signs, of the following diagram of long exact sequences that appears in the proof of Poincaré duality:

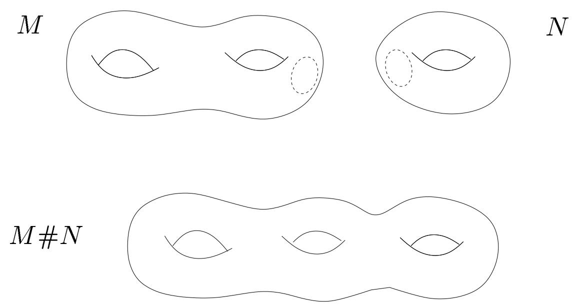
$$\begin{tikzcd}
{} \arrow[r] & H^k(U \cup V) \arrow[r] \arrow[d] & H^k(U) \oplus H^k(V) \arrow[r] \arrow[d] & H^k(U \cap V) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H_c^{d-k}(U \cup V)^* \arrow[r] & H_c^{d-k}(U)^* \oplus H_c^{d-k}(V)^* \arrow[r] & H_c^{d-k}(U \cap V)^* \arrow[r] & {}
\end{tikzcd}$$

4. Consider the following two subdivisions of the square $[0,1] \times [0,1]$:



(a) Verify that only one of these subdivisions induces a triangulation of the 2-torus $\mathbb{T}^2$;
(b) Compute r_0, r_1 and r_2 for this triangulation.

5. Let M and N be connected compact manifolds of dimension d. Let $M\#N$ be the **connected sum** of M and N, i.e., the manifold obtained by gluing M and N along the boundary of open sets $U \subset M$ and $V \subset N$ both diffeomorphic to the ball $\{x \in \mathbb{R}^d : ||x|| < 1\}$: Show that the Euler characteristics satisfy:



$$\chi(M\#N) = \chi(M) + \chi(N) - \chi(\mathbb{S}^d).$$

6. Let M be a connected manifold of dimension d, which is not orientable. Show that $H_c^d(M) = 0$ as follows. Let $\widetilde{M}$ denote the set of orientations for all the tangent spaces T_pM:

$$\widetilde{M} = \{(p, [\mu_p]) : [\mu_p] \text{ is an orientation for } T_pM\}.$$

One calls $\widetilde{M}$ the **orientation cover of** M. Show that:

(a) $\pi : \widetilde{M} \to M$, $(p, [\mu_p]) \mapsto p$, is double covering of M.

(b) For the unique smooth structure on $\widetilde{M}$ for which $\pi : \widetilde{M} \to M$ is a local diffeomorphism, show that $\widetilde{M}$ is a connected orientable manifold.

(c) Show that the map $\Phi : \widetilde{M} \to \widetilde{M}$, $(p, [\mu_p]) \mapsto (p, -[\mu_p])$ is a diffeomorphism that changes orientation and satisfies:

$$\pi = \pi \circ \Phi, \quad \Phi \circ \Phi = \text{id}.$$

(d) Show that $\widetilde{\omega} \in \Omega^k(\widetilde{M})$ is of the form $\widetilde{\omega} = \pi^*\omega$, for some $\omega \in \Omega^k(M)$, if and only if $\Phi^*\widetilde{\omega} = \widetilde{\omega}$.

(e) Conclude that one must have $H_c^d(M) = 0$.

7. Let $M_1, M_2, \dots,$ be manifolds of finite type of dimension d and consider the disjoint union of the M_i:

$$M = \bigcup_{i=1}^{+\infty} M_i.$$

Show that:

(a) The cohomology of M is the direct product:

$$H^k(M) = \prod_{i=1}^{+\infty} H^k(M_i);$$

(b) The cohomology of M with compact support is the direct sum:

$$H_c^k(M) = \bigoplus_{i=1}^{+\infty} H_c^k(M_i);$$

Conclude that there exists an isomorphism:

$$H^k(M) \simeq (H_c^{d-k}(M))^*,$$

but that $H_c^{d-k}(M)$ may not be isomorphic to $H^k(M)^*$.

8. Compute $H^k(M)$ and $H_c^k(M)$ for the following manifolds:
   (a) Möbius band;
   (b) Klein bottle;
   (c) The d-torus;
   (Answer: $\dim H^k(\mathbb{T}^d) = \binom{d}{k}$.)
   (d) Complex projective space;
   (Answer: $\dim H^{2k}(\mathbb{P}^d(\mathbb{C})) = 1$ if $2k \leq d$, and 0 otherwise.)

Lecture 22. The Degree and the Index

We saw in the previous lecture that a connected manifold M of dimension d is orientable if and only if $H_c^d(M) \simeq \mathbb{R}$. Notice that a choice of orientation for M determines a generator of $H_c^d(M)$. In fact, in this case, integration gives an isomorphism $H_c^d(M) \simeq \mathbb{R}$ by:

$$H_c^d(M) \to \mathbb{R}, \ [\omega] \mapsto \int_M \omega.$$

By the way, this isomorphism is just Poincaré duality, since M being connected $H^0(M)$ is the space of constant functions in M. In the sequel, we will often use the same symbol μ to denote the orientation and the generator $\mu \in H_c^d(M)$ that corresponds to the constant function 1.

Let $\Phi : M \to N$ be a proper map between connected, oriented manifolds of the same dimension: $\dim M = \dim N = d$. The canonical isomorphisms $H_c^d(M) \simeq \mathbb{R}$ and $H_c^d(N) \simeq \mathbb{R}$ give a representation of the induced map in cohomology

$$\Phi^* : H_c^d(N) \to H_c^d(M)$$

as a real number which one calls the *degree of the map*. In other words:

Definition 22.1. *Let $\Phi : M \to N$ be a proper map between connected, oriented manifolds of the same dimension d. The* ***degree*** *of Φ is the unique real number* $\deg \Phi$ *such that:*

$$\int_M \Phi^*\omega = \deg \Phi \int_N \omega,$$

for every differential form $\omega \in \Omega_c^d(N)$.

Notice that if μ_M and μ_N are orientations of M and N then the degree of a proper map $\Phi : M \to N$ is given by:

$$\Phi^* \mu_N = (\deg \Phi) \mu_M,$$

where, according to the convention above, the symbols μ_M and μ_N represent also generators of $H_c^d(M)$ and $H_c^d(N)$ determined by the orientations.

In the sequel. for simplicity, we will consider only the case where both manifolds are compact. You may wish to try to extend the results below to any proper map. Our aim is to give a geometric characterization of the degree of map, which allows also for its computation.

We start with the following property:

Proposition 22.2. *Let $\Phi : M \to N$ be a smooth map between compact, connected, oriented manifolds of the same dimension d. If Φ is not surjective then* $\deg \Phi = 0$*.*

Proof. Let $q_0 \in N - \Phi(M)$. Since $\Phi(M)$ is closed, there is an open neighborhood of q_0 such that $U \subset N - \Phi(M)$. Let $\omega \in \Omega_c^d(N)$ have its support in U be such that $\int_N \omega \neq 0$. Then:

$$0 = \int_M \Phi^* \omega = \deg \Phi \int_N \omega,$$

hence $\deg \Phi = 0$. □

We can now give a geometric interpretation of the degree of a map. This interpretation also shows that the degree is always an integer, something which is not obvious from our definition of the degree.

Theorem 22.3. *Let $\Phi : M \to N$ be a smooth map between compact, connected, oriented manifolds of the same dimension d. Let $q \in N$ be a regular value of Φ and for each $p \in \Phi^{-1}(q)$ define*

$$\operatorname{sgn}_p \Phi \equiv \begin{cases} 1 & \text{if } \mathrm{d}_p\Phi : T_pM \to T_qN \text{ preserves orientations,} \\ -1 & \text{if } \mathrm{d}_p\Phi : T_pM \to T_qN \text{ switches orientations.} \end{cases}$$

Then [9]*:*

$$\deg \Phi = \sum_{p \in \Phi^{-1}(q)} \operatorname{sgn}_p \Phi.$$

In particular, the degree is an integer.

Proof. Let q be a regular value of Φ. Then $\Phi^{-1}(q)$ is a discrete subset of M which, by compactness, must be finite: $\Phi^{-1}(q) = \{p_1, \dots, p_N\}$. We need the following lemma:

Lemma 22.4. *There exists a neighborhood V of q and disjoint neighborhoods $U_1, \dots, U_n$ of $p_1, \dots, p_N$ such that*

$$\Phi^{-1}(V) = U_1 \cup \dots \cup U_N.$$

[9]If $\Phi^{-1}(q)$ is empty then we convention that the sum is zero.

Assuming this lemma to hold, let V and $U_1, \dots, U_N$ be as in its statement. Since each p_i is a regular point, we can assume, additionally that V is the domain of a chart $(y^1, \dots, y^d)$ in N and that the U_i's are domains of charts in M, such that the restrictions $\Phi|_{U_i}$ are diffeomorphisms.

Let $\omega \in \Omega^d(N)$ be a form:

$$\omega = f \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^d,$$

where $f \geq 0$ has $\operatorname{supp} f \subset V$. Obviously, we have

$$\operatorname{supp} \Phi^*\omega \subset U_1 \cup \cdots \cup U_N,$$

so we find:

$$\int_M \Phi^*\omega = \sum_{i=1}^N \int_{U_i} \Phi^*\omega.$$

Since each $\Phi|_{U_i}$ is a diffeomorphism, the change of variables formula gives:

$$\int_{U_i} \Phi^*\omega = \pm \int_V \omega = \pm \int_N \omega,$$

where the sign is positive if $\Phi|_{U_i}$ preserves orientations and negative otherwise. Since $\Phi|_{U_i}$ preserves orientations if $\operatorname{sgn}_{p_i} \Phi > 0$ and switches orientations if $\operatorname{sgn}_{p_i} \Phi < 0$, we conclude that

$$\int_M \Phi^*\omega = \sum_{i=1}^N \operatorname{sgn}_{p_i} \Phi \int_N \omega,$$

as claimed.

To finish the proof it remains to prove the lemma. Let $O_1, \dots, O_N$ be any disjoint open neighborhoods of $p_1, \dots, p_N$, and W a compact neighborhood of q. The set $W' \subset M$ defined by:

$$W' = \Phi^{-1}(W) - (O_1 \cup \cdots \cup O_N),$$

is compact. Hence, $\Phi(W')$ is a closed set which does not contain q. Therefore, there exists an open set $V \subset W - \Phi(W')$, containing q, and we have $\Phi^{-1}(V) \subset O_1 \cup \cdots \cup O_N$. If we let $U_i = O_i \cap \Phi^{-1}(V)$, we see that the lemma holds. $\square$

The degrees of two homotopic maps coincide, since homotopic maps induce the same map in homotopy. This is a very useful fact in computing degrees, and can be explored to deduce global properties of manifolds. A classic illustration of this is given in the next example.

Example 22.5.

Consider the antipodal map $\Phi : \mathbb{S}^d \to \mathbb{S}^d$, $p \mapsto -p$. For the canonical orientation of the sphere $\mathbb{S}^d$ defined by the form

$$\omega = \sum_{i=1}^{d+1} (-1)^i x^i \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^{d+1}.$$

we see that Φ preserves or switches orientations if d is odd or or even. Since $\Phi^{-1}(q)$ has only a point, we conclude that

$$\deg \Phi = (-1)^{d-1}.$$

By the way, we could also compute the degree directly from the definition, since we have

$$\int_{\mathbb{S}^d} \Phi^* \omega = (-1)^{d-1} \int_{\mathbb{S}^d} \omega.$$

We claim that we can use this fact to show that every vector field on a even dimensional sphere vanishes at some point. In fact, let $X \in \mathfrak{X}(\mathbb{S}^{2d})$ be a nowhere vanishing vector field. Then for each $p \in \mathbb{S}^{2d}$ there exists a semi-circle γ_p joining p to $-p$ with tangent vector $X(p)$. It follows that the map $H : \mathbb{S}^{2d} \times [0,1] \to \mathbb{S}^{2d}$ given by

$$H(p,t) = \gamma_p(t),$$

is a homotopy between Φ and the identity map. Hence,

$$-1 = \deg \Phi = \deg \mathrm{id} = 1,$$

a contradiction.

You should notice that, in contrast, any odd degree $\mathbb{S}^{2d-1} \subset \mathbb{R}^{2d}$ admits the vector field:

$$X = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2} + \cdots + x^{2d} \frac{\partial}{\partial x^{2d-1}} - x^{2d-1} \frac{\partial}{\partial x^{2d}},$$

which is a nowhere vanishing vector field.

As another application of degree theory, we will introduce now the index of a vector field, which will eventually lead to a geometric formula for the Euler characteristic of a manifold, known as the Poincaré-Hopf Theorem.

Consider first a vector field defined in some open set $U \subset \mathbb{R}^d$ which has an isolated zero at $x_0 \in U$. This means that we have a map $X : U \to \mathbb{R}^d$ which vanishes at x_0 and is non-zero is some deleted neighborhood $V - \{x_0\}$. In other words, there exists $\varepsilon > 0$ such that $\bar{B}_\varepsilon(x_0) \subset U$, the closed ball of radius ε centered at x_0, does not contain another zero of X. If

$$S_\varepsilon := \partial \bar{B}_\varepsilon(x_0)$$

is the sphere of radius ε centered at x_0, we can define the **Gauss map** $G : S_\varepsilon \to \mathbb{S}^{d-1}$ by:

$$G(x) = \frac{X(x)}{||X(x)||}.$$

The **index** of X at x_0 is the degree of the Gauss map:

$$\mathrm{ind}_{x_0} X \equiv \deg G,$$

where on each sphere we consider the orientation induced from the canonical orientation on $\mathbb{R}^d$.

Our next result shows that the degree is independent of ε and is a diffeomorphism invariant:

Proposition 22.6. *(i) For all ε small enough the degree of the Gauss map is the same, so the degree is independent of ε.*

(ii) Let X and X' be vector fields in $U, U' \subset \mathbb{R}^d$ and $\Phi : U \to U'$ a diffeomorphism. If X is Φ-related with X' and x_0 is an isolated zero of X, then

$$\operatorname{ind}_{x_0} X = \operatorname{ind}_{\Phi(x_0)} X'.$$

Proof. We can assume that $\Phi(x_0) = x_0 = 0$ and that U is convex.

Assume first that Φ preserves orientations. Then the map

$$H(t,x) = \begin{cases} \frac{1}{t}\Phi(tx), & \text{if } t > 0, \\ \\ \Phi'(x), & \text{if } t = 0. \end{cases}$$

is a homotopy between Φ' and Φ, consisting of diffeomorphismos that fix the origin. Since Φ' is homotopic to the identity, via diffeomorphismos that fix the origin, we see that there exists a homotopy, via diffeomorphismos that fix the origin, between Φ and the identity. Hence, we conclude that the Gauss maps of X and X' are homotopic, so that the indices of X and X' coincide.

To deal with the case where Φ switches orientations, it is enough to consider the case where Φ is any reflection. In this case the vector fields X and X' are related by:

$$X' = \Phi \circ X \circ \Phi^{-1}.$$

The corresponding Gauss maps are also related by:

$$G' = \Phi \circ G \circ \Phi^{-1},$$

and, hence, their degrees coincide. □

The proposition allows us to define the index for a vector field on a manifold:

Definition 22.7. *If $X \in \mathfrak{X}(M)$ is a vector field with an isolated zero x_0, the **index of** X at $p_0 \in M$, is the number*

$$\operatorname{ind}_{p_0} X \equiv \operatorname{ind}_0 \phi_* X|_U,$$

where (U, ϕ) is any coordinate system centered at p_0.

We will see in the next set of lectures that we have the following famous result:

Theorem 22.8 (Poincaré-Hopf)**.** *Let $X \in \mathfrak{X}(M)$ is a vector field on a compact manifold with a finite number of zeros $\{p_1, \dots, p_N\}$. Then:*

$$\chi(M) = \sum_{i=1}^{N} \operatorname{ind}_{p_i} X.$$

For now we limit ourselves to explain how one can compute the index of a vector field. Let X be a vector field in a manifold M and let $p_0 \in M$ be a zero of X. The zero section $Z \subset TM$ and the fiber $T_{p_0}M \subset TM$ intersect transversely at $0 \in T_{p_0}M$:

$$T_0(TM) = T_{p_0}Z \oplus T_{p_0}(T_{p_0}M) \simeq T_{p_0}M \oplus T_{p_0}M.$$

Under this decomposition, the differential $\mathrm{d}_{p_0}X : T_{p_0}M \to T_0(TM)$ has first component the identity (since X is a section), while the second component is a linear map $T_{p_0}M \to T_{p_0}M$ which will be denoted also by $\mathrm{d}_{p_0}X$, and called the **linear approximation** to X at the zero p_0.

Definition 22.9. *A zero p_0 of a vector field $X \in \mathfrak{X}(M)$ is called a* ***non-degenerate zero*** *if the the linear approximation $\mathrm{d}_{p_0}X : T_{p_0}M \to T_{p_0}M$ is invertible.*

Non-degenerate zeros are always isolated and their index can be computed very easily:

Proposition 22.10. *Let $p_0 \in M$ be a non-degenerate zero of a vector field $X \in \mathfrak{X}(M)$. Then p_0 is an isolated zero and:*

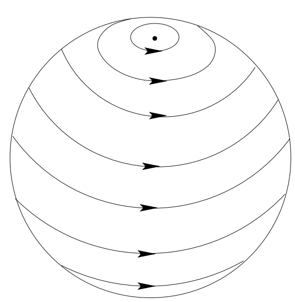
$$\operatorname{ind}_{p_0} X = \begin{cases} +1, & \text{if } \det \mathrm{d}_{p_0}X > 0, \\ \\ -1, & \text{if } \det \mathrm{d}_{p_0}X < 0. \end{cases}$$

Proof. Choose local coordinates (U, ϕ) centered at p_0. The vector field $(\phi)_*X|_U$ has an associated Gauss map $G : S_\varepsilon \to \mathbb{S}^{d-1}$ which is a diffeomorphism. Moreover, this diffeomorphism preserves (switches) orientations if and only if $\det \mathrm{d}_{p_0}X > 0$ (respectively, < 0). Hence the result follows from Theorem 22.3. $\square$

Example 22.11.
Em $\mathbb{R}^3$, with coordinates (x, y, z), consider the vector fuel

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

This vector field is tangent to the sphere $\mathbb{S}^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ and hence defines a vector field $X \in \mathfrak{X}(\mathbb{S}^2)$, with exactly two zeros: the north pole p_N and the south pole p_S.



The projection $\phi : (x, y, z) \mapsto (x, y)$ give a system of coordinates $\mathbb{S}^2$ centered at p_N (and also p_S), and we have:

$$\phi_* X = v \frac{\partial}{\partial u} - u \frac{\partial}{\partial v}.$$

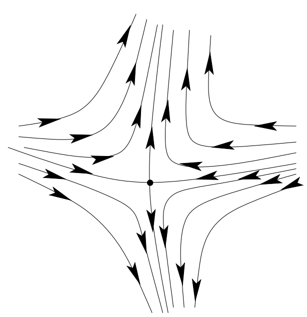
where (u, v) are coordenates in $\mathbb{R}^2$. Since the map $(u, v) \mapsto (v, -u)$ has differential

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

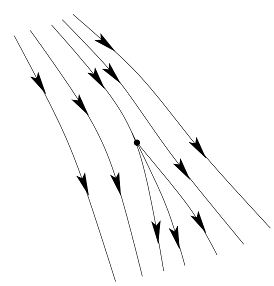
we conclude that p_N and p_S are non-degenerate zeros and:

$$\operatorname{ind}_{p_N} X = \operatorname{ind}_{p_S} X = 1.$$

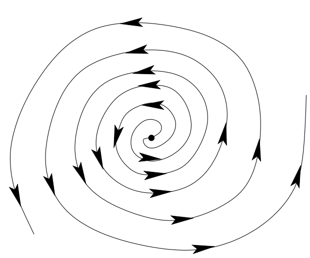
In some simple cases it is possible to determine the index of a vector field from its phase portrait, even if the zeros are degenerate. The next figure illustrates some examples of planar vector fields with a zero and the value of its index. You should try to check that the degree of the corresponding Gauss maps is indeed the integer in each figure.



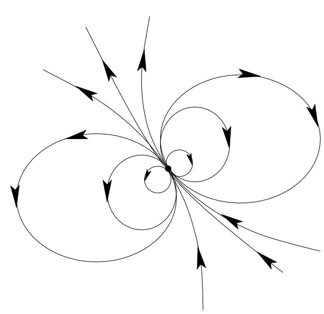
$\operatorname{ind}_{p_0} X = -1$



$\operatorname{ind}_{p_0} X = 0$



$\operatorname{ind}_{p_0} X = 1$



$\operatorname{ind}_{p_0} X = 2$

Homework.

1. Let $\Phi : \mathbb{C} \to \mathbb{C}$ be a polynomial map of degree d. Find $\deg \Phi$.

2. Show that for a smooth manifold M of dimension $d > 0$ the identity map $M \to M$ is never homotopic to a constant map. Use this fact to prove that there is no retraction of the closed unit ball $\bar{B}_1(0) \subset \mathbb{R}^d$ on its boundary.

   Hint: If there was such a retraction $r : \bar{B}_1(0) \to \mathbb{S}^{d-1}$ consider the map $H(x,t) = r(rx)$.

3. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix with integer entries. Identifying $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, consider the map $\Phi : \mathbb{T}^2 \to \mathbb{T}^2$ defined by:
$$\Phi([x,y]) = [ax + by, cx + dy].$$
   Determine $\deg \Psi$.

4. Let $X : \mathbb{R}^d \to \mathbb{R}^d$ be a vector field with an isolated zero at 0 and for any $\varepsilon > 0$ let
$$S_\varepsilon := \partial \bar{B}_\varepsilon(x_0)$$
   be the sphere of radius ε centered at x_0. Show that degree of the Gauss map $G : S_\varepsilon \to \mathbb{S}^{d-1}$
$$G_\varepsilon(x) = \frac{X(x)}{||X(x)||}.$$
   is the same all ε, so the degree is independent of ε.

