CLASSICAL FIELD THEORY IN THE TIME OF COVID-19
PROBLEM SHEET III

Problem 1.

WRAPPING-CHARGE EXTENSIONS OF THE GLOBAL-SYMMETRY ALGEBRA IN THE TWO-DIMENSIONAL σ-MODEL ON THE CYLINDER

Consider the cylinder $\Sigma \equiv \mathbb{R}\times\mathbb{S}^1 \ni (t,\phi) \equiv \sigma$ (termed the **worldsheet** in this context), with the non-compact direction timelike and the compact one spacelike with respect to the global metric

$$\eta(\sigma) = -\mathsf{d}t\otimes\mathsf{d}t + \mathsf{d}\phi\otimes\mathsf{d}\phi\,,$$

as the spacetime of the two-dimensional theory of embeddings $x\ :\ \Sigma\longrightarrow M$ of Σ in a metric manifold (M,g) (termed the **target space**) endowed with a de Rham 3-cocycle $\mathrm{H}\in Z^3_{\mathrm{dR}}(M)$ with periods $\mathrm{Per}(\mathrm{H})\subset 2\pi\mathbb{Z}$, the theory being defined by the principle of least action for the Dirac–Feynman amplitude

$$\mathcal{A}_{\mathrm{DF}}\ :\ C^\infty(\Sigma,M)\longrightarrow \mathrm{U}(1)\ :\ x\longmapsto \mathcal{A}_{\mathrm{metr}}[x]\cdot\mathcal{A}_{\mathrm{top}}[x]\equiv\mathcal{A}_{\mathrm{DF}}[x]$$

whose first (metric) factor takes the form

$$\mathcal{A}_{\mathrm{metr}}[x] = \exp\big(-\tfrac{\mathrm{i}}{\hbar}\,\mu\int_\Sigma \mathrm{Vol}(\Sigma,\eta)\,x^*\mathrm{g}\big(\eta^*\big)\,\big)\equiv\exp\big(-\tfrac{\mathrm{i}}{\hbar}\,\mu\int_\Sigma\mathcal{L}_{\mathrm{metr}}(\sigma,x,\partial x)\big)$$

written in terms of a (mass) parameter $\mu\in\mathbb{R}_{>0}$, the metric volume form $\mathrm{Vol}(\Sigma,\eta)$ on Σ, and the bi-vector

$$\eta^*(\sigma) = -\tfrac{1}{2}\,\tfrac{\partial}{\partial t}\otimes\tfrac{\partial}{\partial t}+\tfrac{1}{2}\,\tfrac{\partial}{\partial\phi}\otimes\tfrac{\partial}{\partial\phi}\,,$$

dual to η, whereas the second (topological, or Wess–Zumino) term

$$\mathcal{A}_{\mathrm{top}}[x] = \mathrm{Hol}_{\mathcal{G}}\big(x(\Sigma)\big)$$

is a **Cheeger–Simons differential character of degree 2 modulo** $2\pi\mathbb{Z}$ termed the **surface holonomy** the **1-gerbe** $\mathcal{G}$ of the curvature $\mathrm{curv}(\mathcal{G})=\mathrm{H}$, *i.e.*, a(n abelian-)group homomorphism

$$\mathrm{Hol}_{\mathcal{G}}\ :\ Z_2(M)\longrightarrow\mathrm{U}(1)$$

(where the abelian group $Z_2(M)$ of 2-cycles (that is closed submanifolds) in M, with the disjoint union as the binary operation) with the property – crucial to our definition –

$$\forall_{c\in C_3(M)}\ :\ \mathrm{Hol}_{\mathcal{G}}(\partial c) = \exp\big(\tfrac{\mathrm{i}}{\hbar}\int_c\mathrm{curv}(\mathcal{G})\big)$$

