CLASSICAL FIELD THEORY IN THE TIME OF COVID-19
PROBLEM SHEET V

Consider a two-dimensional metric manifold (Σ,γ) without boundary, $\partial\Sigma=\emptyset$, to be thought of as the spacetime of a field theory with the field bundle $\Sigma\times M\longrightarrow\Sigma$ whose typical fibre M supports a metric tensor (field) g and a de Rhama 3-coboundary $\mathrm{H}=\mathsf{d}\mathrm{B}\in\Omega^3(M)$, and with the (Polyakov) action functional given by the formula

$$S_{\mathrm{P}}[x]=-\tfrac{1}{2}\int_\Sigma \mathrm{Vol}(\Sigma)\sqrt{|\det\gamma|}\,\gamma^{-1}\lrcorner\, x^*\mathrm{g}+\int_\Sigma x^*\mathrm{B}\,,\qquad x\in[\Sigma,M]\,,$$

in which $\mathrm{Vol}(\Sigma)$ is the standard volume form on Σ. Let G be the Lie group (with the tangent Lie algebra $\mathfrak{g}$) of those autodiffeomorhisms M, acting as

$$\lambda_\cdot\ :\ \mathrm{G}\times M\longrightarrow M\ :\ (g,m)\longmapsto\lambda_g(m)\,,$$

that preserve the metric g and the potential B,

$$\forall_{g\in\mathrm{G}}\ :\ \big(\qquad\lambda_g^*\mathrm{g}=\mathrm{g}\qquad\wedge\qquad\lambda_g^*\mathrm{B}=\mathrm{B}\qquad\big)\,.$$

Consider "small" gauge transformations (from the connected component of the group unit in G)

$$\exp\big(\varepsilon\,\Lambda(\cdot)\big)\ :\ \Sigma\longrightarrow\mathrm{G}\,,\qquad\qquad\Lambda(\cdot)\ :\ \Sigma\longrightarrow\mathfrak{g}\,,\qquad\varepsilon\approx 0$$

in the gauge determined by the *trivial* principal bundle $\mathrm{G}\longrightarrow\Sigma\times\mathrm{G}\longrightarrow\Sigma$ over Σ. How does the value of the action functional S_{P} change upon replacement of the lagrangean field $x(\cdot)\in[\Sigma,M]$ with its gauge transform $\lambda_{\exp(\varepsilon\,\Lambda(\cdot))}(x(\cdot))$ to the *first* order in the small ε? What should be the transformation properties of the correction

$$\alpha\in\Gamma\big(\mathsf{T}^*\Sigma\otimes_\Sigma\mathsf{T}M\restriction_{x(\Sigma)}\big)$$

of the exterior derivative $\mathsf{d}x$ present in the Polyakov action functional for the corrected derivative,

$$Dx:=\mathsf{d}x+\alpha\,,$$

to define, upon substitution for the original one in S_P, a functional invariant with respect to the above "small" gauge transformations? Does the result hold if we replace the invariance condition for B with the weaker *quasi*-invariance condition:

$$\forall_{g\in\mathrm{G}}\ :\ \lambda_g^*\mathrm{B}=\mathrm{B}+\mathsf{d}\kappa_g\,?$$