5. Identify $M = \mathbb{R}^2$ with the field of complex numbers $\mathbb{C}$. Show that the polynomial map $z \mapsto z^k$ defines a vector field in $\mathbb{R}^2$ which has a zero at the origin of index k. How would you change $z \mapsto z^k$ to obtain a vector field with a zero of index $-k$?

6. Find the index of the zeros of the following vector fields in $\mathbb{R}^2$:
   (a) $x\frac{\partial}{\partial x} \pm y\frac{\partial}{\partial y}$;
   (b) $(x^2y + y^3)\frac{\partial}{\partial x} - (x^3 + xy^2)\frac{\partial}{\partial y}$;

7. Show that a vector field on a compact, oriented, surface of genus g, where $g \neq 1$, must have at least one zero.

8. Consider the vector field $X \in \mathbb{S}^{2d}$ obtained by restriction of the vector field in $\mathbb{R}^{2d+1}$:
$$X = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2} + \cdots + x^{2d}\frac{\partial}{\partial x^{2d-1}} - x^{2d-1}\frac{\partial}{\partial x^{2d}}.$$
   Show that X induces a vector field $\overline{X}$ in $\mathbb{P}^{2d}$. Use this vector field and the Poincaré-Hopf theorem to compute the Euler characteristic of $\mathbb{P}^{2d}$.

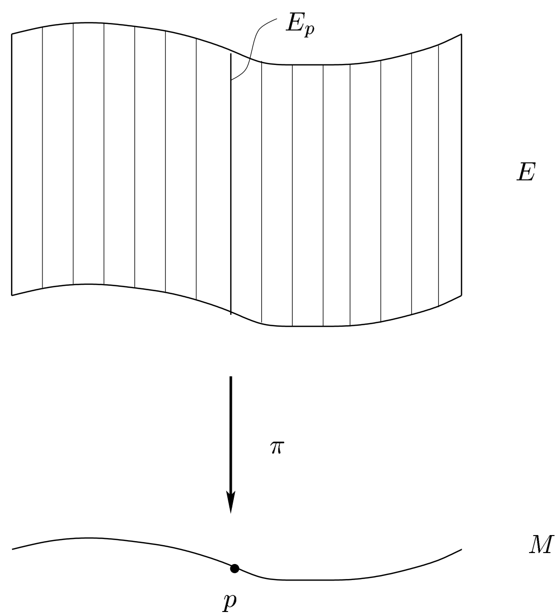
Part 4. Fiber Bundles

We have seen already several examples of fiber bundles, such as the tangent bundle, the cotangent bundle or the exterior bundles. So far, we have used the concept of a bundle in a more or less informal way. We will see now that one can understand many global properties of manifolds by studying more systematically fibre bundles and their properties.

The main notions and concepts to retain from the next series of lectures are the following:

- Lecture 23: The notion of a *vector bundle* and the basic constructions with these bundles, such as the sum, tensor product and exterior product. produtos tensoriais, produtos exteriores, etc.
- Lecture 24: Two import an invariants of vector bundles: the *Thom class* and the *Euler class*. The relationship between the Euler class of the tangent bundle and the Euler characteristic and, as a consequence, the *Poincaré-Hopf Theorem*.
- Lecture 25: The *pull-back* of vector bundles and the *classification of vector bundles*, which shows that every vector bundle is the pull-back of a *universal vector bundle*.
- Lecture 26: The concept of a *connection* in a vector bundle, which allows one to differentiate sections of the vector bundle along vector fields in the basisand hence compare different fibers.
- Lecture 27: The *curvature* of a connection and the *holonomy* of a connection, which give rise to invariants, characterizing the global structure of a vector bundle with a connection.
- Lecture 28: The *Chern-Weil theory* of characteristics classes of real vector bundles (*Pontrjagin classes*) and complex vector bundles (*Chern classes*).
- Lecture 29: The abstract notion of a *fibre bundle* and of a *principal fibre bundle*. The constructions of the *associated bundles*.
- Lecture 30: The *classification of principal bundles*, *connections in principal bundles* and *characteristics classes of principal bundles*.

A vector bundle is a collection $\{E_p\}_{p \in M}$ of vector spaces parameterized by a manifold M. The union of these vector spaces is a manifold E, and the map $\pi : E \to M$, $\pi(E_p) = p$ must satisfy a local trivialization condition. You should be able to recognize all these properties in the tangent bundle of a manifold.



In order to formalize this concept, let $\pi : E \to M$ be a smooth map between differentiable manifolds. A **trivializing chart** of dimension r for π is a pair (U, ϕ), where $U \subset M$ is open and $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ is a diffeomorphism, such that we have a commutative diagram:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^r \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

In this diagram, $\pi_1 : U \times \mathbb{R}^r \to U$ denotes the projection in the first factor.

Let $E_p = \pi^{-1}(p)$ be the fiber over $p \in U$. We define a diffeomorphism $\phi^p : E_p \to \mathbb{R}^r$ as the composition:

$$\phi^p : E_p \xrightarrow{\phi} \{p\} \times \mathbb{R}^r \longrightarrow \mathbb{R}^r \ .$$

Hence, if $\mathbf{v} \in E_p$, we have

$$\phi(\mathbf{v}) = (p, \phi^p(\mathbf{v})).$$

Notice that since each ϕ^p is a diffeomorphism, we can use ϕ^p to transport the vector space structure of $\mathbb{R}^r$ to E_p. Given two trivializing charts whose domains intersect we would like that the induced vector space structures on the fibers coincide. This leads to the following definition:

Definition 23.1. *A* ***vector bundle structure*** *of rank r over a manifold M is a triple $\xi = (\pi, E, M)$, where $\pi : E \to M$ is a smooth map admitting a collection of trivializing charts $\mathcal{C} = \{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ of dimension r, satisfying the following properties:*

(i) $\{U_\alpha : \alpha \in A\}$ *is an open cover of* M*:* $\bigcup_{\alpha\in A} U_\alpha = M$*;*
(ii) The charts are compatible: for any $\alpha, \beta \in A$ and every $p \in U_\alpha \cap U_\beta$, the ***transition functions*** *$g_{\alpha\beta}(p) \equiv \phi^p_\alpha \circ (\phi^p_\beta)^{-1} : \mathbb{R}^r \to \mathbb{R}^r$ are linear isomorphisms;*
(iii) The collection $\mathcal{C}$ is maximal: if (U, ϕ) is a trivializing chart of dimension r with the property that for every $\alpha \in A$, the maps $\phi^p \circ (\phi^p_\alpha)^{-1}$ e $\phi^p_\alpha \circ (\phi^p)^{-1}$ are linear isomorphisms, then $(U, \phi) \in \mathcal{C}$.

We call $\xi = (\pi, E, M)$ a ***vector bundle*** *of rank r.*

For a vector bundle $\xi = (\pi, E, M)$ we will use the following notations:

- E is call the **total space**, M is called the **basisspace**, and π the **projection** of ξ.
- A collection of charts satisfying (i) and (ii) is called an **atlas of the vector bundle** or a **trivialization** of ξ.

An atlas of a vector bundle defines a vector bundle, since every atlas is contain in a unique maximal atlas. As we have already remarked, (ii) implies that the fiber E_p has a vector space structure such that for any trivializing chart (U, ϕ) the map $\phi^p : E_p \to \mathbb{R}^r$ is a linear isomorphism.

In the definition above of a vector bundle all maps are C^∞. Of course, one can also define C^k-vector bundles over C^k-manifold or even topological manifolds. On the other hand, one can define complex vector bundles over smooth manifolds by replacing $\mathbb{R}^r$ by $\mathbb{C}^r$ (note that the basisis still a real smooth manifold). In these notes, unless otherwise mentioned, all vector bundles are real and C^∞.

Let $\xi = (\pi, E, M)$ be a vector bundle and $U \subset M$ an open set. A map $s : U \to E$ is called a **section** over U if $\pi \circ s$ is the identity in U. The sections over U form a real vector space which will be denoted by $\Gamma_U(E)$. If $\operatorname{rank} \xi = r$ a collection $s_1, \dots, s_r$ of sections over U is called a **frame** in U if, for every $p \in U$, the sections $\{s_1(p), \dots, s_r(p)\}$ form a basis for E_p. When $U = M$ we call a section over M a **global section** of E and we write $\Gamma(E)$ instead of $\Gamma_M(E)$.

Definition 23.2. *Let $\xi_1 = (\pi_1, E_1, M_1)$ and $\xi_2 = (\pi_2, E_2, M_2)$ be two vector bundles. A* ***morphism of vector bundles*** *is a smooth map $\Psi : E_1 \to E_2$ which maps the fibers of ξ_1 linearly in the fibers of ξ_2, i.e., Ψ covers a smooth* *map* $\psi : M_1 \to M_2$*:*

$$\begin{tikzcd} E_1 \arrow[r, "\Psi"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"'] \\ M_1 \arrow[r, "\psi"] & M_2 \end{tikzcd}$$

and, for which $p \in M_1$*, the map of the fibers*

$$\Psi^p \equiv \Psi|_{(E_1)_p} : (E_1)_p \to (E_2)_{\psi(p)}$$

is a linear transformation.

In this way we have the category of vector bundles.

Often we will be interested in vector bundles over the *same* basisand morphisms over the identity (i.e., the map $\psi : M \to M$ is the identity). These also form a category.

Two vector bundles $\xi_1 = (\pi_1, E_1, M_1)$ and $\xi_2 = (\pi_2, E_2, M_2)$ are called:

- **equivalent** if there exist morphisms $\Psi : \xi_1 \to \xi_2$ and $\Psi' : \xi_2 \to \xi_1$ which are inverse to each other, i.e., an isomorphism in the category of vector bundles. Then Ψ covers a diffeomorphism $\psi : M_1 \to M_2$ and each fiber map $\Psi^p : (E_1)_p \to (E_2)_{\psi(p)}$ is a linear isomorphism.
- **isomorphic** if $M_1 = M_2 = M$ and there exist morphisms $\Psi : \xi_1 \to \xi_2$ and $\Psi' : \xi_2 \to \xi_1$, covering the identity which are inverse to each other, i.e., an isomorphism in the subcategory of vector bundles over the same base). In this case, Ψ covers the identity $\psi =\mathrm{id}_M$ and each fiber map $\Psi^p : (E_1)_p \to (E_2)_p$ is a linear isomorphism.

Examples 23.3.

1. *Obviously, for any smooth manifold* M*, we have the vector bundles* TM*,* T^*M *and* $\wedge^k(T^*M)$*. The sections of these bundles are the vector fields and the differential forms that we have studied before. If* $\Psi : M \to N$ *is a smooth map, its differential* $\mathrm{d}\Psi : TM \to TN$ *is a morphism of vector bundles (note, however, that the transpose* $(\mathrm{d}_x\Phi)^*$*, in general,* is not *a vector bundle morphism).*
2. *The* ***trivial vector bundle*** *of rank* r *over* M *is the vector bundle* $\varepsilon^r_M = (\pi, M \times \mathbb{R}^r, M)$*, where* $\pi : M \times \mathbb{R}^r \to M$ *is the projection in the first factor. Note that* $\Gamma(\varepsilon^r_M) = C^\infty(M; \mathbb{R}^r)$*. In geral, a vector bundle* ξ *over* M *of rank* r *is said to be trivial if it is isomorphic to* ε^r_M*. We leave it as an exercise to show that a vector bundle is trivial if and only if it admits a global frame. A* ***parallelizable manifold*** *is a manifold* M *for which* TM *is a trivial vector bundle. For example, any Lie group* G *is parallelizable, but* $\mathbb{S}^2$ *is not parallelizable (actually, one can show that* $\mathbb{S}^d$ *is parallelizable if and only if* $d = 0, 1, 3$ *and* 7*).*
3. *A* r*-dimensional distribution* D *in a manifold* M*, defines a vector bundle over* M *of rank* r*. The fibers are the subspaces* $D_p \subset T_pM$*. You should verify that the local triviality condition holds. A section of this vector bundle is a vector field tangent to the distribution.*

4. A vector bundle of rank 1 is usually refer to as a ***line bundle****. For example, a non-vanishing vector field defines a line bundle which is always trivial. More generally a rank 1 distribution defines a line bundle which is trivial if and only if the distribution is generated by a single vector field.*

5. For another example of a line bundle let

$$E = \{([x], \mathbf{v}) \in \mathbb{P}^d \times \mathbb{R}^{d+1} : \mathbf{v} = \lambda x, \text{ for some } \lambda \in \mathbb{R}\}.$$

In other words, a element of E is a pair $([x], \mathbf{v})$ where $[x]$ is a line through the origin in $\mathbb{R}^{d+1}$ and $\mathbf{v}$ is a point in this line. The projection $\pi : E \to \mathbb{P}^d$ is given by $\pi([x], \mathbf{v}) = [x]$. In other to check the local triviality, let $V \subset \mathbb{S}^d$ be an open set such that if $x \in V$ then $-x \notin V$. The corresponding open set in projective space is denoted $U = \{[x] : x \in V\} \subset \mathbb{P}^d$. Then the map defined by:

$$\psi : U \times \mathbb{R} \to \pi^{-1}(U), \quad \psi([x], t) = ([x], tx), \forall x \in V,$$

is a diffeomorphism and its inverse $\phi = \psi^{-1}$ defines a trivializing chart over U. The family of all such charts (U, ϕ) form an atlas of a vector bundle over $\mathbb{P}^d$. This vector bundle is called the ***canonical line bundle*** *over $\mathbb{P}^d$ and is denoted γ_d^1.*

One can describe a vector bundles through its transition functions: let $\xi = (\pi, E, M)$ be a rank r vector bundle. If (U_α, ϕ_α) and (U_α, ϕ_α) are trivializing charts let $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r)$ be the transition function

$$p \mapsto g_{\alpha\beta}(p) \equiv \phi_\alpha^p \circ (\phi_\beta^p)^{-1},$$

so that:

$$\phi_\alpha \circ (\phi_\beta)^{-1}(p, \mathbf{v}) = (p, g_{\alpha\beta}(p) \cdot \mathbf{v}).$$

These transition functions satisfy:

$$g_{\alpha\beta}(p) g_{\beta\gamma}(p) = g_{\alpha\gamma}(p), \quad (p \in U_\alpha \cap U_\beta \cap U_\gamma). \tag{23.1}$$

If $\alpha = \beta = \gamma$, this condition reduces to:

$$g_{\alpha\alpha}(p) = I, \quad (p \in U_\alpha),$$

and when $\gamma = \alpha$ we obtain:

$$g_{\beta\alpha}(p) = g_{\alpha\beta}(p)^{-1}, \quad (p \in U_\alpha \cap U_\beta).$$

The family $\{g_{\alpha\beta}\}$ depends on the choice of trivializing charts. However, we have:

Lemma 23.4. *Let ξ and η be vector bundles over M with trivializations $\{\phi_\alpha\}$ and $\{\phi'_\alpha\}$ subordinated to the same open covering $\{U_\alpha\}$ of M. Denote by $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ the corresponding transition functions. If ξ is isomorphic to η, then there exist smooth maps $\lambda_\alpha : U_\alpha \to GL(r)$ such that:*

$$g'_{\alpha\beta}(p) = \lambda_\alpha(p) \cdot g_{\alpha\beta}(p) \cdot \lambda_\beta^{-1}(p), \quad (p \in U_\alpha \cap U_\beta). \tag{23.2}$$

Proof. Let $\Psi : \xi \to \eta$ be an isomorphism. For each U_α we define smooth maps $\lambda_\alpha : U_\alpha \to GL(r)$ by:

$$\lambda_\alpha(p) = \phi'^p_\alpha \circ \Psi \circ (\phi^p_\alpha)^{-1}.$$

If $p \in U_\alpha \cap U_\beta$, we have:

$$\begin{aligned} g'_{\alpha\beta}(p) &= \phi'^p_\alpha \circ (\phi'^p_\beta)^{-1} \\ &= \lambda_\alpha(p) \circ \phi^p_\alpha \circ (\phi^p_\beta)^{-1} \circ (\lambda_\beta(p))^{-1} \\ &= \lambda_\alpha(p) \circ g_{\alpha\beta}(p) \circ \lambda_\beta(p)^{-1}. \end{aligned}$$

□

For a manifold M we call a family of maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r)$ a **cocycle** subordinated to the open cover $\{U_\alpha\}_{\alpha\in A}$ of M, if they satisfy (23.1). Two cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ subordinated to the same cover are said to be **equivalents** if there exist smooth maps $\lambda_\alpha : U_\alpha \to GL(r)$ of classe C^∞, satisfying (23.2).

We saw above that (i) a trivialization of a vector bundle determines a cocycle, and that (ii) two trivializations of isomorphic vector bundles subordinated to the same cover determine equivalent cocycles. Note that if two cocycles are subordinated to different covers we can always refine the covers to obtain cocycles subordinated to the same cover. Moreover, we have the following converse:

Proposition 23.5. *Let $\{g_{\alpha\beta}\}$ be a cocycle subordinates to an open cover $\{U_\alpha\}$ of M. There exists a vector bundle $\xi = (\pi, E, M)$, that admits a trivialization $\{\phi_\alpha\}$ for which the transition functions are the $\{g_{\alpha\beta}\}$. Two equivalent cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ determine isomorphic vector bundles.*

Proof. Given a cocycle $\{g_{\alpha\beta}\}$, subordinated to the cover $\{U_\alpha\}$ of M, we construct the manifold E as the quotient:

$$E = \bigcup_{\alpha\in A} (U_\alpha \times \mathbb{R}^r) \Big/ \sim$$

where $\sim$ is the equivalence relation defined by:

$$(p, \mathbf{v}) \sim (q, \mathbf{w}) \text{ iff } \begin{cases} p = q \text{ and} \\ \exists \alpha, \beta \in A : \; g_{\alpha\beta}(p) \cdot \mathbf{v} = \mathbf{w}. \end{cases}$$

The projection $\pi : E \to M$ is the obvious map:

$$\pi([p, \mathbf{v}]) = p.$$

It is easy to see that the maps $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$ given by:

$$\phi_\alpha([p, \mathbf{v}]) = (p, \mathbf{v}),$$

give trivializing charts. The corresponding transition functions are exactly the $\{g_{\alpha\beta}\}$. Denote this vector bundle by $\xi = (\pi, E, M)$

If $\{g'_{\alpha\beta}\}$ is another cocycle equivalent to $\{g_{\alpha\beta}\}$ through the family $\{\lambda_\alpha\}$ and $\xi' = (\pi', E', M)$ denotes the vector bundle associated with $\{g'_{\alpha\beta}\}$, we have a vector isomorphism $\Psi : \xi \to \xi'$, which in each open set $\pi^{-1}(U_\alpha)$ is given by:

$$\Psi([p, \mathbf{v}]) = [p, \lambda_\alpha(p) \cdot \mathbf{v}].$$

The details are left as an exercise. $\square$

Let us now turn to constructions with vector bundles. We have the following general principle: *for every functorial construction with vector spaces there is a similar construction with vector bundles.* This principle can actually be made precise, but instead of following the abstract route, we will describe now explicitly the constructions that are most relevant to us.

Subbundles and quotients. Every vector bundle $\xi = (\pi, E, M)$ can be restricted to a submanifold $N \subset M$. The **restriction** ξ_N is the vector bundle with total space:

$$E_N = \{E_p : p \in N\},$$

and projection $\pi_N : E_N \to N$ the restriction of π to E_N. The restriction is an example of a vector subbundle:

Definition 23.6. *A vector bundle $\eta = (\tau, F, N)$ is called a **vector subbundle** of a vector bundle $\xi = (\pi, E, M)$ if F is a submanifold of E, and the inclusion $F \hookrightarrow E$ is a morphism of vector bundles.*

If $\Psi : \eta \to \xi$ is a morphism of vector bundles covering the identity, in general, its image and its kernel *are not* vector subbundles. However, this will be the case if the morphism of vector bundles $\Psi : (\pi, E, M) \to (\tau, F, M)$ has **constant rank** k, i.e., if all linear maps $\Psi^p : E_p \to F_p$ have the same rank k. For a constant rank morphism we can define the following vector subbundles over M:

- The **kernel** of Ψ is the vector subbundle $\operatorname{Ker}\Phi \subset E$ whose total space is $\{\mathbf{v} \in E : \Phi(\mathbf{v}) = 0\}$;
- The **image** of Ψ is the vector subbundle $\operatorname{Im}\Phi \subset F$ whose total space is $\{\Phi(\mathbf{v}) \in F : \mathbf{v} \in E\}$;
- The **co-kernel** of Ψ is the vector bundle $\operatorname{coKer}\Phi$ whose total space is the quotient $F/\sim$, where $\sim$ the equivalence relation $\mathbf{w}_1 \sim \mathbf{w}_2$ if and only if $\mathbf{w}_1 - \mathbf{w}_2 = \Phi(\mathbf{v})$, for some $\mathbf{v} \in E$.

Note that if Ψ is a *monomorphism* (i.e., each Ψ^p is injective) or if Ψ is an *epimorphism* (i.e., each Ψ^p is surjective) then Ψ has constant rank. Therefore, the kernel, image and cokernel of monomorphisms and epimorphisms are vector subbundles. We leave the details as an exercise.

The notions associated with exact sequences can be easily extended to vector bundles and morphisms of constant rank. For example, a short exact sequence of vector bundles is a sequence

$$0 \longrightarrow \xi \xrightarrow{\Phi} \eta \xrightarrow{\Psi} \theta \longrightarrow 0$$

where Φ is a monomorphism, Ψ is an epimorphism and $\operatorname{Im}\Phi = \operatorname{Ker}\Psi$. In this case, we have vector bundle isomorphisms $\xi \simeq \operatorname{Ker}\Psi$ and $\theta \simeq \operatorname{coKer}\Psi$, and we say that θ is the **quotient vector bundle** of the monomorphism Φ.