(here, $C_3(M)$ is the (abelian) group of 3-cochains in M). The theory is called the (**monophase**) **two-dimensional non-linear bosonic σ-model** and it is to be understood as a theory of fields with a local (on M) presentation $(x^\mu)^{\mu\in\overline{1,D}}$, $D\equiv\dim M$, where the x^μ are local coordinates on the target space. As for the topological term, we may always assume hereunder that

$$\mathrm{Hol}_{\mathcal{G}}\big(x(\Sigma)\big) =_{\mathrm{loc.}} \exp\big(\tfrac{\mathrm{i}}{\hbar}\int_\Sigma x^*\mathrm{B}\big)$$

for $\mathrm{B}\in\Omega^2(\mathcal{O})$ a locally smooth primitive of H on some open set $\mathcal{O}\subset M$, *i.e.*, $\mathrm{H}=_{\mathrm{loc.}}\mathsf{d}\mathrm{B}$. This is to be compared with the situation encountered in the discussion of the lagrangean model of propagation of a massive charged point-like particle in externeal fields: gravitational and electromagnetic.

Perform the symmetry analysis of the field theory introduced above by solving the following sub-problems:

(i) Find the Euler–Lagrange equations for the (local) fields x^μ. What do they describe if $\mathrm{H}\equiv 0$?

(ii) Derive the presymplectic form of the field theory, expressing it in terms of the variables: $\underline{x}^\mu\equiv x^\mu\restriction_{\Sigma_t}$ (the restriction of a classical field configuration to the Cauchy hypersurface given by the equitemporal slice $\Sigma_t\equiv\{t\}\times\mathbb{S}^1\cong\mathbb{S}^1$) and the *kinetic* momentum $\mathrm{p}_\mu := \frac{\partial\mathcal{L}_{\mathrm{metr}}}{\partial\partial_t x^\mu}$.

(iii) Check that flows of vector fields $\mathcal{K}\in\Gamma(\mathrm{T}M)$ that are Killing for g and satisfy the **strong invariance indentity**

(1) $$\mathcal{K}\lrcorner\,\mathrm{H} = -\mathsf{d}\kappa$$

for some $\kappa\in\Omega^1(M)$ are (global) symmetries of the theory.

(iv) Find the Noether charges $Q_{\mathcal{K}}$ for the above symmetries. In so doing, lift the Killing fields $\mathcal{K}$ to vector fields $\widetilde{\mathcal{K}}[\underline{x},\mathrm{p}] = \mathcal{K}[\underline{x}] + \Delta_\mu[\underline{x},\mathrm{p}]\,\frac{\delta}{\delta \mathrm{p}_\mu}$ on the space of states by solving the constraints $\mathscr{L}_{\widetilde{\mathcal{K}}}\theta \overset{!}{=} 0$ for θ the so-called (**kinetic-**)**action 1-form** with a local presentation $\theta[\underline{x},\mathrm{p}] = \mathrm{p}_\mu\,\delta\underline{x}^\mu$. Interpret the constraints (differential-geometrically).

(v) Prove that Killing vector fields satisfying Eq. (1) span a Lie subalgebra $\mathfrak{g}_\sigma$ in the Lie algebra $(\Gamma(\mathsf{T}M),[\cdot,\cdot])$. Demonstrate that the Lie algebra spanned by the extensions $\widetilde{\mathcal{K}}$ is (identically) isomorphic with $\mathfrak{g}_\sigma$.

(vi) Fix a basis $\{\mathcal{K}_A\}_{A\in\overline{1,d}}$, $d\equiv\dim\mathfrak{g}_\sigma$ in $\mathfrak{g}_\sigma$ and denote as κ_X, $X = X^A\,\mathcal{K}_A$ the 1-forms on M which satisfy the identities

$$\mathsf{d}\kappa_X = -X \lrcorner \mathrm{H}\,.$$

Identify a potential source of a (wrapping-charge) central extension of $\mathfrak{g}_\sigma$ furnished by the Poisson algebra of the Noether charges $\{Q_A\equiv Q_{\mathcal{K}_A}\}_{A\in\overline{1,d}}$ whenever

$$\big(H^1(M,\mathbb{R})\ni\big)[\mathscr{L}_X\kappa_Y - \kappa_{[X,Y]}]_{\mathrm{dR}}\not\equiv 0\,,\qquad X,Y\in\mathfrak{g}_\sigma$$

(in the standard de Rham cohomology).

Problem 2.

Central extensions and Lie-algebra cohomology

Let $(\mathcal{P},\Omega)$ be the (pre)symplectic space of states of a field theory, and let $\{\widetilde{\mathcal{K}}_A\}_{A\in\overline{1,N}}$ be a basis in the $\mathbb{R}$-linear space of vector fields on $\mathcal{P}$ generating the respective one-parameter families of symmetry transformations of the field theory, spanning a Lie subalgebra ($[\cdot,\cdot]$ is the Lie bracket of vector fields on $\mathcal{P}$)

$$\Big(\mathfrak{s}:=\bigoplus_{A=1}^N\big\langle\widetilde{\mathcal{K}}_A\big\rangle_{\mathbb{R}}\,,[\cdot,\cdot]\restriction_{\mathfrak{s}\times\mathfrak{s}}\Big)\subset\big(\Gamma(\mathsf{T}\mathcal{P}),[\cdot,\cdot]\big)$$

with structure equations

$$[\widetilde{\mathcal{K}}_A,\widetilde{\mathcal{K}}_B] = f_{AB}{}^C\,\widetilde{\mathcal{K}}_C\,.$$

To these vector fields, we have associated the corresponding Noether charges $\{Q_A\}_{A\in\overline{1,N}}\subset C^\infty(\mathcal{P},\mathbb{R})$ that solve the *defining* equations

(2) $$\delta Q_A = -\widetilde{\mathcal{K}}_A\lrcorner\,\Omega\,,\qquad A\in\overline{1,N}\,.$$

Our field-theoretic considerations, backed up by concrete examples, have led us to contemplate Poisson relations for the charges $\widetilde{Q}_A = -Q_A$ in the general form

$$\{\widetilde{Q}_A,\widetilde{Q}_B\}_\Omega = f_{AB}{}^C\,\widetilde{Q}_C + C_{AB}\,\mathbf{1}\,,$$

where the $C_{AB} = -C_{BA}(\not\equiv 0)$ are constants (this is emphasised by inserting the constant function $\mathbf{1}\in C^\infty(\mathcal{P},\mathbb{R})$ in the above equation). In this manner, we have obtained a Lie subalgebra

$$\Big(\widehat{\mathfrak{s}}:=\bigoplus_{A=1}^N\big\langle\widetilde{Q}_A\big\rangle_{\mathbb{R}}\oplus\langle\mathbf{1}\rangle_{\mathbb{R}}\,,\{\cdot,\cdot\}_\Omega\restriction_{\widehat{\mathfrak{s}}\times\widehat{\mathfrak{s}}}\Big)\subset\big(C^\infty(\mathcal{P},\mathbb{R}),\{\cdot,\cdot\}_\Omega\big)\,.$$

Indeed, we have, trivially,

$$\{\widetilde{Q}_A,\mathbf{1}\}_\Omega = 0 = \{\mathbf{1},\mathbf{1}\}_\Omega\,,$$

and so the Poisson bracket closes on the subspace $\widehat{\mathfrak{s}}$, and the bracket is a Lie bracket by construction (due to the closedness of the presymplectic form Ω). We shall, next, try to understand the relation between $\mathfrak{s}$ and $\widehat{\mathfrak{s}}$ conceptually, with view to systematising our knowledge on field-theoretic realisations of lagrangean symmetries.

Thus, note that the Jacobi identity

$$\mathrm{Jac}_{\widehat{\mathfrak{s}}}\big(\widetilde{Q}_A,\widetilde{Q}_B,\widetilde{Q}_C\big) = 0$$

implies the relations

(3) $$f_{AB}{}^D\,C_{DC} + f_{CA}{}^D\,C_{DB} + f_{BC}{}^D\,C_{DA} = 0\,,\qquad A,B,C\in\overline{1,N}\,.$$