An important example of a quotient vector bundle is obtained by taking a vector subbundle $\xi = (\tau, F, M) \subset \eta = (\pi, E, M)$: the inclusion is a monomorphism of vector bundles, hence we can form its quotient, which we denote by η/ξ. Notice that the fibers of η/ξ are the quotient vector spaces E_p/F_p.

EXAMPLE 23.7.
*Let M be a manifold and $N \subset M$ a submanifold. The tangent bundle TN is a vector subbundle of $T_N M$. The quotient bundle $\nu(N) \equiv T_N M/TN$ is usually called the **normal bundle to** N in M.*

*More generally, let $\mathcal{F}$ be a foliation of M. Then $\mathcal{F}$ gives rise to the vector subbundle $T\mathcal{F} \subset TM$. The quotient bundle $\nu(\mathcal{F}) \equiv TM/T\mathcal{F}$ is usually called the **normal bundle of** $\mathcal{F}$ in M. If L is a leaf of $\mathcal{F}$ the restriction of $\nu(\mathcal{F})$ to L is the normal bundle $\nu(L)$.*

Sums and tensor products. Let $\xi = (\pi, E, M)$ and $\eta = (\tau, F, M)$ be vector bundles over the same manifold M. The **Whitney sum** or **direct sum** of ξ and η is the vector bundle $\xi \oplus \eta$ whose total space is:

$$E \oplus F = \{(\mathbf{v}, \mathbf{w}) \in E \times F : \pi(\mathbf{v}) = \tau(\mathbf{w})\},$$

and whose projection is:

$$E \oplus F \to M, \ (\mathbf{v}, \mathbf{w}) \mapsto \pi(\mathbf{v}) = \tau(\mathbf{w}).$$

Note that the fiber of $\xi \oplus \eta$ over $p \in M$ is the direct sum $E_p \oplus F_p$. The local triviality condition is easily verified: if $\{\phi_\alpha\}$ and $\{\psi_\alpha\}$ are trivializations of ξ and η, subordinated to the same covering, with corresponding cocycles $\{g_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$, then we have the trivialization of $\xi \oplus \eta$ given by $\{(\phi_\alpha \times \psi_\alpha)|_{E\oplus F}\}$, to which corresponds the cocycle defined by:

$$g_{\alpha\beta} \oplus h_{\alpha\beta} = \begin{bmatrix} g_{\alpha\beta} & 0 \\ 0 & h_{\alpha\beta} \end{bmatrix}.$$

Similarly, we can define:

- The **tensor product** $\xi \otimes \eta$: the fibers are the tensor products $E_p \otimes F_p$ and the transition functions are $g_{\alpha\beta} \otimes h_{\alpha\beta}$.
- The **dual vector bundle** ξ^*: the fibers are the dual vector spaces E_p^* and the transition functions are inverse transpose maps $(g_{\alpha\beta}^t)^{-1}$.
- The **exterior product** $\wedge^k \xi$: the fibers are the exterior products $\wedge^k E_p$ and the transition functions are the exterior powers $\wedge^k g_{\alpha\beta}$.
- The **Hom(ξ, η)-bundles**: the fibers are the space of all linear morphisms $\operatorname{Hom}(E_x, F_x)$. We leave as an exercise to show that there is a natural isomorphism $\operatorname{Hom}(\xi, \eta) \simeq \xi^* \otimes \eta$.

Orientations. A vector bundle $\xi = (\pi, E, M)$ of rank r is called an **orientable vector bundle** if the exterior product $\wedge^r \xi$ has a section which never vanishes. Note that this section corresponds to a smooth choice of an orientation in each vector space E_p. We call an **orientation** for ξ an equivalence class $[s]$, where two non-vanishing sections $s_1, s_2 \in \Gamma(\wedge^r \xi)$ are equivalent if and only if $s_2 = f s_1$ for some smooth positive function $f \in C^\infty(M)$. We leave as an exercise to check that ξ is orientable if and only if it admits a trivialization $\{\phi_\alpha\}$ for which the associated cocycle $\{g_{\alpha\beta}\}$ takes values in $GL^+(r)$, the group of invertible $r \times r$ matrices with positive determinant:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL^+(r) \subset GL(r).$$

When $\xi = TM$, this notion of orientation corresponds to the notion of orientation for M that we have studied before. The possible orientations for ξ, E and M are related as follows:

Lemma 23.8. *If $\xi = (\pi, E, M)$ is an orientable vector bundle. Then M is an orientable manifold if and only if E is also an orientable manifold.*

The proof is elementary and is left as an exercise.

Riemmanian structures. A **Riemann structure** in a vector bundle $\xi = (\pi, E, M)$ is a choice of an inner product $\langle\ ,\ \rangle : E_p \times E_p \to \mathbb{R}$ in each fiber which varies smoothly, i.e., for any sections $s_1, s_2 \in \Gamma(E)$ the map $p \mapsto \langle s_1(p), s_2(p) \rangle$ is smooth. This condition is equivalent to say that the section of the vector bundle $\otimes^2 \xi$ defined by $\langle\ ,\ \rangle$ is smooth.

You should check that ξ has a Riemann structure if and only it it admits a trivialization $\{\phi_\alpha\}$ where the associated cocycle $\{g_{\alpha\beta}\}$ take values in $O(r)$:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to O(r) \subset GL(r).$$

It is easy to see, using a partition of unity, that every vector bundle admits a Riemann structure. Underlying this fact, is the polar decomposition of $GL(r)$:

$$GL(r) = O(r) \times \{\text{positive definite symmetric matrices}\}.$$

If $\xi = (\pi, E, M)$ is a vector bundle and $\langle\ ,\ \rangle$ is a Riemann structure in ξ, then for any vector subbundle $\eta = (\tau, F, N)$ we can define the **orthogonal vector bundle** $\eta^\perp$ over N as the subbundle of ξ with total space $F^\perp$, where

$$F_p^\perp \equiv \{\mathbf{v} \in E_p : \langle \mathbf{v}, \mathbf{w} \rangle = 0, \forall \mathbf{w} \in F_p\}.$$

When $M = N$, we obtain:

$$\xi = \eta \oplus \eta^\perp.$$

in this case $\eta^\perp \simeq \xi/\eta$, since the natural projection $\xi \to \xi/\eta$ restricts to an isomorphism on $\eta^\perp$.

HOMEWORK.

1. Show that a vector bundle is trivial if and only it admits a global frame.

2. Let $G_r(\mathbb{R}^d)$ be the Grassmannian manifold of r-planes in $\mathbb{R}^d$. Consider the submanifold $E \subset G_r(\mathbb{R}^d) \times \mathbb{R}^d$ defined by:

$$E = \{(S, x) : S \text{ is a subspace of } \mathbb{R}^d \text{ and } x \in S\},$$

and the smooth map $\pi : E \to G_r(\mathbb{R}^d)$ given by:

$$\pi(S, x) = S.$$

Show that $\gamma_d^r = (\pi, E, G_r(\mathbb{R}^d))$ is a vector bundle of rank k. It is called the **canonical bundle** over $G_r(\mathbb{R}^d)$.

3. Let $\Psi : \eta \to \xi$ be a morphism of vector bundles which covers the identity. Show that the kernel and the image of Ψ are vector subbundles if the rank of the linear maps Ψ^p is constant. Give counter-examples when the rank is not constant.

4. Consider a short exact sequence of vector bundles

$$0 \longrightarrow \xi \longrightarrow \eta \xrightarrow{\Psi} \theta \longrightarrow 0$$

Show that this sequence always *splits*, i.e., there exists a monomorphism of vector bundles $\Phi : \theta \to \eta$ such that $\Psi \circ \Phi = \mathrm{id}_\theta$.

5. Given two vector bundles $\xi = (\pi, E, M)$ and $\eta = (\tau, F, M)$, show that there exists a vector bundle $\mathrm{Hom}(\xi, \eta)$ whose fibers are the vector spaces $\mathrm{Hom}(E_x, F_x)$. Find the transition function of $\mathrm{Hom}(\xi, \eta)$ in terms of the transition functions of ξ and η and verify that there exists a natural isomorphism $\mathrm{Hom}(\xi, \eta) \simeq \xi^* \otimes \eta$.

6. If $\xi = (\pi, E, M)$ is an orientable vector bundle. Show that M is orientable if and only if E is an orientable manifold.

7. Let $\xi = (\pi, E, M)$ be a vector bundle. Show that there exists an isomorphism of vector bundles

$$T_M E \simeq \xi \oplus TM,$$

which is natural: if $\Psi : (\pi, E, M) \to (\tau, F, N)$ is a morphism of vector bundles which covers the map $\psi : M \to N$, then the following diagram commutes:

$$\begin{tikzcd}
T_M E \arrow[r, "\mathrm{d}\Psi"] \arrow[d, "\simeq"'] & T_N F \arrow[d, "\simeq"] \\
\xi \oplus TM \arrow[r, "\Psi \oplus \mathrm{d}\psi"'] & \eta \oplus TN
\end{tikzcd}$$

8. For a vector bundle ξ show that the following statements are equivalent:
   (a) ξ is orientable;
   (b) There exists a trivialization of ξ for which the transition functions take values in $GL^+(r)$;
   (c) There exists a trivialization of ξ for which the transition functions take values in $SO(r)$.

The homotopy invariance of de Rham cohomology relied crucially on the following isomorphisms

$$H^\bullet(M \times \mathbb{R}^r) \simeq H^\bullet(M),$$
$$H_c^\bullet(M \times \mathbb{R}^r) \simeq H_c^{\bullet-r}(M).$$

One can interpret these isomorphisms as relating the cohomology of the total space of the trivial bundle with the cohomology of the base. More generally, we have:

Proposition 24.1. *For any vector bundle $\xi = (\pi, E, M)$:*

$$H^\bullet(E) \simeq H^\bullet(M).$$

Proof. Let $s : M \to E$ be the zero. Its image is a deformation retract of E. Therefore, by homotopy invariance we see that $s^* : H^\bullet(E) \to H^\bullet(M)$ is an isomorphism. □

One may guess that the statement for compactly supported cohomology also generalizes, but the following example shows that one must be careful:

EXAMPLE 24.2.
Let $M = \mathbb{S}^1$ and consider the non-trivial line bundle $\pi : E \to \mathbb{S}^1$. One can realize this line bundle, for example, by considering the central circle in the Möbius band and the fibers to be the transverse lines. Since E is a non-oriented manifold of dimension 2, we have $H_c^2(E) = 0$. On the other hand,

$$H_c^{2-1}(\mathbb{S}^1) = H^1(\mathbb{S}^1) \simeq \mathbb{R} \neq 0.$$

When E and M are both orientable we do have:

Proposition 24.3. *Let $\xi = (\pi, E, M)$ be a vector bundle of rank r, where E and M are both orientable of finite type. Then:*

$$H_c^\bullet(E) \simeq H_c^{\bullet-r}(M).$$

Proof. Since E and M are both orientable of finite type we can apply Poincaré duality to conclude:

$$\begin{aligned} H_c^\bullet(E) &\simeq H^{d+r-\bullet}(E) && \text{(by Poincaré duality for } E), \\ &\simeq H^{d+r-\bullet}(M) && \text{(by Proposition 24.1)}, \\ &\simeq H_c^{\bullet-r}(M) && \text{(by Poincaré duality for } M). \end{aligned}$$

□

When M is a compact manifold the total space of any vector bundle over M is of finite type. Therefore:

Corollary 24.4 (Thom Duality)**.** *Let $\xi = (\pi, E, M)$ be an orientable vector bundle of rank r and M an orientable compact manifold. Then:*

$$H_c^\bullet(E) \simeq H^{\bullet - r}(M).$$

The map giving the isomorphism in Thom duality is a push-forward map

$$\pi_* : \Omega_c^\bullet(E) \to \Omega^{\bullet - r}(M)$$

and is called **integration along the fibers**. If we unwind the proof above, we can find an explicit description of π_* as follows. We start by covering M by trivializing oriented charts (U_α, ϕ_α) for ξ, where each U_α is the domain of a chart $(x^1, \dots, x^d)$ of M. We obtain a system of coordinates $(x^1, \dots, x^d, t^1, \dots, t^r)$ for the total space E with domain $\pi^{-1}(U_\alpha)$, where $(t^1, \dots, t^r)$ are linear coordinates on the fibers. If ω is some differential form in E, then $\omega_\alpha = \omega|_{\pi^{-1}(U_\alpha)}$ is a linear combination of two kinds of forms:

- $f(x,t)(\pi^*\theta) \wedge dt^{i_1} \wedge \cdots \wedge dt^{i_k}$, with $k < r$;
- $f(x,t)(\pi^*\theta) \wedge dt^1 \wedge \cdots \wedge dt^r$;

where θ is a differential form in M and $f(x,t)$ has compact support. The map $\pi_* : \Omega_c^\bullet(E) \to \Omega^{\bullet - r}(M)$ is zero on forms of the first kind, and on forms of the second kind it is integration along the fibers:

$$f(x,t)(\pi^*\theta) \wedge dt^1 \wedge \cdots \wedge dt^r \longmapsto \theta \int_{\mathbb{R}^r} f(x, t^1, \dots, t^r) dt^1 \cdots dt^r.$$

Since two such system of coordinates in the fibers, say $(t^1, \dots, t^r)$ and $(\bar{t}^1, \dots, \bar{t}^r)$, are related by an element of $GL(r)^+$, we obtain $\pi_*\omega_\alpha = \pi_*\omega_\beta$, whenever $U_\alpha \cap U_\beta \neq \emptyset$.

With this explicit description, it is now easy to check that:

Proposition 24.5. *If $\pi_* : \Omega_c^\bullet(E) \to \Omega^{\bullet - r}(M)$ denotes integration along the fibers, then:*

(i) π_* is a cochain map:

$$\mathrm{d}\pi_* = \pi_*\mathrm{d};$$

(ii) for any $\theta \in \Omega^*(M)$ and $\omega \in \Omega_c^\bullet(E)$ the following ***projection formula*** *holds:*

$$\pi_*(\pi^*\theta \wedge \omega) = \theta \wedge \pi_*\omega. \tag{24.1}$$

Remark 24.6. The explicit description of integration along the fibers also shows how one should proceed when one has a vector bundle over a non-comptac manifold. One considers compactly supported forms in the vertical direction: the complex $\Omega_{cv}^*(E)$ is defined by differential forms ω in E such that $\operatorname{supp}\omega \cap \pi^{-1}(K)$ is compact for every compact set $K \subset M$. Hence, the restriction of ω to each fiber E_p has compact support and we can still define integration along the fibers by the same formula. If one assumes that E is orientable, we obtain Thom duality:

$$H_{cv}^\bullet(E) \simeq H^{\bullet - r}(M).$$

In order to simplify the presentation, we will consider only Thom duality in the case where M is compact.

Let M be an oriented compact, connected, manifold. Let $d = \dim M$ and denote the orientation by μ. We know that there exists a canonical generator in cohomology which we also denote by $\mu \in H_c^d(M)$: the class μ is represented by any top degree form $\omega \in \Omega_c^d(M)$ such that:

$$\int_M \omega = 1.$$

In fact μ is the image of 1 under Poincaré duality $H^0(M) \simeq H_c^d(M)$. On the other hand, we can use Thom duality in much the same way to obtain a canonical class in $H_c^r(E)$:

Definition 24.7. *The **Thom class** of an oriented vector bundle $\xi = (\pi, E, M)$ over a compact, connected, oriented manifold M is the image of 1 under Thom duality $H^0(M) \simeq H_c^r(E)$. We will denote this class by $U \in H_c^r(E)$.*

The Thom class allows one to write, in a more or less explicit way, the inverse to the integration along fibers $\pi_* : H_c^\bullet(E) \to H^{\bullet - r}(M)$. In fact, since $\pi_* U = 1$, the projection formula (24.1) shows that the linear map $H^\bullet(M) \to H_c^{\bullet + r}(E)$ defined map por:

$$(\pi_*)^{-1}([\omega]) = [\pi^* \omega] \cup U.$$

is an inverse to π_*.

The following result gives an alternative characterization of the Thom class:

Theorem 24.8. *The Thom class of an oriented vector bundle $\xi = (\pi, E, M)$ over an oriented, compact, connected manifold is the unique class $U \in H_c^r(E)$ whose pull-back to each fiber E_p is the canonical generator of $H_c^r(E_p)$, i.e.,*

$$\int_{E_p} i^* U = 1, \quad \forall p \in M,$$

where $i : E_p \hookrightarrow E$ is the inclusion.

Proof. Since $\pi_* U = 1$, we see that the restriction $i^* U$ to each fiber E_p is a compactly supported form with $\int_{E_c} i^* U = 1$.

Conversely, let $U' \in H_c^r(E)$ be a class such the restriction $i^* U' \in H_c^r(E_p)$ is the canonical generator, for each $p \in M$. By the projection formula (24.1), we obtain

$$\pi_*(\pi^* \theta \wedge U') = \theta \wedge \pi_* U' = \theta, \quad \forall \theta \in H^\bullet(M).$$

Hence, $\theta \mapsto \pi^* \theta \wedge U'$ inverts π_*, and the image of 1, which is U', coincides with the Thom class. $\square$

The Thom class of a vector bundle $\xi = (\pi, E, M)$ is an invariant of the bundle, but it lies in the cohomology of the total space. We can use a global section to obtain an invariant which lies in the cohomology of the base:

Definition 24.9. *Let $\xi = (\pi, E, M)$ be an oriented vector bundle of rank r over an oriented, compact, connected manifold M. The* ***Euler class*** *of ξ is the class $\chi(\xi) \in H^r(M)$ defined by:*

$$\chi(\xi) \equiv s^* U,$$

where U is the Thom class of ξ and $s : M \to E$ is any *section global of ξ.*

Note that a vector bundle always as a global section, namely the zero section. On the other hand, if $s_0, s_1 : M \to E$ are two global sections then $H(p,t) = ts_1(p) + (1-t)s_0(p)$ is an homotopy between them, so we have $[s_0^* U] = [s_1^* U]$. This shows that the definition above of the Euler class makes sense.

The following proposition lists some properties of the Euler class. We leave its proof for the exercises:

Proposition 24.10. *Let $\xi = (\pi, E, M)$ be an oriented vector bundle of rank r over an oriented, compact, connected manifold M. Then:*

(i) If $\Psi : \eta \to \xi$ is a morphism of vector bundles of rank r, preserving orientations, which covers a map $\psi : N \to M$, then: $\chi(\eta) = \psi^* \chi(\xi)$.
(ii) If $\bar{\xi}$ denotes the vector bundle ξ with the opposite orientation then $\chi(\bar{\xi}) = -\chi(\xi)$.
(iii) If rank r is odd, then $\chi(\xi) = 0$.
(iv) If $\xi' = (\pi', E', M)$ is another oriented vector bundle of rank r' over M, then $\chi(\xi \oplus \xi') = \chi(\xi) \cup \chi(\xi')$.

The Euler class of a vector bundle is *an obstruction* to the existence of a non-vanishing global section. In fact, we have:

Theorem 24.11. *Let $\xi = (\pi, E, M)$ be an oriented vector bundle of rank r over an oriented, compact, connected manifold M. If ξ admits a non-vanishing section then $\chi(\xi) = 0$.*

Proof. Let $s : M \to E$ be a non-vanishing section. If $\omega \in \Omega_c^r(E)$ is a compactly supported form representing the Thom class, i.e., $U = [\omega]$, then there exists a $c \in \mathbb{R}$ such that the image of the section cs does not intersect $\operatorname{supp} \omega$. Hence:

$$\chi(\xi) = (cs)^* U = [(cs)^* \omega] = 0.$$

□

Note, however, that in general the Euler class is not the only obstruction to the existence of a non-vanishing global section: there are examples of vector bundles ξ with $\chi(\xi) = 0$, but where any global section has a zero.

The name *Euler class* is related with the special case where $\xi = TM$. Recalling the notion of index of an isolated zero of a vector field from Lecture 22), we have:

Theorem 24.12. *Let M an oriented, compact, connected manifold of dimension d. For any vector field $X \in \mathfrak{X}(M)$ with a finite number of zeros $\{p_1, \ldots, p_N\}$, one has:*

$$\chi(TM) = \left(\sum_{i=1}^{N} \operatorname{ind}_{p_i} X \right) \mu \in H^d(M),$$

where $\mu \in H^d(M)$ the class defined by the orientation of M.

Proof. Let $\omega \in \Omega_c^d(TM)$ be a compactly supported form representing the Thom class. Denote the sum of the indices of the zeros by σ. We need to show that:

$$\int_M X^*\omega = \sigma.$$

Choose coordinate systems (U_i, ϕ_i) centered at p_i and denote by B_i the closed balls:

$$B_i = \phi_i^{-1}(\{x \in \mathbb{R}^d : ||x|| < 1\}).$$

We can assume that $X_p \notin \operatorname{supp} \omega$, for all $p \notin \bigcup_{i=1}^N B_i$. Therefore, it is enough to verify that:

$$\int_{B_i} X^*\omega = \operatorname{ind}_{p_i} X.$$

and we leaved this check to the exercises. □

An immediate corollary is:

Corollary 24.13. *Let X and Y be vector fields with a finite number of zeros on an oriented, compact, connected manifold M. The sum of the indices of the zeros of X coincides with the sum of the indices of the zeros of Y.*

We must have already guessed that we have:

Theorem 24.14 (Poincaré-Hopf)**.** *Let M be an oriented, compact, connected manifold of dimension d. Then for any vector field $X \in \mathfrak{X}(M)$ with a finite number of zeros $\{p_1, \ldots, p_N\}$, we have:*

$$\chi(M) = \sum_{i=1}^{N} \operatorname{ind}_{p_i} X.$$

In particular, $\chi(TM) = \chi(M)\mu$, where $\mu \in H^d(M)$ is the orientation class.