Consider the (abstract vector-space) dual $\mathfrak{s}^*$ of $\mathfrak{s}$ and introduce a basis

$$\mathcal{B}^*\equiv\{\kappa^A\}^{A\in\overline{1,N}}$$

of the former vector space dual to the basis

$$\mathcal{B}\equiv\{\widetilde{\mathcal{K}}_A\}_{A\in\overline{1,N}}$$

of $\mathfrak{s}$, determined by the relations

$$\kappa^A\big(\widetilde{\mathcal{K}}_B\big)=\delta^A_{\ B}\,.$$

The constants C_{AB} give rise to a 2-form on $\mathfrak{s}$ given by

$$\Theta:=C_{AB}\,\kappa^A\wedge\kappa^B\in\mathfrak{s}^*\wedge\mathfrak{s}^*$$

that we shall call the **extension 2-cocycle**. The consistency conditions (3) can now be written concisely as

$$\Theta\big([\widetilde{\mathcal{K}}_A,\widetilde{\mathcal{K}}_B],\widetilde{\mathcal{K}}_C\big)+\Theta\big([\widetilde{\mathcal{K}}_C,\widetilde{\mathcal{K}}_A],\widetilde{\mathcal{K}}_B\big)+\Theta\big([\widetilde{\mathcal{K}}_B,\widetilde{\mathcal{K}}_C],\widetilde{\mathcal{K}}_A\big)=0\,.$$

Taking into account the tri-$\mathbb{R}$-linearity of the right-hand side, we may rewrite the above relation as

$$\forall_{\mathcal{U},\mathcal{V},\mathcal{W}\in\mathfrak{s}}\ :\ \Theta\big([\mathcal{U},\mathcal{V}],\mathcal{W}\big)+\Theta\big([\mathcal{W},\mathcal{U}],\mathcal{V}\big)+\Theta\big([\mathcal{V},\mathcal{W}],\mathcal{U}\big)=0\,.$$

The left-hand side of the above equality is a 3-form on $\mathfrak{s}$ fully determined by Θ (and the structure of $\mathfrak{s}$ itself) that we shall denote as

$$\delta^{(2)}_{\mathfrak{s}}\Theta(\mathcal{U},\mathcal{V},\mathcal{W}):=\Theta\big([\mathcal{U},\mathcal{V}],\mathcal{W}\big)+\Theta\big([\mathcal{W},\mathcal{U}],\mathcal{V}\big)+\Theta\big([\mathcal{V},\mathcal{W}],\mathcal{U}\big)$$

and call the **3-coboundary** of Θ. Thus, the 3-coboundary of the extension 2-cocycle vanishes identically.

In the next step, recall our simple test of 'Lie-algebraic triviality' of the one-dimensional extension $\widehat{\mathfrak{s}}$ of $\mathfrak{s}$: If we can shift the Noether charges $\widetilde{Q}_A$ by the respective constants $\Delta_A\in\mathbb{R}$,

$$\widetilde{Q}_A\longmapsto\widetilde{Q}_A-\Delta_A\,\mathbf{1}=:\widehat{Q}_A\,,$$

as allowed by the defining Eq. (2), in such a manner that the Poisson relations

$$\{\widehat{Q}_A,\widehat{Q}_B\}_\Omega=f_{AB}^{\ \ C}\,\widehat{Q}_C$$

hold true for the shifted charges, we are right to consider the extension 'trivial'. For this to be the case, we need

$$C_{AB}=-f_{AB}^{\ \ C}\,\Delta_C\,,\qquad A,B\in\overline{1,N}\,,$$

which we can rephrase in terms of the 1-form on $\mathfrak{s}$ given by

$$\mu:=\Delta_A\,\kappa^A\in\mathfrak{s}^*$$

as

$$\forall_{\mathcal{U},\mathcal{V}\in\mathfrak{s}}\ :\ \Theta(\mathcal{U},\mathcal{V})=-\mu\big([\mathcal{U},\mathcal{V}]\big)\,.$$

The right-hand side of the above equality is a 2-form on $\mathfrak{s}$ fully determined by μ (and the structure of $\mathfrak{s}$ itself) that we shall denote as

$$\delta^{(1)}_{\mathfrak{s}}\mu(\mathcal{U},\mathcal{V}):=-\mu\big([\mathcal{U},\mathcal{V}]\big)$$

and call the **2-coboundary** of μ. We may now restate our condition of 'triviality' in a simple manner: The extension is trivial if the extension 2-cocycle is a 2-coboundary. We then have

$$\Theta=\delta^{(1)}_{\mathfrak{s}}\mu\qquad\Longrightarrow\qquad\delta^{(2)}_{\mathfrak{s}}\Theta\equiv\delta^{(2)}_{\mathfrak{s}}\big(\delta^{(1)}_{\mathfrak{s}}\mu\big)\equiv\big(\delta^{(2)}_{\mathfrak{s}}\circ\delta^{(1)}_{\mathfrak{s}}\big)\mu\equiv0\,.$$

Clearly, the identity

$$\delta^{(2)}_{\mathfrak{s}}\circ\delta^{(1)}_{\mathfrak{s}}\equiv0$$

follows from the Jacobi identity for the underlying Lie algebra $\mathfrak{s}$.

The scheme uncovered above sounds (or at least should sound) familiar to those of us who have come across a serious discussion of the exterior algebra $(\Omega^\bullet(M)\equiv\bigoplus_{k=0}^{\dim M}\Omega^k(M),\mathsf{d})$ of differential forms on a manifold M. Indeed, in that context, we encounter the de Rham coboundary operator(s)

$$\mathsf{d}^{(k)}_{\rm dR}\equiv\mathsf{d}\ :\ \Omega^k(M)\longrightarrow\Omega^{k+1}(M)\,,\qquad k\in\overline{0,\dim M}\,,$$

with the understanding that $\Omega^{\dim M+1}\equiv\mathbf{0}$. The fundamental property

$$\mathsf{d}^{(k+1)}_{\rm dR}\circ\mathsf{d}^{(k)}_{\rm dR}\equiv0$$

leads to the emergence of the (real) **de Rham cohomology**

$$H^\bullet_{\rm dR}(M)=\bigoplus_{k=0}^{\dim M}H^k(M)\,,$$

with the **$(k+1)$-th de Rham cohomology group** defined as the (abelian) quotient group

$$H^{k+1}(M)=\ker\mathsf{d}^{(k+1)}_{\rm dR}/{\rm im}\,\mathsf{d}^{(k)}_{\rm dR}\,,\qquad k\in\overline{0,\dim M}\,,\qquad\qquad H^0(M)\equiv\ker\mathsf{d}^{(0)}_{\rm dR}\big(\equiv\mathbb{R}^{|\pi_0(M)|}\big)\,.$$

As we have seen before, the de Rham cohomology and its geometrisations are relevant to the description and study of the dynamics of systems endowed with topological charge (*e.g.*, electromagnetically charged point-like particles). Now, in the field-theoretic context of interest, natural question arise: Does our construction extend to a fully fledged cohomology on $\mathfrak{g}$? If so, does the component of that cohomology discovered in our simple considerations effectively quantify (physically motivated) extensions of a given Lie algebra (of symmetries)? Answering these mutually entangled questions constitutes the goal of the present Problem.

(i) **The cohomology of a Lie algebra with values in a module.**
We begin by introducing an ancillary concept in

Definition 1. Let $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra over a base field $\mathbb{K}$. A (**left**) **$\mathfrak{g}$-module** is a pair $(((V,+_V,\mathsf{P}_V\equiv -(\cdot),\bullet\longmapsto 0_V),\rhd_{\mathbb{K}}),\rhd.)$ composed of a $\mathbb{K}$-linear space $((V,+_V,\mathsf{P}_V,\bullet\longmapsto 0_V),\rhd_{\mathbb{K}})$ (here, $\rhd_{\mathbb{K}}$ is the action of the base field $\mathbb{K}$ on the abelian group V) endowed with a bi-$\mathbb{K}$-linear mapping

$$\ell.\;:\;\mathfrak{g}\times V\longrightarrow V\;:\;(X,v)\longmapsto X\rhd_{\mathfrak{g}}v\equiv\ell_X(v)$$

satisfying – for any $X_1,X_2\in\mathfrak{g}$ and $v\in V$ – the identity

$$[X_1,X_2]_{\mathfrak{g}}\rhd_{\mathfrak{g}}v=X_1\rhd_{\mathfrak{g}}(X_2\rhd_{\mathfrak{g}}v)-X_2\rhd_{\mathfrak{g}}(X_1\rhd_{\mathfrak{g}}v)\,.$$

In what follows, we write $\rhd\equiv\rhd_{\mathfrak{g}}$ (whenever it does not lead to confusion) to unclutter the notation.