Proof. By the corollary above it is enough to construct a vector field X in M, with a finite number of zeros, for which the equality holds. For that, we fix a triangulation $\{\sigma_1, \ldots, \sigma_{r_d}\}$ of M, and we construct a vector field X with the following properties:

(a) X has exactly one zero p_i in each face of the triangulation.
(b) The zero p_i is non-degenerate and $\operatorname{ind}_{p_i} X = (-1)^k$, where k is the dimension of the face.

Hence, if r_k is the number of faces of dimension k, we have

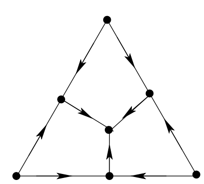
$$\sum_{i=1}^{N} \operatorname{ind}_{p_i} X = r_0 - r_1 + \cdots + (-1)^d r_d,$$

so the result follows from Euler's Formula (Theorem 21.5).

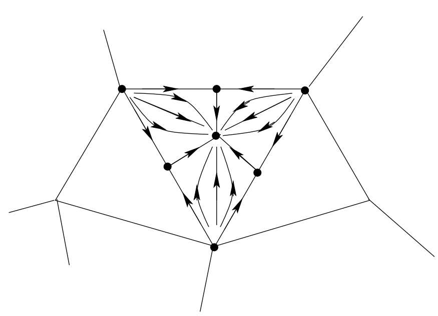
We construct X describing its phase portrait in each face:

- In each face of dimension 0, the vector field X has a zero.
- In each face of dimension 1, we put a zero in the center of the face and connect it by orbits to the zeros in the vertices, as in the following figure:

- In each face of dimension 2, we put a zero in the center of the face and connect it by heteroclinic orbits to the zeros in the faces of dimension 1, as in the following figure:



  Then we complete the phase portrait of X in the face of dimension 2, so that the zero in its interior becomes an attractor of the vector field restricted to the face:



- In general, once one has constructed the phase portrait in the faces of dimension $k-1$, we construct the phase portrait in a face of dimension k, putting a zero in the center of the face and connecting it by heteroclinic orbits to the zeros in the faces of dimension $k-1$. We then complete the phase portrait so that the new zero is an attractor of the vector field restricted to the face of dimension k.

The vector field one constructs in this way has exactly one zero in each face. Moreover, we can assume that they are non-degenerate zeros. For a zero p_i in the face of dimension k, the linearization of the vector field at p_i is a real matrix with k eingenvalues with negative real part, corresponding to the directions along the face, and $n-k$ eingenvalues with positive real part, corresponding to the directions normal to the face. The determinant of this matrix if $(-1)^k$. Hence, we have that $\operatorname{ind}_{p_i} X = (-1)^k$, so the vector field X satisfies (a) and (b). □

Remark 24.15. We remarked above that there exists vector bundles with $\chi(\xi) = 0$, but where every section has a zero. However, in the case of the tangent bundle the Euler class $\chi(TM)$ (and hence the Euler number $\chi(M)$) is the only obstruction: an exercise in this section shows that $\chi(TM) = 0$ if and only if é there exists a non-vanishing vector field in M.

HOMEWORK.

1. Let $\xi = (\pi, E, M)$ be an orientable vector bundle over a compact manifold. Show that the projection formula holds:
$$\pi_*(\pi^*\theta \wedge \omega) = \theta \wedge \pi_*\omega, \quad (\theta \in \Omega^*(M),\ \omega \in \Omega^\bullet_c(E)).$$

2. Let $E_1 \to M$ and $E_2 \to M$ be oriented vector bundles over an oriented, compact, connected manifold M. Consider their Whitney sum and the natural projections:
$$\begin{tikzcd} & E_1 \oplus E_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ E_1 & & E_2 \end{tikzcd}$$
Show that the Thom classes of E_1, E_2 and $E_1 \oplus E_2$ are related by:
$$U_{E_1 \oplus E_2} = \pi_1^* U_{E_1} \wedge \pi_2^* U_{E_2}.$$

3. Let $\xi = (\pi, E, M)$ and $\xi' = (\pi', E', M')$ be oriented vector bundles over an oriented, compact, connected manifold M. Show that:
$$\chi(\xi \oplus \xi') = \chi(\xi) \cup \chi(\xi'),$$
where on the Whitney sum $\xi \oplus \xi'$ we take the direct sum of the orientations.

4. Let $\xi = (\pi, E, M)$ and $\eta = (\tau, F, N)$ be oriented vector bundles of rank r, where M and N are oriented, compact, connected manifolds. If $\Psi : \eta \to \xi$ is a vector bundle morphism which preserves orientations, covering a map $\psi : N \to M$, show that:
$$\chi(\eta) = \psi^* \chi(\xi).$$
Use this property to conclude that:
   (a) If $\bar{\xi}$ denotes the vector bundle ξ with the opposite orientation then $\chi(\bar{\xi}) = -\chi(\xi)$.
   (b) If $\operatorname{rank} \xi$ is odd, then $\chi(\xi) = 0$.

5. Complete the proof of Theorem 24.12.

6. Let M be a compact manifold of dimension d. One can show that:
   (a) If $p_1, \dots, p_N \in M$ there exists an open set $U \subset M$, diffeomorphic to the ball $\{x \in \mathbb{R}^d : ||x|| < 1\}$, such that $p_1, \dots, p_n \in U$.
   (b) If $\psi : \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ is a map with degree zero, then it is homotopic to the constant map.

   Use these facts to show that if $\chi(M) = 0$, then there exists a nowhere vanishing vector field in M.

Lecture 25. Pull-backs and the Classification of Vector Bundles

We now turn to the global characterization of vector bundles. In this characterization the *pull-back* of vector bundles under a smooth map plays a crucial role.

Definition 25.1. *Let $\psi : M \to N$ be a smooth map and $\xi = (\pi, E, N)$ a vector bundle over N of rank r. The **pull-back** of ξ by ψ is the vector bundle $\psi^*\xi = (\hat{\pi}, \psi^*E, M)$ of rank r, with total space given by:*

$$\psi^*E = \{(p, \mathbf{v}) \in M \times E : \psi(p) = \pi(\mathbf{v})\},$$

and projection defined by:

$$\hat{\pi} : \psi^*E \to N, \ (p, \mathbf{v}) \mapsto p.$$

Note that the fiber of $\psi^*\xi$ over p is a copy of the fiber of ξ over $\psi(p)$. Therefore the pull-back of ξ by ψ is a vector bundle for which we take a copy of the fiber of ξ over q for each point in the preimage $\psi^{-1}(q)$.

We still need to check that the construction in the definition above does indeed produce a vector bundle. First of all, note that

$$\psi^*E = (\psi \times \pi)(\Delta),$$

where $\Delta \subset N \times N$ is the diagonal. Since $\pi : E \to N$ is a submersion, we have that $(\psi \times \pi) \pitchfork \Delta$, so E is a manifold. To cheek local triviality of $\psi^*\xi$, let $\{\phi_\alpha\}$ be a trivialization of ξ, subordinated to the open cover $\{U_\alpha\}$ of N. We obtain a trivialization $\{\widetilde{\phi}_\alpha\}$ for $\psi^*\xi$, subordinated to the open cover $\{\psi^{-1}(U_\alpha)\}$ of M, where

$$\begin{aligned} \widetilde{\phi}_\alpha : \hat{\pi}^{-1}(\psi^{-1}(U_\alpha)) &\to \psi^{-1}(U_\alpha) \times \mathbb{R}^r \\ (p, \mathbf{v}) &\longmapsto (p, \phi_\alpha^{\psi(p)}(\mathbf{v})). \end{aligned}$$

Moreover, if $\{g_{\alpha\beta}\}$ is the cocycle of ξ associated with the trivialization $\{\phi_\alpha\}$, then $\{\psi^* g_{\alpha\beta}\} = \{g_{\alpha\beta} \circ \psi\}$ is the cocycle of $\psi^*\xi$ associated with the trivialization $\{\widetilde{\phi}_\alpha\}$.

Let us notice now that the map

$$\Psi : \psi^*\xi \to \xi \ (p, \mathbf{v}) \mapsto \mathbf{v},$$

is a morphism of vector bundles covering ψ. Hence, the pull-back construction allows us to complete the following commutative diagram of morphisms of vector bundles:

$$\begin{tikzcd}
\psi^* E \arrow[r, dashed, "\Psi"] \arrow[d, dashed, "\hat{\pi}"'] & E \arrow[d, "\pi"] \\
M \arrow[r, "\psi"] & N
\end{tikzcd}$$

In fact, we have the following universal property which characterizes the pull-back up to isomorphism:

Proposition 25.2. *Let $\psi : M \to N$ be a smooth map, $\eta = (\tau, F, M)$ and $\xi = (\pi, E, N)$ vector bundles and $\Phi : \eta \to \xi$ a morphism of vector bundles covering ψ. Then there exists a unique morphism of vector bundles $\tilde{\Phi} : \eta \to \psi^*\xi$, covering the identity, which makes the following diagram commutative:*

$$\begin{tikzcd}
F \arrow[dr, dashed, "\tilde{\Phi}"] \arrow[drr, bend left, "\Phi"] \arrow[ddr, bend right, "\tau"'] & & \\
& \psi^* E \arrow[r, "\Psi"] \arrow[d, "\hat{\pi}"'] & E \arrow[d, "\pi"] \\
& M \arrow[r, "\psi"'] & N
\end{tikzcd}$$

Moreover, $\tilde{\Phi}$ is an isomorphism if an only if $\Phi^p : F_p \to E_{\psi(p)}$ is an isomorphism for all $p \in M$.

Let $\xi = (\pi, E, N)$ and $\eta = (\tau, F, N)$ be vector bundles over N, and let $\Phi : \xi \to \eta$ be a morphism of vector bundles covering the identity. If $\psi : M \to N$ is a smooth map, then we have a morphism of vector bundles $\psi^*(\Phi) : \psi^*\xi \to \psi^*\eta$, defined by:

$$\psi^*(\Phi)(p, \mathbf{v}) = (p, \Phi(\mathbf{v})).$$

Obviously, this morphism makes the following diagram commute:

$$\begin{tikzcd}
\psi^* E \arrow[rr, "\psi^*(\Phi)"] \arrow[dr] \arrow[dd] & & \psi^* F \arrow[dr] \arrow[dd] & \\
& E \arrow[rr, "\Phi"] \arrow[dd] & & F \arrow[dd] \\
M \arrow[rr] \arrow[dr, "\psi"'] & & M \arrow[dr, "\psi"'] & \\
& N \arrow[rr] & & N
\end{tikzcd}$$

We can now list the main properties of the pull-back of vector bundles:

Proposition 25.3. *Let $\psi : M \to N$ be a smooth map. Then:*

(i) The pull-back of the trivial vector bundle is the trivial vector bundle: $\psi^*(\varepsilon^r_N) = \varepsilon^r_M$.
(ii) If $\phi : Q \to M$ is a smooth map, then $(\psi \circ \phi)^*\xi = \phi^*(\psi^*\xi)$, for any vector bundle ξ over N.
(iii) The pull-back of the identity morphism is the identity: $\psi^*(id_\xi) = id_{\psi^*\xi}$.
(iv) If $\Phi : \xi \to \eta$ and $\Psi : \eta \to \theta$ are morphisms of vector bundles over the identity, then $\phi^*(\Psi \circ \Phi) = \phi^*(\Psi) \circ \phi^*(\Phi)$.

The results above show that if we fix manifolds M and N, as well as a smooth map $\psi : M \to N$, then we have:

- The pull-back defines a covariant functor from the category of vector bundles over N to the category of vector bundles over M.

Let us denote by $\mathrm{Vect}_r(M)$ the set of isomorphism classes of vector bundles of rank r over a manifold M. There is a distinguish point in $\mathrm{Vect}_r(M)$, namely the class of the the trivial vector bundles. Given a smooth map $\psi : M \to N$, the pull-back $\psi^* : \mathrm{Vect}_r(N) \to \mathrm{Vect}_r(M)$ preserves this distinguished point, so we conclude also:

- The pull-back defines a contravariant functor from the category of smooth manifolds to the category of sets with a distinguished point.

The fundamental property of the pull-back of vector bundles is the following:

Theorem 25.4 (Homotopy invariance). *If ψ and $\phi : M \to N$ are homotopic maps and ξ is a vector bundle over N, then the pull-backs $\psi^*\xi$ and $\phi^*\xi$ are isomorphic vector bundles.*

Proof. Let $H : M \times [0,1] \to N$ be an homotopy between ϕ and ψ. We have:

$$\phi^*\xi = H_0^*\xi = H^*\xi|_{M\times\{0\}},$$
$$\psi^*\xi = H_1^*\xi = H^*\xi|_{M\times\{1\}}.$$

Hence, it is enough to show that for any vector bundle η over $M \times [0,1]$, the restrictions $\eta|_{M\times\{0\}}$ and $\eta|_{M\times\{1\}}$ are isomorphic.

One can show that:

(a) a morphism of vector bundles of class C^0, covering a map of class C^∞, can be approximated by a morphism of classe C^∞, which covers the same map.
(b) a morphism which is close enough to an isomorphism, is also an isomorphism.

Hence, it is enough to proof that for any vector bundle $\eta = (\pi, E, M \times [0,1])$, there exists a C^0-morphism of vector bundles $\Delta : \eta \to \eta$, covering the smooth map

$$\delta : M \times [0,1] \to M \times [0,1], \ (p,t) \mapsto (t,1),$$

and such that the induced maps in the fibers are isomorphisms. In order to construct Δ, we will use the following lemma, whose proof is left as an exercise:

Lemma 25.5. *Let η be a vector bundle over $M \times [0,1]$. There exists an open cover $\{U_\alpha\}_{\alpha\in A}$ of M such that the restrictions $\eta|_{U_\alpha\times[0,1]}$ are trivial vector bundles.*

Now choose a locally finite countable open cover $\{U_k\}_{k\in\mathbb{N}}$ of M such that the restrictions $\eta|_{U_k\times[0,1]}$ are trivial. Let us denote the trivializing maps ϕ_k by:

$$\begin{tikzcd} E|_{U_k\times[0,1]} \arrow[rr, "\phi_k"] \arrow[dr, "\pi"'] & & (U_k\times[0,1])\times\mathbb{R}^r \arrow[dl, "\pi_1"] \\ & U_k\times[0,1] & \end{tikzcd}$$

Denote by $\{\rho_k\}_{k\in\mathbb{N}}$ an *envelope of unity* subordinated to the cover $\{U_k\}_{k\in Nn}$, i.e., a collection of continuous maps $\rho_k : M \to \mathbb{R}$ such that $0 \le \rho_k \le 1$, $\operatorname{supp}\rho_k \subset U_k$ and, for all $p \in M$,

$$\max\{\rho_k(p) : k \in \mathbb{N}\} = 1.$$

Such an envelope of unity can be constructed starting with a partition of unity $\{\theta_k\}$ and defining:

$$\rho_k(p) \equiv \frac{\theta_k(p)}{\max\{\theta_k(p) : k \in \mathbb{N}\}}.$$

For each $k \in \mathbb{N}$ we define vector bundle morphisms $\Delta_k : \eta \to \eta$ by:

(a) Δ_k cover the map $\delta_k : M \times [0,1] \to M \times [0,1]$ given by:

$$\delta_k(p,t) = (p, \max(\rho_k(p), t)).$$

(b) In $\pi^{-1}(U_k \times [0,1])$, Δ_k is defined by:

$$\Delta_k(\phi_k^{-1}(p,t,\mathbf{v})) \equiv \phi_k^{-1}(p, \max(\rho_k(p), t), v),$$

and Δ_k is the identity outside $\pi^{-1}(U_k \times [0,1])$.

Finally, one defines $\Delta : \eta \to \eta$ by:

$$\Delta = \cdots \circ \Delta_k \circ \cdots \circ \Delta_1.$$

Since each $p \in M$ has a neighborhood which intersects a finite number of open sets U_k, this is a well-defined vector bundle morphism $\Delta : \eta \to \eta$ which locally is an the composition of vector bundle which isomorphisms on the fibers. Hence, Δ is a vector bundle isomorphism which covers $\delta : M \times [0,1] \to M \times [0,1]$. $\square$

Corollary 25.6. *Any vector bundle over a contractible manifold is trivial.*

Proof. Let $\xi = (\pi, E, M)$ be a vector bundle and let $\phi : M \to \{*\}$ and $\psi : \{*\} \to M$ be smooth maps such that $\psi \circ \phi$ is homotopic to id_M. The Theorem shows that:

$$\xi \simeq (\psi \circ \phi)^* \xi \simeq \phi^*(\psi^* \xi).$$

Since $\psi^*\xi$ is a vector bundle over a set which consist of a single point, it is the trivial vector bundle. Hence $\xi \simeq \phi^*(\psi^*\xi)$ is a trivial vector bundle. $\square$

Hence, when M is contractible the space $\mathrm{Vect}_r(M)$ consisting of isomorphism classes of vector bundles of rank r over M has only one point.

EXAMPLE 25.7.

Let $M = \mathbb{S}^1$. Given a line bundle $\xi = (\pi, E, \mathbb{S}^1)$, we can cover $\mathbb{S}^1$ by the two contractible open sets $U = \mathbb{S}^1 - \{p_N\}$ and $V = \mathbb{S}^1 - \{p_S\}$. By the corollary, over each open set U and V the vector bundle trivializes: $\phi_U : E|_U \simeq U \times \mathbb{R}$ and $\phi_V : E|_V \simeq V \times \mathbb{R}$. Therefore, the line bundle is completely characterized by the transition function $g_{UV} : U \cap V \to \mathbb{R}$, so that:

$$\phi_V \circ \phi_U^{-1} : U \times \mathbb{R} \to V \times \mathbb{R}, \ (p, v) \mapsto (p, g_{UV}(p)v).$$

The intersection $U \cap V$ has two connected components, and we leave it as an exercise to check that if $g_{UV}(x)$ has the same sign in both components, then ξ is trivial, while if $g_{UV}(x)$ has the opposite signs in the two components then the line bundle is isomorphic to the line bundle whose total space is the Möbis band. In other words, the the space $\mathrm{Vect}_1(\mathbb{S}^1)$ *consisting of isomorphism classes of line bundles over $\mathbb{S}^1$ has two points.*

The problem of determining $\mathrm{Vect}_k(M)$ can be reduced to a problem in homotopy theory, as we now briefly indicate.

Recall that γ_n^r denotes the **canonical bundle** over the Grassmannian $G_r(\mathbb{R}^n)$ (Lecture 23, Exercise 2): the total space of γ_n^r is defined by:

$$E = \{(S, x) : S \text{ is } r\text{-dimensional subspace of } \mathbb{R}^n \text{ and } x \in S\},$$

and the projection $\pi : E \to \mathcal{G}_r(\mathbb{R}^n)$ is given by $\pi(S, x) = S$. The canonical bundle is a subbundle of the trivial vector bundle $\varepsilon^n_{G_r(\mathbb{R}^n)}$. The **universal bundle** is the quotient vector bundle obtained from the short exact sequence of vector bundles:

$$0 \longrightarrow \gamma_n^{n-r} \longrightarrow \varepsilon^n_{G_{n-r}(\mathbb{R}^n)} \longrightarrow \eta_n^r \longrightarrow 0$$

Note that the fiber of η_n^r over the $n-r$-dimensional subspace $S \in G_{n-r}(\mathbb{R}^n)$ is the quotient R^n/S. The reason for the name *universal* is justified by the following proposition:

Proposition 25.8. *Let M be a smooth manifold and ξ a rank r vector bundle over M. If ξ admits n global sections $s_1, \dots, s_n$ which generate E_p for all $p \in M$, then there exists a smooth map $\psi : M \to G_{n-r}(\mathbb{R}^n)$ such that:*

$$\xi \simeq \psi^*(\eta_n^r).$$

Proof. Let V be the n-dimensional vector space with basis $\{s_1, \ldots, s_n\}$. Since the sections s_i generate E_p, for each $p \in M$, there exists a linear surjective map

$$V \xrightarrow{\mathrm{ev}_p} E_p \longrightarrow 0 \ .$$

The kernel $\operatorname{Ker} \mathrm{ev}_p$ of this map is a subspace of V of codimension r. On the other hand, the fiber of the universal bundle η_n^r over the Grassmannian $G_{n-r}(V)$ is $V/\operatorname{Ker} \mathrm{ev}_p \simeq E_p$. Hence, if we define a smooth map by

$$\psi : M \to G_{n-r}(V),\ p \mapsto \operatorname{Ker} \mathrm{ev}_p,$$

then $\xi \simeq \psi^*(\eta_n^r)$. Now, if we identify V with $\mathbb{R}^n$ and $G_{n-r}(V)$ with $G_{n-r}(\mathbb{R}^n)$, the result follows. □

A map $\psi : M \to G_{n-r}(\mathbb{R}^n)$ such that $\xi \simeq \psi^*\eta_n^r$ is called a **classifying map** for the vector bundle ξ.

Proposition 25.9. *Let M be a manifold which admits a good cover with k open sets and let ξ be a vector bundle over M of rank r. If*

$$n \geq \max\{rk, \dim M + r - 1\},$$

then:

(i) There exist classifying maps $\psi : M \to G_{n-r}(\mathbb{R}^n)$ for ξ;
(ii) Any two classifying maps are homotopic.

Proof. Let $U_1, \ldots, U_k$ be a good cover of M. We claim that ξ admits $n = rk$ global sections $s_1, \ldots, s_n$ which generate E_p, for all $p \in M$, so that (i) follows from Proposition 25.8. To see this, we observe that since each U_i is contractible, the restriction $\xi|_{U_\alpha}$ is trivial. Hence, we can choose a basis of local sections $\{s_1^\alpha, \ldots, s_r^\alpha\}$ for $\Gamma(\xi|_{U_\alpha})$. Therefore, if we choose a partition of unity $\{\rho_\alpha\}$ subordinated to the cover $\{U_\alpha\}$, we can define the rk global sections $\tilde{s}_i^\alpha := \rho_\alpha s_i^\alpha$, which generate E_p, for all $p \in M$.

Let us sketch a proof of (ii). If $\psi : M \to G_{n-r}(\mathbb{R}^n)$ is a classifying map, the isomorphism $\Psi : \xi \simeq \psi^*\eta_n^r$ gives a vector bundle map from the trivial vector bundle to ξ, covering the identity:

$$\begin{tikzcd}
M \times \mathbb{R}^n \arrow[rr, "\hat{\Psi}"] \arrow[dr] & & E \arrow[dl] \\
& M &
\end{tikzcd}$$

which is surjective on the fibers. Conversely, given such an epimorphism $\hat{\Psi}$ we can associate to it the classifying map:

$$\psi : M \to G_{n-r}(\mathbb{R}^n), \quad p \mapsto \operatorname{Ker} \hat{\Psi}_p.$$

Therefore, to give a classifying map $\psi : M \to G_{n-r}(\mathbb{R}^n)$ is the same as giving a section $\hat{\Psi}$ of the vector bundle $\mathrm{End}(M \times \mathbb{R}^n, E)$, which takes values in the submanifold $\mathrm{Epi}(M \times \mathbb{R}^n, E)$ consisting of linear maps bewteen the fibers which are surjective. Hence, assume we are given two sets of such sections

$\hat{\Psi}_0$ and $\hat{\Psi}_0$, yielding classifying maps $\psi_0, \psi_1 : M \to G_{n-r}(\mathbb{R}^n)$. We obtain a section $\hat{\Psi}$ of the bundle $\mathrm{End}(M \times \mathbb{R}^n, E)$, defined in $M \times \{0,1\}$, and which takes values in $\mathrm{Epi}(M \times \mathbb{R}^n, E)$. We need to show that we can extend $\hat{\Psi}$ to section in $M \times [0,1]$, which takes values in $\mathrm{Epi}(M \times \mathbb{R}^n, E)$. The associated family of classifying maps define the desired homotopy $\phi_t : M \to G_{n-r}(\mathbb{R}^n)$.

Of course, there is no problem in extending the section $\hat{\Psi}$ to a section of $\mathrm{End}(M \times \mathbb{R}^n, E)$, defined in $M \times [0,1]$. If this section is transverse to the submanifold consisting of linear maps between the fibers whose rank is less than r, then we can perturb it to a section extending the original section, and which takes values in $\mathrm{Epi}(M \times \mathbb{R}^n, E)$. Since the codimension of the submanifold consisting of linear maps between the fibers whose rank is less than r is $n - r + 1$, it is enough to assume that:

$$\dim M < n - r + 1 \Longleftrightarrow n > \dim M + r - 1.$$

□

Denote by $[M, N]$ the set of homotopy classes of maps $\phi : M \to N$. We obtain:

Theorem 25.10 (Classification of vector bundles)**.** *Let M be a manifold which admits a good open cover with k open sets. For every $n \geq rk$, there exists a bijection:*

$$\mathrm{Vect}_r(M) \simeq [M, G_{n-r}(\mathbb{R}^n)].$$

Proof. We saw above that the homotopy class of a classifying map for ξ is determined by the isomorphism class of ξ, so we have a well-defined map:

$$f : \mathrm{Vect}_r(M) \to [M : G_{n-r}(\mathbb{R}^n)].$$

On the other hand, by the homotopy invariance of the pull-backs, we conclude that the pull-back of the universal bundle induces a map

$$g : [M : G_{n-r}(\mathbb{R}^n)] \to \mathrm{Vect}_r(M), \; \psi \mapsto \psi^* \eta_n^r.$$

We leave as an exercise to show that the maps f and g are inverse to each other, so the result follows. □

This result reduces the classification of vector bundles to a homotopy issue. We illustrate this in the next example, which assumes some knowledge of homotopy theory.

Example 25.11.

Let us recall that if X is a path connected topological space then the free homotopies and the homotopies based at $x_0 \in X$ are related by:

$$\pi_k(X, x)/\pi_1(X, x) \simeq [\mathbb{S}^k, X],$$

where the quotient is the orbit space for the natural action of $\pi_1(X, x)$ in $\pi_k(X, x)$. Therefore, we have:

$$\mathrm{Vect}_r(\mathbb{S}^k) = [\mathbb{S}^k, G_{n-r}(\mathbb{R}^n)] \simeq \pi_k(G_{n-r}(\mathbb{R}^n))/\pi_1(G_{n-r}(\mathbb{R}^n)),$$

for n large enough. On the other, since the Grassmannian is a homogeneous space:

$$G_{n-r}(\mathbb{R}^n) = O(n)/(O(n-r) \times O(r)),$$

and $\pi_k(O(n)/O(n-r)) = 0$, if n is large enough, the long exact sequence in homotopy yields:

$$\pi_k(G_{n-r}(\mathbb{R}^n)) = \pi_{k-1}(O(r)),$$

for n large enough. Hence, we conclude that:

$$\mathrm{Vect}_r(\mathbb{S}^k) = \pi_{k-1}(O(r))/\pi_0(O(r)) = \pi_{k-1}(O(r))/\mathbb{Z}_2.$$

In order to understand this quotient, we need to figure out the action of $\pi_0(O(r))$ on $\pi_{k-1}(O(r))$. If $g \in O(r)$, the action by conjugation of g in $O(r)$ induces an action in homotopy:

$$i_g : O(n) \to O(n),\ i_g(h) = ghg^{-1} \implies (i_g)_* : \pi_k(O(r)) \to \pi_k(O(r)).$$

If g_1 and g_2 being to the same connected component, then $(i_{g_1})_* = (i_{g_2})_*$. Hence, we obtain an action of $\pi_0(O(r)) = \mathbb{Z}_2$ on $\pi_{k-1}(O(r))$, which is precisely the action above.

For example, if r is odd then $-I$ represents the non-trivial class in $\pi_0(O_r)$. Since the action by conjugation of $-I$ is trivial, we conclude that

$$\mathrm{Vect}_r(\mathbb{S}^k) = \pi_{k-1}(O(r)), \text{ if } r \text{ is odd.}$$

For example, we have:

$$\mathrm{Vect}_3(\mathbb{S}^4) = \pi_3(SO(3)) = \pi_3(\mathbb{S}^3) = \mathbb{Z}.$$

On the other hand, when r is even, the action maybe non-trivial. Take for example $r = 2$, so we have $\pi_1(O(2)) = \mathbb{Z}$. The action of $\pi_0(O_2) = \mathbb{Z}_2$ in $\mathbb{Z}$ is just $\pm 1 \cdot n = \pm n$. Hence, we have

$$\mathrm{Vect}_2(\mathbb{S}^k) = \pi_{k-1}(O(2))/\mathbb{Z}_2 = \pi_{k-1}(\mathbb{S}^1)/\mathbb{Z}_2 = \begin{cases} \mathbb{Z}/\mathbb{Z}_2 & \text{if } k = 2, \\ 0 & \text{if } k \geq 3. \end{cases}$$

Remark 25.12. If a manifold is not of finite type, there still exists a classification of vector bundles over M. In this case, we need to consider the the space:

$$\mathbb{R}^\infty = \bigoplus_{d=0}^{\infty} \mathbb{R}^d,$$

which is the direct limit of the increasing sequence of vector spaces:

$$\cdots \subset \mathbb{R}^d \subset \mathbb{R}^{d+1} \subset \mathbb{R}^{d+2} \subset \cdots$$

In $\mathbb{R}^\infty$, we can consider the set of subspaces of dimension r, i.e., the Grassmannian:

$$\tilde{G}_r(\mathbb{R}^\infty) = G_{\infty-r}(\mathbb{R}^\infty) = \{S \subset \mathbb{R}^\infty : \text{ subspace of codimension } r\}.$$

Over this infinite Grassmannian there is a vector bundle $\eta^r_\infty = (\pi, E, \tilde{G}_r(\mathbb{R}^\infty))$, called the **universal bundle of rank** r. It has total space:

$$E = \{(S, x) : S \subset \mathbb{R}^\infty \text{ subspace of codimension } r,\ x \in \mathbb{R}^\infty/S\},$$

and projection:

$$\pi : E \to \tilde{G}_r(\mathbb{R}^\infty),\ (S, x) \mapsto S.$$

One can show that a vector bundle of rank r over a manifold M is isomorphic to the pull-back $\psi^*\eta^r_\infty$, for some classifying map $\psi : M \to \tilde{G}_r(\mathbb{R}^\infty)$. Hence, for *any* manifold M, we have a bijection:

$$\mathrm{Vect}_r(M) \simeq [M, \tilde{G}_r(\mathbb{R}^\infty)].$$

Homework.

1. Give a proof of the universal property of pull-backs (Proposition 25.2). Show that this property characterizes the pull-back of vector bundles up to isomorphism.

2. Verify the properties of the pull-back of vector bundles given by Proposition 25.3.

3. Let ξ be a vector bundle over $M \times [0, 1]$. Show that there exists an open cover $\{U_\alpha\}_{\alpha\in A}$ of M such that the restrictions $\xi|_{U_\alpha\times[0,1]}$ are trivial.
Hint: Show that if ξ is a vector bundle over $M \times [a, c]$ which is trivial when restricted to both $M \times [a, b]$ and $M \times [b, c]$, for some $a < b < c$, then ξ is a trivial vector bundle.

4. Let $\xi = (\pi, E, M)$ be a vector bundle and $N \subset M$ a closed submanifold. Show that every section $s : N \to E$ over N, admits an extension to a section $\tilde{s} : U \to E$ definided over an open set $U \supset N$.

5. Determine $\mathrm{Vect}_1(\mathbb{S}^1)$ without using the classification of vector bundles.

6. Determine $\mathrm{Vect}_r(\mathbb{S}^1)$, $\mathrm{Vect}_r(\mathbb{S}^2)$ and $\mathrm{Vect}_r(\mathbb{S}^3)$.

Lecture 26. Connections and Parallel Transport

In general, there is no natural way to differentiate sections of a vector bundle. The reason is that, in general, there is no way of comparing fibers of a vector bundle over different points of the base. We will now see how to fix this.

Definition 26.1. *A* ***connection on a vector bundle*** $\xi = (\pi, E, M)$ *is a map*

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E),\ (X, s) \mapsto \nabla_X s,$$

which satisfies the following properties:

(i) $\nabla_{X_1+X_2}s = \nabla_{X_1}s + \nabla_{X_2}s$*;*
(ii) $\nabla_X(s_1 + s_2) = \nabla_X s_1 + \nabla_X s_2$*;*
(iii) $\nabla_{fX}s = f\nabla_X s$*;*
(iv) $\nabla_X(fs) = f\nabla_X s + X(f)s$*.*

Properties (iii) a (iv) show that a connection ∇ is local. Hence a connection ∇ can be restrict to any open set $U \subset M$, yielding a connection in $\xi|_U$. On the other hand, a map $X \mapsto \nabla_X$ is $C^\infty(M)$-linear, hence, for any section s definided in a neighborhood U of $p \in M$ and any $\mathbf{v} \in T_pM$, we can define

$$\nabla_{\mathbf{v}} s \equiv \nabla_X s(p) \in E_p,$$

where X is any vector field defined in a neighborhood of p such that $X_p = \mathbf{v}$. Note, however, that $\nabla_{\mathbf{v}} s$ depends on the values of s is a neighborhood of p, not only on $s(p)$ (property (iv) in the definition).

Let $U \subset M$ be an open set where ξ trivializes, so we can choose a basis of sections $\{s_1, \dots, s_r\}$ for $\xi|_U$. Any other section s of $\xi|_U$ is a linear combination:

$$s = f_1 s_1 + \cdots + f_r s_r.$$

for unique smooth functions $f_i \in C^\infty(U)$. The connection ∇ is then completely determined by its effect on the sections s_i: for any vector field $X \in \mathfrak{X}(M)$, by property (iv), we have:

$$\nabla_X s = \sum_{a=1}^{r} f_a \nabla_X s_a + X(f_i) s_a.$$

Moreover, if additionally U is the domain of some coordinate system $(x^1, \dots, x^d)$ of M, we find:

$$\nabla_{\frac{\partial}{\partial x^i}} s_a = \sum_{b=1}^{r} \Gamma_{ia}^{b} s_b, \quad (i = 1, \dots, d,\ a = 1, \dots, r),$$

for unique functions $\Gamma_{ia}^{b} s_b \in C^\infty(U)$. One calls $\Gamma_{ia}^{b} s_b$ the **Christoffel symbols** of the connection relative to the coordinate systems and basis of local sections.

We can also organize the Christoffel symbols as a $r \times r$ matrix of differential forms in U defined by:

$$\omega_a^b = \sum_{i=1}^{r} \Gamma_{ia}^{b} \mathrm{d}x^i.$$

One calls $\omega = [\omega_a^b]$ the **connection 1-form**. By property (iii) in the definition of a connection, it is independent of the choice of local coordinates. Exercise 3, in the Homework, discusses how it depends on the choice of trivializing sections.

Example 26.2.
Recall that the vector bundle $\xi = (\pi, E, M)$ of rank r is trivial if and only if it admits a basis of global sections $\{s_1, \dots, s_r\}$. For each choice of basis, we can define a connection in ξ by setting:

$$\nabla_X s_i = 0, \quad (a = 1, \dots, r).$$

Note that this connection depends on the choice of trivializing sections.

The collection of all connections on a fixed vector bundle ξ has an affine structure: if $f \in C^\infty(M)$ is any smooth function, ∇_1 and ∇_2 are connections, then the affine combination

$$f\nabla_1 + (1-f)\nabla_2,$$

also defines a connection in ξ. It is this fact that allows us to show that:

Proposition 26.3. *Every vector bundle $\xi = (\pi, E, M)$ admits a connection.*

Proof. Let $\{U_\alpha\}$ be an open cover of M by trivializing open sets. The previous example shows that in each U_α we can choose a connection ∇^α. We define a connection ∇ in M "gluing" these connections: if $\{\rho_\alpha\}$ is a partition of unity subordinated to the cover $\{U_\alpha\}$, then

$$\nabla \equiv \sum_\alpha \rho_\alpha \nabla^\alpha,$$

defines a connection in ξ. □

If one starts with vector bundles with a connection, the usual constructions lead to vector bundles with connections:

Proposition 26.4. *Let ξ and ξ' be vector bundles over M, furnished with connections ∇ and ∇'. Then the associated bundles $\xi \oplus \xi'$, ξ^* and $\wedge^k \xi$, have induced connections satisfying:*

$$\begin{aligned}
\nabla_X(s_1 \oplus s_2) &= \nabla_X s_1 \oplus \nabla_X s_2,\\
\nabla_X(s_1 \wedge \cdots \wedge s_k) &= \nabla_X s_1 \wedge \cdots \wedge s_k + \cdots + s_1 \wedge \cdots \nabla_X \wedge s_k\\
X(\langle s, \eta\rangle) &= \langle \nabla_X s, \eta\rangle + \langle s, \nabla_X \eta\rangle.
\end{aligned}$$

If $\psi : N \to M$ is a smooth map, then $\psi^*\xi$ has a connection induced from ∇ such that:

$$(\nabla_v \psi^* s) = \psi^*(\nabla_{\mathrm{d}_p\psi(v)} s), \quad \forall v \in T_pN, s \in \Gamma(\xi).$$

We leave the proof for the exercises.

Connections can be used to compare different fibers of a vector bundle. Let $\xi = (\pi, E, M)$ be a vector bundle with a connection ∇. If $c : [0,1] \to M$ is a smooth curve then the pullback bundle $c^*\xi$ has an induced connection which we still denote by ∇. Notice that a section s of the bundle $c^*\xi$ is just a section of ξ along c, i.e., a smooth map $s : [0,1] \to E$ such that $\pi(s(t)) = c(t)$, for all $t \in [0,1]$.

Definition 26.5. *The* ***covariant derivative*** *of a section along a curve c is the section along c given by:*

$$\frac{Ds}{Dt} \equiv \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} s.$$

A section along c is called a ***parallel section*** *if it has vanishing covariant derivative: $\frac{Ds}{Dt} = 0$*

Choose local coordinates $(U, x^1, \dots, x^d)$ and trivializing sections $\{s_1, \dots, s_r\}$ over U. Given a curve $c(t)$ we let $c^i(t) = x^i(c(t))$, an for a section s along c we write $s(t) = \sum_a v^a(t) s_a(c(t))$. Then the covariant derivative along c has components:

$$\left(\frac{Ds}{Dt}\right)^a = \frac{\mathrm{d}v^a(t)}{\mathrm{d}t} + \sum_{ib} \frac{\mathrm{d}c^i(t)}{\mathrm{d}t} \Gamma^a_{ib}(c(t)) v^b(t), \quad (a = 1, \dots, r). \tag{26.1}$$

Remark 26.6. Not that even for a constant curve $c(t) = p_0$, the covariant derivative along c is not zero! In fact, in this case, a section along c is just a curve $s : [0,1] \to T_{p_0}M$ in the tangent space at p_0 and the the covariant derivative is the usual derivative of this curve.

The well known results about existence and uniqueness of solutions of linear ordinary equations with time dependent coefficients, we obtain:

Lemma 26.7. *For any curve $c : [0,1] \to M$ and any $v_0 \in E_{c(0)}$, there exists a unique parallel section s along c with initial condition $s(0) = v_0$.*

Under the conditions of this lemma, we say that the vectors $s(t) \in E_{c(t)}$ are obtained by parallel transport along c. We denote by $\tau_t : E_{c(0)} \to E_{c(t)}$ the operation of **parallel transport** defined by $\tau_t(v_0) = s(t)$.

Proposition 26.8. *Let $\xi = (\pi, E, M)$ be a vector bundle with a connection ∇ and let $c : [0,1] \to M$ be a smooth curve. Then:*

(i) Parallel transport $\tau_t : E_{c(0)} \to E_{c(t)}$ along c is a linear isomorphism.
(ii) If $\mathbf{v} = c'(0) \in T_{c(0)}M$ is the vector tangent to c, then:

$$\nabla_{\mathbf{v}} s = \lim_{t \to 0} \frac{1}{t} \left(\tau_t^{-1}(s(c(t))) - s(c(0)) \right),$$

for any section $s \in \Gamma(\xi)$.

Proof. Since the differential equation defining parallel transport is linear, it depends linear on the initial conditions, so τ_t is linear. On the other hand, τ_t is invertible, since its inverse is parallel transport along the curve $\bar{c} : [0,t] \to M$, given by $\bar{c}(\varepsilon) = c(t - \varepsilon)$. We leave the proof of (ii) as an exercise. $\square$

Consider now the tangent bundle $\xi = TM$ of a manifold M. For a connection ∇ in TM, the notions above have a more geometric meaning. For example, in $M = \mathbb{R}^d$, there is a canonical connection ∇ in $T\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^d$, which corresponds to the usual directional derivative. A vector field X (i.e., a section of TM) is parallel for this connection along a curve $c(t)$ if and only if the vectors $X_{c(t)}$ are parallel in the usual sense.

For a connection in the tangent bundle TM there are additional notions that do not make sense for connections on a general vector bundle. This is because a connection in TM differentiates vector fields along vector fields, so we have a more symmetric situation. Here is a first example:

Definition 26.9. *Let ∇ be a connection in TM. A* ***geodesic*** *is a curve $c(t)$ for which its derivative $\dot{c}(t)$ (a vector field along $c(t)$) is parallel, i.e., we have:*

$$\frac{D\dot{c}}{Dt}(t) = 0.$$

If we choose local coordinates $(U, x^1, \dots, x^d)$, we have trivializing vector fields $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^d}\}$ for $TM|_U$, and we can write:

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x^k}.$$

The equations for the components $c^i(t) = x^i(c(t))$ of a geodesic $c(t)$ in local coordinates are:

$$\frac{\mathrm{d}^2 c^k(t)}{\mathrm{d}t^2} = -\sum_{ij} \Gamma^k_{ij}(c(t)) \frac{\mathrm{d}c^i(t)}{\mathrm{d}t} \frac{\mathrm{d}c^j(t)}{\mathrm{d}t}, \quad (k = 1, \dots, n).$$

Using these equations, it should be clear that given $p_0 \in M$ and $\mathbf{v} \in T_{p_0}M$, there exists a unique geodesic $c(t)$ such that $c(0) = p_0$ and $\dot{c}(0) = \mathbf{v}$. This geodesic is defined for $0 \leq t < \varepsilon$, and if we choose $\mathbf{v}$ sufficiently small we can assume that $\varepsilon > 1$. In this case, we set::

$$\exp_{p_0}(\mathbf{v}) \equiv c(1).$$

In this way, we obtain the **exponential map** $\exp_{p_0} : U \to M$, which is defined in an open neighborhood $U \subset T_{p_0}M$ of the origin.

Another notion which only makes sense for connections ∇ in TM is the **torsion of a connection**: this is the map $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by:

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

One checks that T is $C^\infty(M)$-linear in both arguments, so it defines a morphism of vector bundles $T : TM \oplus TM \to TM$. One calls T the **torsion tensor** of the connection. A **symmetric connection** is a connection ∇ whose torsion is zero.