◇

Next, we generalise the previously contemplated algebraic concept in

Definition 2. Let $(\mathfrak{g},[\cdot,\cdot\}_{\mathfrak{g}})$ and $(\mathfrak{a},[\cdot,\cdot\}_{\mathfrak{a}})$ be two Lie algebras over a common base field $\mathbb{K}$. A **central extension of $\mathfrak{g}$ by $\mathfrak{a}$** is a Lie algebra $(\widetilde{\mathfrak{g}},[\cdot,\cdot]_{\widetilde{\mathfrak{g}}})$ over $\mathbb{K}$ described by the short exact sequence of Lie algebras

$$\mathbf{0}\longrightarrow\mathfrak{a}\xrightarrow{\jmath}\widetilde{\mathfrak{g}}\xrightarrow{\pi}\mathfrak{g}\longrightarrow\mathbf{0}\,,\tag{4}$$

written in terms of an Lie-algebra monomorphism $\jmath$ and of a Lie-algebra epimorphism π, and such that $\jmath(\mathfrak{a})\subset\mathfrak{z}(\widetilde{\mathfrak{g}})$ (the centre of $\widetilde{\mathfrak{g}}$). Hence, in particular, $\mathfrak{a}$ is necessarily commutative, that is $[\cdot,\cdot\}_{\mathfrak{a}}\equiv 0$.

Whenever π admits a **section**, *i.e.*, there exists a Lie-algebra homomorphism

$$\sigma\;:\;\mathfrak{g}\longrightarrow\widetilde{\mathfrak{g}}$$

with the property

$$\pi\circ\sigma=\mathrm{id}_{\mathfrak{g}}\,,$$

the central extension is said to **split**.

An equivalence of central extensions $\widetilde{\mathfrak{g}}_A, A\in\{1,2\}$ of $\mathfrak{g}$ by $\mathfrak{a}$ is represented by a commutative diagram

$$\begin{array}{ccccccccc} & & & & \widetilde{\mathfrak{g}}_1 & & & & \\ & & & \nearrow & & \searrow & & & \\ \mathbf{0} & \longrightarrow & \mathfrak{a} & & \downarrow\cong & & \mathfrak{g} & \longrightarrow & \mathbf{0}\;, \\ & & & \searrow & & \nearrow & & & \\ & & & & \widetilde{\mathfrak{g}}_2 & & & & \end{array}$$

in which the vertical arrow is a Lie-algebra isomorphism.

◇

Finally, we generalise our physically motivated construction of coboundary operators and identify the ensuing cohomology in

Definition 3. Let $(\mathfrak{g},[\cdot,\cdot\}_{\mathfrak{g}})$ be a Lie algebra over a base field $\mathbb{K}$ and let $(V,\ell.)$ be a $\mathfrak{g}$-module. A **p-cochain on $\mathfrak{g}$ with values in V** (also termed a **V-valued p-form on $\mathfrak{g}$**) is a p-linear map $\underset{(p)}{\varphi}:\mathfrak{g}^{\times p}\longrightarrow V$ that is totally skew-symmetric, *i.e.*, for any $X_i\in\mathfrak{g}$, $i\in\overline{1,p}$, it satisfies

$$\forall_{j\in\overline{1,p-1}}\;:\;\underset{(p)}{\varphi}(X_1,X_2,\ldots,X_{j-1},X_{j+1},X_j,X_{j+2},X_{j+3},\ldots,X_p)=-\underset{(p)}{\varphi}(X_1,X_2,\ldots,X_p)\,.$$