The next proposition gives a characterization of the torsion in terms of the covariant derivative. For that we choose $\phi : [0,1] \times [0,1] \to M$ an injective immersion (i.e., a parameterized surface) and we denote the parameters by (x, y). This section gives rise to:

$$\frac{\partial \phi}{\partial x} \equiv \phi_*(\frac{\partial}{\partial x}), \quad \frac{\partial \phi}{\partial y} \equiv \phi_*(\frac{\partial}{\partial y})$$

which are vector fields along the curves obtained by freezing x and y. Hence, we can take the covariant derivatives:

- $\frac{D}{Dx}\frac{\partial \phi}{\partial y}$ the covariant derivative along the curve $t \mapsto \phi(t, y)$ at $t = x$;
- $\frac{D}{Dy}\frac{\partial \phi}{\partial x}$ the covariant derivative along the curve $t \mapsto \phi(x, t)$ at $t = y$;

We have:

Proposition 26.10. *Consider a parameterized surface $\phi : [0,1] \times [0,1] \to M$. The torsion of a connection ∇ in TM satisfies:*

$$\frac{D}{Dx}\frac{\partial\phi}{\partial y} - \frac{D}{Dy}\frac{\partial\phi}{\partial x} = T(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial x}).$$

Proof. The proof is immediate if one computes both sides in local coordinates. □

The most classical example of a connection is the *Levi-Civita connection* in the tangent bundle of a Riemannian manifold, which we now describe. We start with a definition:

Definition 26.11. *Let ξ be a vector bundle over M with a fiber metric $\langle\ ,\ \rangle$. A connection in ξ is said to be* ***compatible with the metric*** *if*

$$X(\langle s_1, s_2\rangle) = \langle \nabla_X s_1, s_2\rangle + \langle s_1, \nabla_X s_2\rangle,$$

for every vector field $X \in \mathfrak{X}(M)$ and every pair of sections $s_1, s_2 \in \Gamma(\xi)$.

For a Riemannian manifold we have a natural choice of compatible metric:

Proposition 26.12. *Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold. There exists a unique symmetric connection in TM compatible with the metric.*

Proof. Let $X, Y, Z \in \mathfrak{X}(M)$ be vector fields in M. The compatibility of ∇ with the metric gives:

$$\begin{aligned} X \cdot \langle Y, Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \\ Y \cdot \langle Z, X\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle, \\ Z \cdot \langle X, Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle. \end{aligned}$$

Adding the first two equations and subtracting the third one, gives:

$$\begin{aligned} X \cdot \langle Y, Z\rangle + Y \cdot \langle Z, X\rangle - Z \cdot \langle X, Y\rangle = 2\langle \nabla_X Y, Z\rangle \\ - \langle X, [Y, Z]\rangle - \langle Y, [Z, X]\rangle - \langle Z, [X, Y]\rangle, \end{aligned}$$

where we have used the symmetry of the connection. This relation shows that the two conditions completely determine the connection by the formula:

$$\begin{aligned} \langle \nabla_X Y, Z\rangle = \frac{1}{2}\left(X \cdot \langle Y, Z\rangle + Y \cdot \langle Z, X\rangle - Z \cdot \langle X, Y\rangle\right) \\ + \frac{1}{2}\left(\langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle\right). \end{aligned}$$

On the other, one checks easily that this formula does define a connection in TM which is symmetric and compatible with the metric. □

The connection in the propositions is known as the **Levi-Civita connection** of the Riemannian manifold. This allows to define parallel transport, geodesics, exponential map, etc., for a Riemannian manifold. The fact that this connection comes from a metric leads to additional properties of these notions (see the Homework).

Homework.

1. Let ξ and ξ' be vector bundles over M, furnished with connections ∇ and ∇'. Show that the associated bundles $\xi \oplus \xi'$, ξ^* and $\wedge^k \xi$, have induced connections satisfying:

$$\begin{aligned}
\nabla_X(s_1 \oplus s_2) &= \nabla_X s_1 \oplus \nabla_X s_2, \\
\nabla_X(s_1 \wedge \cdots \wedge s_k) &= \nabla_X s_1 \wedge \cdots \wedge s_k + \cdots + s_1 \wedge \cdots \nabla_X \wedge s_k \\
X(\langle s, \eta \rangle) &= \langle \nabla_X s, \eta \rangle + \langle s, \nabla_X \eta \rangle.
\end{aligned}$$

Determine the connection 1-form of the pull-back connections in terms of the connection 1-form of the original connections.

2. Let ξ be a vector bundle over M with a connection ∇. If $\psi : N \to M$ is a smooth map, show that $\psi^* \xi$ has a connection induced from ∇ such that:

$$(\nabla_v \psi^* s) = \psi^*(\nabla_{\mathrm{d}_p \psi(v)} s), \quad \forall v \in T_p N, s \in \Gamma(\xi).$$

Determine the connection 1-form of the pull-back connections in terms of the connection 1-form of the original connection.

3. Let $U \subset M$ be an open set where the vector bundle $\xi = (\pi, E, M)$ trivializes, and let $\{s_1, \dots, s_r\}$ and $\{s'_1, \dots, s'_r\}$ be two basis of local sections for ξ. Denote by $A = (a_i^j) \in \mathcal{C}^\infty(U, \mathrm{GL}(r))$ the matrix of change of basis so that $s'_i = \sum_j a_i^j s_j$. If ω and ω' denote the corresponding connection 1-forms show that they are related by:

$$\omega' = A^{-1} \omega A + A^{-1} \mathrm{d}A.$$

4. Deduce formula (26.1) for the local expression of the covariant derivative of a connection.

5. Let $\xi = (\pi, E, M)$ be a vector bundle with a connection ∇. If $c : [0,1] \to M$ is a smooth curve and $\mathbf{v} = c'(0) \in T_{c(0)} M$ show that for any section $s \in \Gamma(\xi)$:

$$\nabla_{\mathbf{v}} s = \lim_{t \to 0} \frac{1}{t} \left(\tau_t^{-1}(s(c(t))) - s(c(0)) \right).$$

6. Let ξ be a vector bundle over M with a fiber metric $\langle\ ,\ \rangle$. Show that ξ has a connection compatible with the metric.

7. Let $\xi = (\pi, E, M)$ be a vector bundle with a fiber metric $\langle\ ,\ \rangle$. For a connection ∇ in ξ, show that the following are equivalent:
   (i) ∇ is compatible with the metric.
   (ii) Parallel transport $\tau_t : E_{c(0)} \to E_{c(t)}$ along any curve c is an isometry.
   (iii) For any basis of orthonormal trivializing sections the connection 1-form $\omega = [\omega_a^b]$ is a skew-symmetric matrix.

8. Let $\langle\ ,\ \rangle$ be a Riemannian metric in M and let $(x^1, \dots, x^d)$ be local coordinates. Find the expression for the Christoffel symbols $\Gamma_{ij}^k(x)$ of the Levi-Civita connection of the metric in terms of the components of the metric $g_{ij}(x) := \langle \frac{\partial}{\partial x^i}\big|_x, \frac{\partial}{\partial x^j}\big|_x \rangle$

We saw in the previous lecture that a trivial vector bundle carries natural connections defined in terms of trivializing sections s_i, for which $\nabla s_i = 0$. In general, for an arbitrary connection, it is not possible to choose a basis of local sections s_i such that $\nabla s_i = 0$. The obstruction is given by the *curvature* of the connection.

If ∇ is a connection in a vector bundle $\xi = (\pi, E, M)$, one defines the **curvature** of ∇ to be the map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(\xi) \to \Gamma(\xi)$ given by:

$$(X, Y, s) \mapsto R(X, Y)s \equiv \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]}s.$$

A simple computation shows that R is $C^\infty(M)$-linear in all the arguments, so we can think of R as a vector bundle map $R : TM \oplus TM \oplus E \to E$. For this reason one also calls R the **curvature tensor**.

The local expression for the curvature in a trivializing open set $U \subset M$ for ξ, in terms of a basis of sections $\{s_1, \dots, s_r\}$ and local coordinates $(x^1, \dots, x^d)$, is given by:

$$R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})s_a = R^b_{ija} s_b,$$

where the components R^b_{ija} can be expressed in terms of the Christoffel symbols Γ^b_{ia} by:

$$R^b_{ija} = \frac{\partial \Gamma^b_{ja}}{\partial x^i} - \frac{\partial \Gamma^b_{ia}}{\partial x^j} + \Gamma^c_{ia}\Gamma^b_{jc} - \Gamma^c_{ja}\Gamma^b_{ic}.$$

We can also codify the curvature in terms of a matrix of differential forms:

$$\Omega^b_a = \sum_{i<j} R^b_{ija} \mathrm{d}x^i \wedge \mathrm{d}x^j,$$

and $\Omega = [\Omega^b_a]$ is called the **curvature 2-form** of the connection. This matrix-valued 2-form is independent of the choice of local coordinates. The dependence on the choice of trivializing sections is discussed in the Homework.

Theorem 27.1. *For a connection in a vector bundle ξ, the connection 1-form ω and the curvature 2-form Ω associated with some trivializing sections, are related by the* ***structure equations****:*

$$\Omega^b_a = \mathrm{d}\omega^b_a + \sum_c \omega^c_a \wedge \omega^b_c,$$

one has the ***Bianchi's identity****:*

$$\mathrm{d}\Omega^b_a = \Omega^c_a \wedge \omega^b_c - \omega^c_a \wedge \Omega^b_c,$$

or equivalently:

$$\Omega = \mathrm{d}\omega + \omega \wedge \omega \quad \textit{and} \quad \mathrm{d}\Omega = \Omega \wedge \omega - \omega \wedge \Omega.$$

Proof. The structure equation follows from the definitions of the connection 1-form and the curvature 2-form. Bianchi's identity follows by taking exterior differentiation of the structure equation. The details are left as an exercise. □

Let us turn now to the geometric interpretation of curvature in term of parallel transport. We choose an injective immersion $\phi : [0,1] \times [0,1] \to M$ (i.e., a parameterized surface) and we denote the parameters by (x, y). We have the vector fields along ϕ given by:

$$\frac{\partial \phi}{\partial x} \equiv \phi_*(\frac{\partial}{\partial x}), \quad \frac{\partial \phi}{\partial y} \equiv \phi_*(\frac{\partial}{\partial y}).$$

Moreover, given a section s of the vector bundle ξ along ϕ, we can introduce the covariant derivatives:

- $\frac{Ds}{Dx}(x, y)$ the covariant derivative along the curve $t \mapsto \phi(t, y)$ in $t = x$;
- $\frac{Ds}{Dy}(x, y)$ the covariant derivative along the curve $t \mapsto \phi(x, t)$ in $t = y$;

We have:

Proposition 27.2. *For any section s of ξ along a parameterized surface $\phi : [0,1] \times [0,1] \to M$, the curvature of the connection satisfies:*

$$\frac{D}{Dx}\frac{Ds}{Dy} - \frac{D}{Dy}\frac{Ds}{Dx} = R(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial x})s.$$

Again the proof is immediate in local coordinates.

A **flat connection** is a connection for which the curvature tensor vanishes. We will often refer to a vector bundle with a flat connection as a **flat bundle**. Clearly, if around each point one can choose coordinates and trivializing sections for which the Christoffel symbols vanish, the connection is flat. The converse is also true, as a consequence of the following local normal form for flat bundles:

Corollary 27.3. *Let $\xi = (\pi, E, M)$ be a vector bundle of rank r with a flat connection ∇. For each $p \in M$, there exists a base of local sections $\{s_1, \dots, s_r\}$ definided in a neighborhood U of p, such that*

$$\nabla_X s_i = 0, \quad \forall X \in \mathfrak{X}(M).$$

Hence, $\xi|_U$ is isomorphic to the trivial vector bundle ε^r_U with the canonical flat connection.

Proof. See Exercise 4 in the Homework. □

In the case of Riemannian manifolds, Corollary 27.3 takes the following more geometric meaning:

Corollary 27.4. *Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold with vanishing curvature tensor: $R = 0$. For each $p \in M$, there exists a neighborhood U of p which is isometric to $\mathbb{R}^d$, with the canonical metric.*

Proof. See Exercise 5 in the Homework. □

The previous results describe flat connections locally. To describe what happens with a flat connection globally, we need to introduce the notion of of *holonomy* of a connection. Given a vector bundle $\xi = (\pi, E, M)$ of rank r with a connection ∇ fix a base point $p_0 \in M$. For each a closed curve $c : [0,1] \to M$ base at p_0, so $c(0) = c(1) = p_0$, parallel transport along the curve $c(t)$ gives a linear isomorphism $H_{p_0}(c) \equiv \tau_1 : E_{p_0} \to E_{p_0}$. We can extend this definition to closed curves which are piecewise smooth in the obvious way. It should be clear that if c_1 and c_2 are piecewise smooth closed curves and $c_1 \cdot c_2$ denotes their concatenation then:

$$H_{p_0}(c_1 \cdot c_2) = H_{p_0}(c_1) \circ H(c_2).$$

When the connection is flat we have:

Lemma 27.5. *If c_0 and c_1 are homotopic closed curves based at p_0, then $H_{p_0}(c_0) = H_{p_0}(c_1)$.*

It follows that we have a group homomorphism $H_{p_0} : \pi_1(M, p_0) \to GL(E_{p_0})$, called the **holonomy representation** of the flat connection ∇, with base point p_0. Note that if $q_0 \in M$ is a different point in the same connected component of M, we can choose a smooth path $c : [0,1] \to M$, connecting p_0 to q_0 (i.e., $c(0) = p_0$ and $c(1) = q_0$). Parallel transport along $c(t)$ gives an isomorphism $\tau : E_{p_0} \to E_{q_0}$ and:

$$H_{q_0} = \tau \circ H_{p_0} \circ \tau^{-1}.$$

Hence, the holonomy representations of different points in the same component are related by conjugacy.

Conversely, the holonomy representation determines the flat bundle:

Theorem 27.6. *Let $\xi = (\pi, E, M)$ be a vector bundle of rank r with a flat connection ∇ over a connected manifold. Then the holonomy of ∇ induces a representation $H : \pi_1(M, x_0) \to GL(\mathbb{R}^r)$. Conversely, every representation of the fundamental group $\pi_1(M, x_0)$ defines a vector bundle of rank r with a flat connection ∇, whose holonomy with base point x_0 coincides with the given representation.*

Proof. We already know that a flat vector bundle induces a representation of the fundamental group. Conversely, given a representation $H : \pi_1(M, p_0) \to GL(\mathbb{R}^r)$ we construct a flat vector bundle as follows: on the one hand, the representation gives an action of $\pi(M, p_0)$ in $\mathbb{R}^r$. On the other hand, the fundamental group $\pi_1(M, p_0)$ acts in the universal cover $\widetilde{M}$ by deck transformations: identifying $\widetilde{M}$ with the set of homotopy classes of paths $[c]$ with initial point $c(0) = p_0$, the action of $\pi_1(M, p_0)$ in $\widetilde{M}$ is given by concatenation:

$$\pi_1(M, p_0) \times \widetilde{M} \to \widetilde{M}, \ ([\gamma], [c]) \mapsto [\gamma \cdot c].$$

Since this action is proper and free, the resulting diagonal action of $\pi_1(M, p_0)$ in $\widetilde{M} \times \mathbb{R}^r$ is also proper and free. Hence, the quotient space $E = (\widetilde{M} \times$

$\mathbb{R}^r)/\pi_1(M, p_0)$ is a manifold, and we have the projection

$$\pi : E \to M, [[c], \mathbf{v}] \mapsto c(1).$$

The triple $\xi = (\pi, E, M)$ is a vector bundle. Moreover, the canonical flat connection in $\widetilde{M} \times \mathbb{R}^r$ induces a connection in ξ for which the holonomy with base point p_0 is precisely $H : \pi_1(M, p_0) \to GL(\mathbb{R}^r)$. $\square$

HOMEWORK.

1. Deduce the structure equation and Bianchi's identity for the connection 1-form and the curvature 2-form of a connection.

2. Let $U \subset M$ be an open set where the vector bundle $\xi = (\pi, E, M)$ trivializes, and let $\{s_1, \ldots, s_r\}$ and $\{s'_1, \ldots, s'_r\}$ be two basis of local sections for ξ. Denote by $A = (a_i^j) \in \mathcal{C}^\infty(U, \mathrm{GL}(r))$ the matrix of change of basis so that $s'_i = \sum_j a_i^j s_j$. If Ω and Ω' denote the corresponding curvature 2-forms show that they are related by:
$$\Omega' = A^{-1}\Omega A.$$

3. Use Exercise 3 in the previous lecture and the previous exercise to show that if ∇ is a flat connection in the vector bundle $\xi = (\pi, E, M)$, then around any point $p \in M$one can find a local basis of flat sections.
HINT: Use the condition $\omega' = 0$ to define a integrable distribution in $U \times \mathrm{GL}(r)$ and apply Frobenius.

4. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Show that there exists a unique connection ∇ in TG, which is invariant under left and right translations, and under inversion. Show that ∇ satisfies the following properties:
   (a) For any left invariant vector fields $X, Y \in \mathfrak{g}$:
   $$\nabla_X Y = \frac{1}{2}[X, Y].$$
   (b) The torsion of ∇ vanishes and its curvature is given by:
   $$R(X, Y) \cdot Z = \frac{1}{4}[[X, Y], Z], \quad \forall X, Y, Z \in \mathfrak{g}).$$
   (c) The exponential map of ∇ at the identity $\exp_e$ coincides with the Lie group exponential map $\exp : \mathfrak{g} \to G$.
   (d) Parallel transport along the curve $c(t) = \exp(tX)$, $X \in \mathfrak{g}$, is given by:
   $$\tau_t(\mathbf{v}) = \mathrm{d}L_{\exp(\frac{t}{2}X)} \cdot \mathrm{d}R_{\exp(\frac{t}{2}X)} \cdot \mathbf{v}, \quad \forall \mathbf{v} \in T_eG.$$
   (e) The geodesics are translations of the 1-parameter subgroups of G.

5. Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold whose curvature tensor vanishes: $R = 0$. Show that for each $p \in M$, there exists a neighborhood U isometric to $\mathbb{R}^d$ with the Euclidean metric.

We saw in the previous lecture that a flat vector bundle is globally characterized by its holonomy representation. The situation in the non-flat case is more complicated but more interesting: we will see now that one can use a connection on a vector bundle to associate to the vector bundle cohomology classes which are invariants of the vector bundle, and which characterize certain properties of the vector bundle up to isomorphism.

Let $\pi : E \to M$ be a vector bundle. We consider differential forms in M with values in E, which we denote by

$$\Omega^\bullet(M;E) = \Omega^\bullet(M) \otimes \Gamma(E).$$

Hence, $\Omega^k(M;E)$ consists of sections of the vector bundle $\wedge^k T^*M \otimes E$, so a differential form of degree k with values in E is a k-multilinear alternating map:

$$\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \Gamma(E).$$

In particular, $\Omega^0(M;E)$ is the space $\Gamma(E)$ of global sections of the vector bundle $\pi : E \to M$.

In order to take the differential of E-valued differential forms we need to choose a connection ∇ in $\pi : E \to M$. Such a connection determines an operator $\mathrm{d}_\nabla : \Omega^0(M;E) \to \Omega^1(M;E)$ through the formula:

$$(\mathrm{d}_\nabla s)(X) = \nabla_X s.$$

The map d_∇ is $\mathbb{R}$-linear and satisfies:

$$\mathrm{d}_\nabla(fs) = \mathrm{d}f \otimes s + f\mathrm{d}_\nabla s.$$

Conversely, every map $\Omega^0(M;E) \to \Omega^1(M;E)$ which is $\mathbb{R}$-linear and satisfies this property defines a connection, so this gives an alternative approach to the theory of connections E.

We have that d_∇ extends, in a unique way, to E-valued differential forms of arbitrary degree:

Proposition 28.1. *Given a connection ∇ and $\omega \in \Omega^k(M;E)$ define $\mathrm{d}_\nabla\omega \in \Omega^{k+1}(M;E)$ by:*

$$\mathrm{d}_\nabla\omega(X_0,\dots,X_k) = \sum_{i=0}^{k+1}(-1)^i \nabla_{X_i}(\omega(X_0,\dots,\widehat{X}_i,\dots,X_k)) + \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_k). \tag{28.1}$$

Then $\mathrm{d}_\nabla : \Omega^\bullet(M;E) \to \Omega^{\bullet+1}(M;E)$ is the unique operator satisfying:

(i) d_∇ is $\mathbb{R}$-linear and satisfies the Leibniz identity:

$$\mathrm{d}_\nabla(\omega \otimes s) = \mathrm{d}_\nabla(\omega) \otimes s + (-1)^{\deg\omega}\omega \wedge \mathrm{d}_\nabla(s), \quad \forall \omega \in \Omega^\bullet(M), s \in \Gamma(E).$$

(ii) For 0-forms, $(\mathrm{d}_\nabla s)(X) = \nabla_X s$.

Note that, in general, $\mathrm{d}_\nabla^2 \neq 0$, so d_∇ is not a differential. In fact, a more or less tedious computation shows that

$$\mathrm{d}_\nabla^2 s(X,Y) = R_\nabla(X,Y)s, \quad \forall X,Y \in \mathfrak{X}(M), s \in \Gamma(E).$$

so d_∇ is a differential if and only if the connection is flat. In this case, one calls the cohomology of the complex $(\Omega^\bullet(M;E), \mathrm{d}_\nabla)$ the **de Rham cohomology of M with coefficients in E** and denotes it by $H^\bullet(M;E)$. Notice that the usual de Rham cohomology corresponds to the case where $E = M \times \mathbb{R}$ is the trivial flat line bundle.

In general, we will have $R \neq 0$, but it will satisfy Bianchi's identity, which in this language can be written in the form:

$$\mathrm{d}_\nabla R_\nabla = 0. \tag{28.2}$$

Note that in this last identity we view the curvature tensor as a bilinear alternating map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \Gamma(\mathrm{End}(E))$, i.e., as a 2-form with values in $\mathrm{End}(E)$. Also, in $\mathrm{End}(E)$ we consider the connection induced by ∇, so the equation makes sense.

The Bianchi identity can be used to define certain cohomology classes. For that we need first to recall the relationship between homogeneous polynomials in a vector space V and multilinear symmetric maps $P : V \times \cdots \times V \to \mathbb{R}$:

(i) Every k-multilinear symmetric map $P : V \times \cdots \times V \to \mathbb{R}$ determines a homogeneous polynomial $\widetilde{P} : \mathfrak{g} \to \mathbb{R}$ of degree k, by the formula:

$$\widetilde{P} : v \mapsto P(v, \dots, v).$$

(ii) Conversely, every homogeneous polynomial $\widetilde{P} : V \to \mathbb{R}$ of degree k determines a k-multilinear symmetric map $P : V \times \cdots \times V \to \mathbb{R}$: if $\xi^1, \dots, \xi^r$ is a base for V^*, then the polynomial $\widetilde{P} : V \to \mathbb{R}$ can be written as:

$$\widetilde{P}(v) = \sum_{i_1 \cdots i_k = 1}^{r} a_{i_1 \cdots i_k} \xi^{i_1} \cdots \xi^{i_k},$$

where the coefficients $a_{i_1 \cdots i_k}$ are symmetric in the indices. Hence, we can define a k-multilinear, symmetric map $P : V \times \cdots \times V \to \mathbb{R}$ by:

$$P(v_1, \dots, v_k) = \sum_{i_1 \cdots i_k = 1}^{r} a_{i_1 \cdots i_k} \xi^{i_1}(v_1) \cdots \xi^{i_k}(v_k).$$

These correspondences are inverse to each other. Under this correspondence the product of k-multilinear, symmetric maps, defined by:

$$P_1 \circ P_2(v_1, \dots, v_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} P_1(v_{\sigma(1)}, \dots, v_{\sigma(k)}) P_2(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}).$$

is mapped to the usual product of polynomials.

We are interested in the case where $V = \mathfrak{g}$ is the Lie algebra of a Lie group G. We will denote by $I^k(G)$ the space of k-multilinear, symmetric maps $P : \mathfrak{g} \times \cdots \times \mathfrak{g} \to \mathbb{R}$ which are invariant under the adjoint action:

$$P(\operatorname{Ad} g \cdot v_1, \dots, \operatorname{Ad} g \cdot v_k) = P(v_1, \dots, v_k), \quad \forall g \in G,\ v_1, \dots, v_k \in \mathfrak{g}.$$

and we let

$$I(G) = \bigoplus_{k=0}^{\infty} I^k(G).$$

Note that $I(G)$ is a ring with the symmetric product. Under the correspondence above, we can identify $I(G)$ with the algebra of polynomials in $\mathfrak{g}$ which are Ad-invariant.

For now, we are only interested in the case where $G = GL(r)$, so that $\mathfrak{g} = \mathfrak{gl}(r)$ is the space of all $r \times r$-matrices. Then the invariance condition is just:

$$P(AX_1A^{-1}, \dots, AX_kA^{-1}) = P(X_1, \dots, X_k), \quad X_1, \dots, X_k \in \mathfrak{gl}(r),$$

which must hold for any invertible matrix $A \in GL(r)$. The key remark is the following:

Proposition 28.2. *Let $\xi = (\pi, E, M)$ be a rank r vector bundle with a connection ∇. Every element $P \in I^k(GL(r))$ determines a map*

$$P : \Omega^\bullet(M; \otimes^k \operatorname{End}(E)) \to \Omega^\bullet(M),$$

which satisfies:

$$\mathrm{d}P = P\mathrm{d}_\nabla.$$

Proof. Note that if $s_1, \dots, s_r$ is a base of local of sections of E then for any section $A \in \Gamma(\operatorname{End}(E))$, we have:

$$As_i = \sum_{j=1}^{r} A_i^j s_j,$$

for some functions A_i^j. Hence, given a $P \in I^k(GL(r))$, we can define a map $P : \Gamma(\otimes^k \operatorname{End}(E)) \to C^\infty(M)$ by:

$$P(A_1 \otimes \cdots \otimes A_k) = P([(A_1)_i^j], \cdots, [(A_k)_i^j]).$$

By the invariance condition, this definition is independent of the choice of base of local of sections. Since a form $\omega \in \Omega^l(M; \otimes^k \operatorname{End}(E))$ is an l-multilinear alternating map $\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \Gamma(\otimes^k \operatorname{End}(E))$, the composition with P determines an l-multilinear alternating map $P \circ \omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$, i.e., an element $P(\omega) \in \Omega^l(M)$. An elementary computation shows that:

$$\mathrm{d}P = P\mathrm{d}_\nabla.$$

∎

If ∇ is a connection in a vector bundle $\pi : E \to M$ of rank r with curvature R, then the k-symmetric power of the curvature is an element $R^k \in \Omega^{2k}(M; \otimes^k \operatorname{End}(E))$. Explicitly, we have:

$$R^k(X_1, \ldots, X_{2K}) = \frac{1}{(2k)!} \sum_{\sigma \in S_{2k}} (-1)^{\operatorname{sgn}\sigma} R(X_{\sigma(1)}, X_{\sigma(2)}) \otimes \cdots \otimes R(X_{\sigma(2k-1)}, X_{\sigma(2k)}).$$

Therefore, if $P \in I^k(GL(r))$, we obtain a differential form $P(R^k) \in \Omega^{2k}(M)$. This form is given explicitly by

$$P(R^k)(X_1, \ldots, X_{2k}) = \frac{1}{(2k)!} \sum_{\sigma \in S_{2k}} (-1)^{\operatorname{sgn}\sigma} P(\Omega(X_{\sigma(1)}, X_{\sigma(2)}), \ldots, \Omega(X_{\sigma(2k-1)}, X_{\sigma(2k)})).$$

It follows that if $P_1 \in I^k(GL(r))$ and $P_2 \in I^l(GL(r))$, then:

$$P_1 \circ P_2(R^{k+l}) = P_1(R^k) \wedge P_2(R^l) \in \Omega^{2(k+l)}(M).$$

The Bianchi identity (28.2) gives:

$$\mathrm{d}P(R^k) = P(\mathrm{d}_\nabla R^k) = kP(R^{k-1}\mathrm{d}_\nabla R) = 0,$$

so $P(R)$ is a closed $2k$-form. Now, we have:

Theorem 28.3 (Chern-Weil). *Let ∇ be a connection in a vector bundle $\pi : E \to M$ of rank r, with curvature R. The map $I(GL(r)) \to H(M)$ defined by:*

$$I^k(GL(r)) \to H^{2k}(M), \ P \longmapsto [P(R^k)],$$

is a ring homomorphism. This homomorphism is independent of the choice of connection.

Proof. All that it remains to be proved is that the homomorphism is independent of the choice of connection. For that we claim that if ∇_0 and ∇_1 are two connections in $\pi : E \to M$, then for all $P \in I^k(GL(r))$ the differential forms $P(R^k_{\nabla_0})$ and $P(R^k_{\nabla_1})$ differ by an exact form.

To prove the claim, consider the projection $p : M \times [0, 1] \to M$. The pull-back bundle p^*E carries a connection ∇ defined by:

$$\nabla := t\nabla_1 + (1 - t)\nabla_0, \quad (t \in [0, 1]).$$

On the other hand, we can consider an operation of *integration along the fibers*:

$$\int_0^1 : \Omega^\bullet(M \times [0, 1]) \to \Omega^{\bullet-1}(M),$$

by setting:

$$(\int_0^1 \omega)(X_1, \ldots, X_{l-1}) = \int_0^1 \omega(\frac{\partial}{\partial t}, X_1, \ldots, X_{l-1})\mathrm{d}t.$$

The **Chern-Simons transgression form** is defined by

$$P(\nabla_0, \nabla_1) \equiv \int_0^1 P(R^k_\nabla) \in \Omega^{2k-1}(M). \tag{28.3}$$

We leave as an exercise to check that:

$$\mathrm{d}P(\nabla_0, \nabla_1) = P(R^k_{\nabla_1}) - P(R^k_{\nabla_0}).$$

This proves the claim, so $P(R^k_{\nabla_1})$ and $P(R^k_{\nabla_0})$ define the same cohomology class. □

The map $I(GL(r)) \to H^\bullet(M)$ is called the **Chern-Weil homomorphism** of the vector bundle $\xi = (\pi, E, M)$. An element in the image of the Chern-Weil homomorphism is called a **characteristic class** of the vector bundle $\xi = (\pi, E, M)$. Because of the next proposition, this class depends only on the isomorphism class of ξ:

Proposition 28.4. *Let $\psi : N \to M$ be a smooth map and let $\xi = (\pi, E, M)$ be a vector bundle of rank r. For every $P \in I^\bullet(GL(r))$,*

$$\phi^* P(R^k_\nabla) = P(R^k_{\phi^*\nabla}),$$

where ∇ is any connection in ξ.

We leave the proof for the Homework.

It remains to find the invariant symmetric multilinear maps or, equivalently, the invariant polynomials. For that, given a matrix $X \in \mathfrak{gl}(r)$ denote by $\sigma_k(X)$ the elementary symmetric function of degree k, so that:

$$\det(I + \lambda X) = I + \lambda\sigma_1(X) + \cdots + \lambda^r \sigma_r(X),$$

for every $\lambda \in \mathbb{R}$. One checks easily that $\sigma_k : \mathfrak{gl}(r) \to \mathbb{R}$ is a homogeneous polynomial of degree k which is Ad-invariant. These elements generate the ring $I(GL(r))$. For example, one has:

$$\begin{aligned}
\sigma_1(X) &= \operatorname{tr} X, \\
\sigma_2(X) &= \frac{1}{2}\left((\operatorname{tr} X)^2 - \operatorname{tr} X^2\right), \\
&\ \ \vdots \\
\sigma_r(X) &= \det X.
\end{aligned}$$

Definition 28.5. *Let $\xi = (\pi, E, M)$ be a vector bundle of rank r. For $k = 1, 2, \ldots,$ the **Pontrjagin classes** of ξ are:*

$$p_k(\xi) = \frac{1}{(2\pi)^{2k}}[\sigma_{2k}(R^{2k})] \in H^{4k}(M),$$

*where R is the curvature of any connection ∇ in ξ. The **total Pontrjagin class** of the vector bundle ξ is:*

$$p(\xi) = 1 + p_1(\xi) + \cdots + p_{[r/2]}(\xi),$$

where $[r/2]$ denotes the largest integer less or equal to $r/2$.

The reason one does not consider the classes $[\sigma_k(R^k)]$ for odd k is that these classes are always zero (see the Homework). The next proposition lists basic properties of these characteristic classes. The proof is immediate from the construction of these classes.

Proposition 28.6. *Let M be a smooth manifold. The Pontrjagin classes satisfy:*

(i) $p(\xi \oplus \eta) = p(\xi) \cup p(\eta)$*, for any vector bundles ξ and η over M;*
(ii) $p(\psi^*\xi) = \psi^* p(\xi)$*, for any vector bundle ξ over M and smooth map $\psi : N \to M$;*
(iii) $p(\xi) = 1$*, if the vector bundle ξ admits a flat connection.*

The Pontrjagin classes $p(TM)$ of the tangent bundle of a manifold M give an important invariant of a smooth manifold. Novikov proved that these classes are in fact topological invariants: two smooth manifolds which are homemorphic have the same Pontrjagin classes $p(TM)$.

Examples 28.7.

1. *Let $M = \mathbb{S}^d \hookrightarrow \mathbb{R}^{d+1}$ and denote by $\nu(\mathbb{S}^d) = T_{\mathbb{S}^d}\mathbb{R}^{d+1}/T\mathbb{S}^d$ the normal bundle of $\mathbb{S}^d$. Notice that the Whitney sum*
$$T\mathbb{S}^d \oplus \nu(\mathbb{S}^d) = T_{\mathbb{S}^d}\mathbb{R}^d,$$
is the trivial vector bundle over $\mathbb{S}^d$. On the other hand, the normal bundle $\nu(\mathbb{S}^d)$ is also trivial, for it is a line bundle which admits a nowhere vanishing section. By property (i) in the Proposition we conclude that $p(T\mathbb{S}^d) = 1$.

2. *Let $M = \mathbb{CP}^d$. Recall that we have $\mathbb{CP}^d = \mathbb{S}^{2d+1}/\mathbb{S}^1$, where $\mathbb{S}^{2d+1} \subset \mathbb{C}^{d+1}$ and $\mathbb{S}^1$ acts by complex multiplication: $\theta \cdot z = e^{i\theta}$. The Euclidean metric in $\mathbb{C}^{d+1} = \mathbb{R}^{2d+2}$ induces a Riemannian metric in $\mathbb{S}^{2d+1}$ which is invariant under the $\mathbb{S}^1$-action. Hence, this induces a Riemannian metric in the quotient $\mathbb{CP}^d = \mathbb{S}^{2d+1}/\mathbb{S}^1$, called the **Fubini-Study metric**.*

   One can use the connection associated with the Fubini-Study metric to compute the Pontrjagin classes $p(T\mathbb{CP}^d)$. For example, in the Homework we sketch how in the case of $\mathbb{CP}^2$ one finds that $p(T\mathbb{CP}^2) = 3[\mu]$, where $[\mu] \in H^4(\mathbb{CP}^2)$ is the class representing the canonical orientation of $\mathbb{CP}^2$ (the orientation induced from the standard orientation of $\mathbb{S}^5$).

So far, all our vector bundles were *real* vector bundles. One can also consider *complex* vector bundles (π, E, M), where the fibers E_x are now complex vector bundles of complex dimension r and the transition functions $g_{\alpha\beta}$ take values in $GL(\mathbb{C}^r)$. Every complex vector bundle of rank r can be viewed as a real vector bundle of rank $2r$, with a complex structure, i.e., an endomorphism of (real) vector bundles $J : \xi \to \xi$ such that $J^2 = -\mathrm{id}$. The complex structure J and the complex structure in the fibers are related by:
$$(a + ib)\mathbf{v} = a\mathbf{v} + bJ(\mathbf{v}), \quad \forall \mathbf{v} \in E.$$

On a complex vector bundle one considers $\mathbb{C}$-connections, i.e., connections ∇ which are compatible with the complex structure: $\nabla J = J\nabla$. This means that for each vector field $X \in \mathfrak{X}(M)$ the map $s \mapsto \nabla_X s$ is $\mathbb{C}$-linear:

$$\nabla_X(\lambda_1 s_1 + \lambda_2 s_2) = \lambda_1 \nabla_X s_1 + \lambda_2 \nabla_X s_2, \quad \forall \lambda_i \in \mathbb{C}, s_i \in \Gamma(\xi).$$

Using such connections, one defines the Chern-Weil homomorphism much the same way as in the real case, obtaining a ring homomorphism

$$I(GL(r,\mathbb{C})) \to H^\bullet(M).$$

Again, the ring of invariant polynomials $I(GL(r,\mathbb{C}))$ is generated by the elementary invariant polynomials: every invariant polynomial in $\mathfrak{gl}(r,\mathbb{C})$ can be expressed as a function of the polynomials $\sigma_1, \dots, \sigma_r$ defined for any element $X \in \mathfrak{gl}(r,\mathbb{C})$ by:

$$\det(I + \lambda X) = I + \lambda \sigma_1(X) + \cdots + \lambda^r \sigma_r(X),$$

for all $\lambda \in \mathbb{C}$. These allow us to define:

Definition 28.8. *Let $\xi = (\pi, E, M)$ be a complex vector bundle of rank r. For $k = 1, \dots, r$ we define the **Chern classes** of ξ by:*

$$c_k(\xi) = \frac{1}{(2\pi i)^k}[\sigma_k(R^k)] \in H^{2k}(M),$$

*where R is the curvature of any connection ∇ in ξ. The **total Chern class** of ξ is sum:*

$$c(\xi) = 1 + c_1(\xi) + \cdots + c_r(\xi) \in H(M).$$

Note that, a priori, the Chern classes are cohomology classes lying in *complex* de Rham cohomology $H^\bullet(M,\mathbb{C})$. However, the normalization factor makes them real cohomology classes. To see this, we use the following lemma:

Lemma 28.9. *Every complex vector bundle $\xi = (\pi, E, M)$ admits a fiber hermitian metric $\langle \cdot, \cdot \rangle$ and a $\mathbb{C}$-connection ∇ compatible with the metric.*

Choosing a connection as in the lemma, for any orthonormal $\mathbb{C}$-basis of local sections $\{s_1, \dots, s_r\}$ of E, the connection 1-form is a unitary matrix. By the structure equation, the curvature 2-form is also unitary so its eigenvalues are purely imaginary. It follows that $i^k \sigma_k(R^k)$ is a real form, so the Chern classes are real cohomology classes.

Similar to the real case, we have the following properties:

Proposition 28.10. *Let M be a smooth manifold. The Chern classes satisfy:*

(i) $c(\xi \oplus \eta) = c(\xi) \cup c(\eta)$, *for any complex vector bundles ξ and η over M;*
(ii) $c(\psi^*\xi) = \psi^* c(\xi)$, *for any complex vector bundle ξ over M and smooth map $\psi : N \to M$;*
(iii) $c(\xi) = 1$, *if ξ is a complex vector bundle which admits a flat connection.*

Remark 28.11. One of the exercises in the Homework asks you to verify that for the canonical (complex) line bundle over $\mathbb{CP}^1 = \mathbb{S}^2$, which we denote by $\gamma_1^1(\mathbb{C})$, the first Chern class is:

$$c_1(\gamma_1^1) = -1,$$

where $1 \in H^2(\mathbb{CP}^1)$ denotes the generator defined by the canonical orientation. One can show that with this *normalization* properties (i)–(iii) above determine completely the Chern classes.

If $\xi = (\pi, E, M)$ is a complex vector bundle then its **complex conjugate** is the complex vector bundle $\bar{\xi}$ which, as a real vector bundle, coincides with ξ, but where the complex structure is the opposite: $J_{\bar{\xi}} = -J_\xi$. Hence, the identity map id: $\xi \to \bar{\xi}$ satisfies:

$$\mathrm{id}(\lambda \mathbf{v}) = \bar{\lambda}\ \mathrm{id}(\mathbf{v}), \quad \forall \mathbf{v} \in E, \lambda \in \mathbb{C}.$$

The proof of the next proposition is left as an exercise:

Proposition 28.12. *Let $\xi = (\pi, E, M)$ be a complex vector bundle. The Chern classes of ξ and $\bar{\xi}$ are related by $c_k(\bar{\xi}) = (-1)^k c_k(\xi)$ so that:*

$$c(\bar{\xi}) = 1 - c_1(\xi) + c_2(\xi) - \cdots + (-1)^r c_r(\xi).$$

Proof. Let ∇ be a $\mathbb{C}$-connection in ξ. It defines also a $\mathbb{C}$-connection in $\bar{\xi}$ which we denote by $\overline{\nabla}$. If one fixes local trivializing sections $\{s_1, \dots, s_r\}$ for ξ, then we have:

$$\nabla_X s_i = \sum_j \omega_i^j(X) s_j, \quad \overline{\nabla}_X s_i = \sum_j \overline{\omega}_i^j(X) s_j.$$

Hence, the curvature 2-forms of these two connections relative to this basis are related by:

$$\Omega_{\overline{\nabla}}(X, Y) = \overline{\Omega_\nabla(X, Y)},$$

and it follows that $\sigma_k(R^k_{\overline{\nabla}}) = \overline{\sigma_k(R^k_\nabla)}$. Therefore, we have:

$$\begin{aligned}
c_k(\bar{\xi}) &= \frac{1}{(2\pi i)^k}[\sigma_k(R^k_{\overline{\nabla}})] \\
&= \frac{1}{(2\pi i)^k}[\overline{\sigma_k(R^k_\nabla)}] \\
&= (-1)^k \overline{\frac{1}{(2\pi i)^k}[\sigma_k(R^k_\nabla)]} \\
&= (-1)^k \frac{1}{(2\pi i)^k}[\sigma_k(R^k_\nabla)] = (-1)^k c_k(\xi).
\end{aligned}$$

□

Let $\xi = (\pi, E, M)$ be a real vector bundle of rank r, and let $M \times \mathbb{C} \to M$ be the trivial *real* vector bundle of rank 2. The tensor product $\xi \otimes \mathbb{C}$ is a real vector bundle of rank 2r, and we can define a vector bundle map $J : \xi \otimes \mathbb{C} \to \xi \otimes \mathbb{C}$ by:

$$J(v \otimes \lambda) = v \otimes i\lambda.$$

Since $J^2 = -\mathrm{id}$, this defines a complex structure in $\xi \otimes \mathbb{C}$. One calls the resulting complex vector bundle $\xi \otimes \mathbb{C}$ the **complexification** of ξ.

The Pontrjagin classes of a real vector bundle ξ can be obtained from the Chern classes of its complexification $\xi \otimes \mathbb{C}$:

Proposition 28.13. *Let ξ be a real vector bundle. Then the Pontrjagin classes of ξ and the Chern classes of of its complexification $\xi \otimes \mathbb{C}$ are related by:*

$$p_k(\xi) = (-1)^k c_{2k}(\xi \otimes \mathbb{C}).$$

The complexification $\xi \otimes \mathbb{C}$ and its conjugate complex vector bundle $\overline{\xi \otimes \mathbb{C}}$ are isomorphic complex vector bundles. An explicit isomorphism is given by the complex conjugation map:

$$\xi \otimes \mathbb{C} \to \overline{\xi \otimes \mathbb{C}},\ v \otimes \lambda \mapsto v \otimes \overline{\lambda}.$$

Hence, by Proposition 28.12, we conclude that

$$c_k(\xi \otimes \mathbb{C}) = 0, \text{ if } k \text{ is odd}.$$

This gives another explanation for why the Pontrjagin classes of a real vector bundle are concentrated in degree $4k$.

Homework.

1. Let ∇ be a connection in a vector bundle $\pi : E \to M$. Show that there exists a unique $\mathbb{R}$-linear operator $\mathrm{d}_\nabla : \Omega^\bullet(M;E) \to \Omega^{\bullet+1}(M;E)$ which satisfies the Leibniz identity:
$$\mathrm{d}_\nabla(\omega \otimes s) = \mathrm{d}_\nabla(\omega) \otimes s + (-1)^{\deg\omega}\omega \wedge \mathrm{d}_\nabla(s), \quad \forall \omega \in \Omega^\bullet(M;E), s \in \Gamma(E).$$
Show that $\mathrm{d}_\nabla \circ \mathrm{d}_\nabla = 0$ if and only if the connection is flat.

2. Let $\pi : E \to M$ be a vector bundle of rank r. Given $P \in I^k(GL(r))$, show that $P : \Omega^\bullet(M; \otimes^k \mathrm{End}(E)) \to \Omega^\bullet(M)$, satisfies: $\mathrm{d}P = P\mathrm{d}_\nabla$.

3. Show that the Chern-Simons transgression form (28.3), satisfies:
$$\mathrm{d}P(\nabla_0, \nabla_1) = P(R^k_{\nabla_1}) - P(R^k_{\nabla_0}).$$

4. Let $\psi : N \to M$ be a smooth map and $\xi = (\pi, E, M)$ a vector bundle of rank r with a connection ∇. Show that for all $P \in I^\bullet(GL(r))$,
$$\phi^* P(R^k_\nabla) = P(R^k_{\phi^*\nabla}).$$

5. Let $\xi = (\pi, E, M)$ be a vector bundle of rank r. Show that for k odd one has $[\sigma_{2k}(R^{2k})] = 0$, for any connection ∇ in ξ with curvature R.
(Hint: Consider some fiber metric in ξ and take a connection ∇ compatible with the metric.)

6. Let $\xi = (\pi, E, M)$ be a complex vector bundle. Show that the Chern classes of ξ and of its conjugate $\bar{\xi}$ are related by $c_k(\bar{\xi}) = (-1)^k c_k(\xi)$.

7. Let $\gamma_1^1(\mathbb{C})$ be the canonical complex line bundle over $\mathbb{P}^1(\mathbb{C}) = \mathbb{S}^2$. Show that $c_1(\gamma_1^1) = -1$, where $-1 \in H^2(\mathbb{P}^1(\mathbb{C}))$ is the canonical generator.

8. Show that the total Chern class of the tangent bundle to $\mathbb{P}^n(\mathbb{C})$ is given by:
$$c(T\mathbb{P}^n(\mathbb{C})) = (1+a)^n,$$
where $a \in H^2(\mathbb{P}^n(\mathbb{C}))$ is an appropriate generator.

9. Let $\sigma_r(X) = \det(X)$. Given a complex vector bundle $\xi = (\pi, E, M)$ of rank r, what is the relationship between the Chern class $\sigma_r(R^{2r}) \in H^{2r}(M)$ and the Euler class $\chi(\xi) \in H^{2r}(M)$?

Lecture 29. Fiber Bundles

Bundles with fiber which are not vector spaces also occur frequently in Differential Geometry. We will study them briefly in these last two lecturers.

Let $\pi : E \to M$ be a surjective submersion. A **trivializing chart** for π with fiber type F is a pair (U, ϕ), where $U \subset M$ is an open set and $\phi : \pi^{-1}(U) \to U \times F$ is a diffeomorphism such that the following diagram commuttes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

where $\pi_1 : U \times F \to U$ denotes the projection in the first factor. If $E_p = \pi^{-1}(p)$ is the fiber over $p \in U$ we obtain a diffeomorphism $\phi^p : E_p \to F$ as the composition of the maps:

$$\phi^p : E_p \xrightarrow{\phi} \{p\} \times F \longrightarrow F\ .$$

Hence, if $\mathbf{v} \in E_p$, we have $\phi(\mathbf{v}) = (p, \phi^p(\mathbf{v}))$. If one is given two trivializing charts (U_α, ϕ_α) and (U_β, ϕ_β) then we have the transition map:

$$\phi_\alpha \circ \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F,\ (p, f) \mapsto (p, \phi_\alpha^p \circ (\phi_\beta^p)^{-1}(f)).$$

This defines the **transition functions** $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Diff}(F)$, where $g_{\alpha\beta}(p) \equiv \phi_\alpha^p \circ (\phi_\beta^p)^{-1}$.

If one is given a covering of M by trivializing charts $\{(U_\alpha, \phi_\alpha) : \alpha \in A\}$, this leads to a cocycle $\{g_{\alpha\beta}\}$ with values in the group $\mathrm{Diff}(M)$. Because this is an infinite dimensional Lie group, we will restrict our attention to fibre bundles which have a finite dimensional structure group $G \subset \mathrm{Diff}(F)$. In other words, we assume that the we have an action of a Lie group G on F and we set:

Definition 29.1. *Let G be a Lie group and $G \times F \to F$ a smooth action. A* ***G-fiber bundle*** *over M with fiber type F is a triple $\xi = (\pi, E, M)$, where $\pi : E \to M$ is a smooth map admitting a collection of trivializing charts $\mathcal{C} = \{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ with fiber type F, satisfying the following properties:*

(i) $\{U_\alpha : \alpha \in A\}$ *is an open cover of* M*:* $\bigcup_{\alpha \in A} U_\alpha = M$*;*

(ii) The charts are compatible: for any $\alpha, \beta \in A$ there are smooth maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ such that the transition functions take the form:

$$(p, f) \mapsto (p, g_{\alpha\beta}(p) \cdot f);$$

(iii) The collection $\mathcal{C}$ is maximal: if (U, ϕ) is a trivializing chart with the property that for every $\alpha \in A$, the maps $p \mapsto \phi^p \circ (\phi^p_\alpha)^{-1}$ and $p \mapsto \phi^p_\alpha \circ (\phi^p)^{-1}$ factor through maps $U \cap U_\alpha \to G$, then $(U, \phi) \in \mathcal{C}$.

We shall use the same notation as in the case of vector bundles, so we have the **total space**, the **base space**, and the **projection** of the G-bundle. A collection of charts that satisfies only (i) and (ii) is called an **atlas of fiber bundle** or a **trivialization** of ξ. We define a **section** over an open set U in the obvious way and we denote the set of all sections over U by $\Gamma_U(E)$. Although a fiber bundle always as local sections, it may fail to have global sections.

Among the most important classes of G-bundles we have:

- **Vector bundles**: In this case the fiber F is a vector space and the structure group is the group of linear invertible transformations $G = GL(V)$. These are precisely the bundles that we have studied in the previous lectures.
- **Principal G-bundles** In this case the fiber F is itself a Lie group G and the structure group is the same Lie group G acting on itself by translations $G \times G \to G$, $(g, h) \mapsto gh$. We shall see that principal bundles play a central role among all G-bundles.

The notion of morphism of G-fiber bundles is similar to the notion of morphisms of vector bundles, where we replace $GL(r)$ by the structure group G.

Definition 29.2. *Let $\xi = (\pi, E, M)$ and $\xi' = (\pi', E', M')$ be two G-bundles with the same fiber F and structure group G. A* ***morphism of G-bundles*** *is a smooth map $\Psi : E \to E'$ mapping fibers of ξ to fibers of ξ', so Ψ cover a smooth map $\psi : M \to M'$:*

$$\begin{tikzcd} E \arrow[r, "\Psi"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "\psi"] & M' \end{tikzcd}$$

and such that for each $p \in M$, the map on the fibers

$$\Psi^p \equiv \Psi|_{E_p} : E_p \to E'_q, \quad (q = \psi(p))$$

satisfies

$$\phi'^q_\beta \circ \Psi^p \circ (\phi^p_\alpha)^{-1} \in G,$$

for any trivializations $\{\phi_\alpha\}$ *of* ξ *and* $\{\phi'_\beta\}$ *of* ξ'.

In this way, we have the category of fiber bundles with fiber type F structure group G. Just like in the case of vector bundles, we shall also distinguish between *equivalence* and *isomorphism* of G-bundles, according to wether the base map is the identity map or not.

The set of transition functions associated with an atlas of a G-fiber bundle completely determined the bundle. The discussion is entirely analogous to the case of vector bundles. First, if $\xi = (\pi, E, M)$ is a G-bundle the transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$, relative to some trivialization $\{(U_\alpha, \phi_\alpha)\}$, satisfy the cocycle condition:

$$g_{\alpha\beta}(p) g_{\beta\gamma}(p) = g_{\alpha\gamma}(p), \quad (p \in U_\alpha \cap U_\beta \cap U_\gamma).$$

We say that two cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ are equivalent if there exist smooth maps $\lambda_\alpha : U_\alpha \to G$ such that:

$$g'_{\alpha\beta}(p) = \lambda_\alpha(p) \cdot g_{\alpha\beta}(p) \cdot \lambda^{-1}_\beta(p), \quad (p \in U_\alpha \cap U_\beta).$$

One checks easily the following analogue of Proposition 23.5:

Proposition 29.3. *Let* M *be a manifold and* G *a Lie group acting on another smooth manifold* F. *Given a cocycle* $\{g_{\alpha\beta}\}$ *with values in* G, *subordinated to a covering* $\{U_\alpha\}$ *of* M, *there exists a* G*-bundle* $\xi = (\pi, E, M)$ *with fiber type* F *which admits an atlas* $\{\phi_\alpha\}$, *for which the transition functions give the cocycle* $\{g_{\alpha\beta}\}$. *Two equivalent cocycles determine isomorphic* G*-bundles.*

Let $\xi = (\pi, E, M)$ be a G-fiber bundle with fiber type F and let $\{g_{\alpha\beta}\}$ be a cocycle associated with a trivialization $\{\phi_\alpha\}$ of ξ. If $H \subset G$ is a Lie subgroup, we say that the **structure group of** ξ **can be reduced to** H if the cocycle is equivalent to a cocycle $\{g'_{\alpha\beta}\}$ where the transition functions take values in H:

$$g'_{\alpha\beta} : U_\alpha \cap U_\beta \to H \subset GL(r).$$

The next examples illustrate how the structure group (and its possible reductions) are intimately related with geometric properties of the bundle.

Examples 29.4.

1. *A fiber bundle* $\xi = (\pi, E, M)$ *with fiber type* F *and structure group* G *is trivial if and only if its structure group can be reduced to the trivial group* $\{e\}$.
2. *We saw before that a vector bundle of rank* r *is orientable if and only if its structure group can be reduced to* $GL^+(r)$. *Similarly, a vector bundles admits a fiber metric if and only if its structure group can be reduced to* $O(r)$ *(and by the polar decomposition, this can always be achieved). A further reduction of its structure group to* $SO(r)$ *amounts to an additional choice of an orientation for the bundle.*

Remark 29.5. In the definition of morphism of G-bundles the choice of a structure group is crucial. For example, a G-bundle may not be isomorphic to the trivial bundle via a morphism of G-bundles, but can be isomorphic to the trivial bundle via a morphism of G'-bundles, where $G' \supset G$ is a structure group containing G as a Lie subgroup. An example is given in the Homework at the end of this lecture.

Principal G-bundles play a special role among all G-bundles because of the following:

- To a G-bundle $\xi_F = (\pi, E, M)$ with fiber type F, we can associate a principal G-bundle $\xi = (\pi, P, M)$: we fix a trivialization $\{\phi_\alpha\}$ of ξ_F, so the associated cocycle $\{g_{\alpha\beta}\}$ takes values in G. Since G acts on itself by translations, this cocycle defines a G-bundle with fibre type G, which is a principal G-bundle.
- To a principal G-bundle $\xi = (\pi, P, M)$, a manifold F and an action of G on F, we can associate a G-bundle $\xi_F = (\pi, E, M)$ with fibre type F: a trivialization $\{\phi_\alpha\}$ of ξ, determines a cocycle $\{g_{\alpha\beta}\}$ with values in G. Since G acts in F, this cocycle defines a G-bundle ξ_F with fiber F.

Principal G-bundles can also be described more succinctly because of the following:

Proposition 29.6. *A fiber bundle $\xi = (\pi, P, M)$ is a principal G-bundle principal if and only if there exists a right action $P \times G \to P$ satisfying the following properties:*

(i) The action is free and proper;
(ii) The quotient P/G is a manifold, $M \simeq P/G$ and $\pi : P \to P/G \simeq M$ is the quotient map;
(iii) The local trivializations (U, ϕ) are G-equivariant: $\phi^p(g \cdot \mathbf{v}) = g \cdot \phi^p(\mathbf{v})$.

Let us explain how given a principal G-bundle $\xi = (\pi, P, M)$ one obtains the right action $P \times G \to P$. For this one chooses a trivializing chart (U, ϕ) and defines the action of G on $\pi^{-1}(U)$ by setting:

$$u \cdot g := \phi^{-1}(p, \phi^p(u)g), \quad (p = \pi(u)).$$

One checks easily that this definition is independent of the choice of trivialization.

Therefore, one can think of a principal G-bundle as being given by a free and proper right action $P \times G \to P$ for then $\xi = (\pi, P, M)$ where $M = P/G$ and $\pi : E \to M$ is the quotient map. Moreover, if $G \times F \to F$ is a smooth action the **associated fiber bundle** $\xi_F = (\pi_F, E, M)$ can be described explicitly as

$$E = P \times_G F,$$

where $P \times_G F$ denotes the quotient space for the action of G in $P \times F$ defined by:

$$(u, f) \cdot g \equiv (u \cdot g, g^{-1} \cdot f)$$

(recall that G acts on the right in P and on the left in F). The projection $\pi_F : E \to M$ is given by: $\pi_F([u, f]) = \pi(u)$.

This descriptions of principal G-bundles and the associated bundles allows us to give many examples of principal G-bundles and fibre bundles.

EXAMPLES 29.7.

1. *For any Lie group G, we have the trivial principal G-bundle $M \times G \to M$. Sections of this bundle are just smooth maps $M \to G$. Moreover, if G acts on some space F, then the associated bundle is also the trivial bundle $M \times F \to M$.*
2. *For any Lie group G and any closed subgroup $H \subset G$, the quotient $G \to G/H$ is a principal H-bundle. For example, if we let $\mathbb{S}^3$ be the group of unit quaternions and let $\mathbb{S}^1 \subset \mathbb{S}^3$ be the subgroup of unit complex numbers, then we obtain a principal $\mathbb{S}^1$-bundle, which is easily seen to be isomorphic to the Hopf fibration.*
3. *If $\pi : \tilde{M} \to M$ is the universal covering space of a manifold M, the triple $(\pi, \tilde{M}, M)$ is a principal bundle with structure group the fundamental group $\pi_1(M)$ (the topology in $\pi_1(M)$ is the discrete topology).*
4. *Let M be a smooth manifold of dimension d. The **frame bundle** is the principal bundle $\pi : F(M) \to M$ with structure group $GL(d)$ whose fiber over $p \in M$ consists of the set of all ordered basis of T_pM:*

$$F(M)_p = \{(\mathbf{v}_1, \dots, \mathbf{v}_r) : \ \mathbf{v}_1, \dots, \mathbf{v}_r \text{ é uma basisof } T_pM\}.$$

The group $GL(d)$ acts on the right on $F(M)$: if $u = (\mathbf{v}_1, \dots, \mathbf{v}_r)$ is a frame and $A = (a_i^j)$ is an invertible matrix, then $u \cdot A = (\mathbf{w}_1, \dots, \mathbf{w}_d)$ the frame:

$$\mathbf{w}_i = \sum_{j=1}^{d} a_i^j \mathbf{w}_j, \qquad (i = 1, \dots, d).$$

This is a proper and free action, hence $F(M)$ is a principal bundle with structure group $GL(d)$.

The group $GL(d)$ acts (on the left) in $\mathbb{R}^d$ by matrix multiplication. Hence, $F(M)$ has an associated fibre bundle with fiber $\mathbb{R}^d$, i.e., a vector bundle. We leave it as an exercise to check that this bundle is canonically isomorphic to the tangent bundle $T(M)$. Similarly, one obtains the cotangent bundle, exterior bundles, tensor bundle, etc., if one considers instead the induced actions of $GL(d)$ in $(\mathbb{R}^d)^*$, $\wedge^k\mathbb{R}^d$, $\otimes^k\mathbb{R}^d$, etc.

More generally, for any (real) vector bundle $\pi : E \to M$ of rank r, one can form the frame bundle $F(E)$, a principal bundle with structure group $GL(r)$. For the usual action of $GL(r)$ on $\mathbb{R}^r$ one obtains an associated bundle to $F(E)$ with fiber $\mathbb{R}^r$, which is canonically isomorphic to the original vector bundle $\pi : E \to M$. Similarly, one can obtained as associated bundles E^*, $\wedge^k E$, $\otimes^k E$, etc.

An entirely similar discussion is valid for complex vector bundles and the bundle of complex frames.

If $\xi = (\pi, P, M)$ is a principal G-bundle and $G \times F \to F$ is a smooth action, then one should expect that *any functorial construction in the associated bundle* $\xi_F = (\pi, E, M)$ *should be expressed in terms of* ξ *and* F. As an example of this principle, for the sections of ξ_F we have

Proposition 29.8. *Let $\xi = (\pi, P, M)$ be a principal G-bundle and $G \times F \to F$ a smooth action. The sections of the associated bundle $\xi_F = (\pi, E, M)$ are in one to one correspondence with the G-equivariant maps $h : P \to F$.*

Proof. The total space of the associated bundle is

$$E = P \times_G F = (P \times F)/G.$$

An element $v \in E_p$ is an equivalence class in $P_p \times_G F$, which can be written as:

$$v = [(u, h_p(u))], \quad \forall u \in P_p,$$

for a unique map $h_p : P_p \to F$ which is G-equivariant:

$$h_p(u \cdot g) = g^{-1} \cdot h_p(u).$$

Hence, a section $s : M \to E$ can be written in the form:

$$s(p) = [(u, h(u))], \quad \forall u \in P \text{ com } \pi(u) = p,$$

where $h : P \to F$ is a G-equivariant map. Conversely, a G-equivariant map $h : P \to F$ determines through this formula a section of ξ_F. $\square$

In order to understand the issue of reduction of the structure group, it is convenient to enlarge the notion of morphism of principal bundles as follows:

Definition 29.9. *Let $\xi' = (\pi', P', M')$ be a principal G'-bundle and $\xi = (\pi, P, M)$ a principal G-bundle. A* ***morphism*** *$\Psi : \xi' \to \xi$ is a pair formed by a smooth map $\Psi : P' \to P$ and a Lie group homomorphism $\Phi : G' \to G$, such that*

$$\Psi(u \cdot g) = \Psi(u)\Phi(g), \forall u \in P', g \in G'.$$

Since a morphism of principal bundles $\Psi : \xi' \to \xi$ takes fibers to fibers, Ψ covers a smooth map $\psi : M' \to M$:

$$\begin{tikzcd} P' \arrow[r, "\Psi"] \arrow[d, "\pi'"'] & P \arrow[d, "\pi"] \\ M' \arrow[r, "\psi"'] & M \end{tikzcd}$$

If $\Psi : P' \to P$ and $\Phi : G' \to G$ are both embeddings, one can identify P' and G' with its images $\Psi(P') \subset P$ and $\Phi(G') \subset G$. We then say that ξ' is a **subbundle** of the principal bundle ξ. When $M' = M$ and $\psi =$id, the sub bundle corresponds to the reduction of the structure group from G to H. In this case, we say that ξ' is a **reduced subbundle** of ξ. You should check that this matches the notion of reduction of the structure group we have introduced before in terms of cocycles.

HOMEWORK.

1. Show that $\xi = (\pi, P, M)$ is a principal G-bundle principal if and only if there exists a right action $P \times G \to P$ satisfying the following properties:
(i) The action is free and proper;
(ii) The quotient P/G is a manifold, $M \simeq P/G$ and $\pi : P \to P/G \simeq M$ is the quotient map;
(iii) The local trivializations (U, ϕ) are G-equivariant: $\phi^p(g \cdot \mathbf{v}) = g \cdot \phi^p(\mathbf{v})$.

2. Give a proof of Proposition 29.3

3. Consider the covering of $M = \mathbb{S}^1$ by the open sets:

$$U_{\pm} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} - \{(\pm 1, 0)\}.$$

Define a cocycle $\{g_{\alpha\beta}\}$ relative to this covering by

$$g_{+-}(x, y) = \begin{cases} I & \text{if } (x, y) \in y > 0, \\ \\ -I & \text{if } (x, y) \in y < 0. \end{cases}$$

where I is the 2×2 identity matrix Show that:
(a) This cocycle defines a G-bundle with fibre type $\mathbb{S}^1$ and structure group $\mathbb{S}^1 = S0(2)$ which is isomorphic (as an $\mathbb{S}^1$-bundle) to the trivial bundle.
(b) This cocycle defines a G-bundle with fibre type $\mathbb{S}^1$ and structure group $\mathbb{Z}_2 = \{I, -I\}$ which is not isomorphic (as a $\mathbb{Z}_2$-bundle) to the trivial bundle.

4. Show that a principal bundle is trivial if and only if it has a global section. (NOTE: This exercise is a very special case of the next exercise.)

5. Let $\xi = (\pi, P, M)$ be a principal G-bundle and $H \subset G$ a closed subgroup. Note that G acts in the quotient G/H hence there is an associate bundle $\xi_{G/H} = (\pi', P \times_G (G/H), M)$. Show that this bundle can be identified with the quotient $(\pi', P/H, M)$, where $\pi' : P/H \to M$ is the map induced in the quotient by $\pi : P \to M$. Show that the following statements are equivalent:
(a) The structure group of ξ can be reduced to H.
(b) The associated bundle $\xi_{G/H}$ has a section, i.e., there exists a map $s : M \to P/H$ such that $\pi' \circ s = \text{id}$.
(c) There exists a G-equivariant map $h : P \to G/H$.

6. Let M be a Riemannian manifold and let $\pi : OF(M) \to M$ be the principal $O(d)$-bundle formed by the orthogonal frames:

$$OF(M)_p = \{(\mathbf{v}_1, \dots, \mathbf{v}_r) : \ \mathbf{v}_1, \dots, \mathbf{v}_r \text{ is an orthonormal base of } T_pM\}.$$

Show that $OF(M)$ is the reduced bundle from $F(M)$, which corresponds to the reduction of the structure group from $GL(d)$ to $O(d)$.