Such maps form a **group of p-cochains on $\mathfrak{g}$ with values in V**, denoted by $C^p(\mathfrak{g},V)$. The family of these groups indexed by $p\in\overline{0,\dim_{\mathbb{K}}\mathfrak{g}}$ forms a bounded complex

$$C^\bullet(\mathfrak{g},V)\qquad:\qquad C^0(\mathfrak{g},V)\xrightarrow{\delta^{(0)}_{\mathfrak{g}}}C^1(\mathfrak{g},V)\xrightarrow{\delta^{(1)}_{\mathfrak{g}}}\cdots\xrightarrow{\delta^{(p-1)}_{\mathfrak{g}}}C^p(\mathfrak{g},V)\xrightarrow{\delta^{(p)}_{\mathfrak{g}}}\cdots\xrightarrow{\delta^{(\dim_{\mathbb{K}}\mathfrak{g}-1)}_{\mathfrak{g}}}C^{\dim_{\mathbb{K}}\mathfrak{g}}(\mathfrak{g},V)$$

with the coboundary operators

$$\delta_{\mathfrak{g}}^{(p)} \,:\, C^p(\mathfrak{g},V) \longrightarrow C^{p+1}(\mathfrak{g},V)$$

determined by the formulæ (written for arbitrary elements $X, X_i \in \mathfrak{g}$, $i \in \overline{1,p+1}$ and $\underset{(p)}{\varphi} \in C^p(\mathfrak{g},V)$ for $p \in \overline{1,\dim_{\mathbb{K}}\mathfrak{g}-1}$)

$$\begin{aligned} \big(\delta_{\mathfrak{g}}^{(0)} \underset{(0)}{\varphi}\big)(X) &:= X \triangleright \underset{(0)}{\varphi}\,, \\ \big(\delta_{\mathfrak{g}}^{(p)} \underset{(p)}{\varphi}\big)(X_1, X_2, \ldots, X_{p+1}) &:= \sum_{i=1}^{p+1} (-1)^{i-1} X_i \triangleright \underset{(p)}{\varphi}(X_1, X_2, \underset{\widehat{i}}{\ldots}, X_{p+1}) + \sum_{1\le i<j\le p+1} (-1)^{i+j} \underset{(p)}{\varphi}([X_i,X_j]_{\mathfrak{g}}, X_1, X_2, \underset{\widehat{i,j}}{\ldots}, X_{p+1})\,, \\ \delta_{\mathfrak{g}}^{(\dim_{\mathbb{K}}\mathfrak{g})} &\equiv 0\,. \end{aligned}$$

We distinguish the **group of p-cocycles**

$$Z^p(\mathfrak{g},V) := \ker \delta_{\mathfrak{g}}^{(p)}\,,$$

and the **group of p-coboundaries**

$$B^p(\mathfrak{g},V) := \mathrm{im}\, \delta_{\mathfrak{g}}^{(p-1)}\,.$$

The **homology groups** of the complex $\big(C^\bullet(\mathfrak{g},V), \delta_{\mathfrak{g}}^{(\bullet)}\big)$ are called the **cohomology groups of $\mathfrak{g}$ with values in V** and denoted by

$$H^p(\mathfrak{g},V) := Z^p(\mathfrak{g},V)/B^p(\mathfrak{g},V)\,, \qquad p \in \overline{1,\dim_{\mathbb{K}}\mathfrak{g}}\,, \qquad\qquad H^0(\mathfrak{g},V) := Z^0(\mathfrak{g},V)\,.$$

◇

(i.1) Write out the first three nontrivial coboundary operators: $\delta_{\mathfrak{g}}^{(p)}$, $p \in \{1,2,3\}$ for $\triangleright$ trivial, *i.e.*, such that $X \triangleright v = 0_V$ for arbitrary $X \in \mathfrak{g}$ and $v \in V$.

(i.2) Prove the identities

$$\delta_{\mathfrak{g}}^{(p)} \circ \delta_{\mathfrak{g}}^{(p-1)} = 0\,, \qquad p \in \overline{1,\dim_{\mathbb{K}}\mathfrak{g}}\,.$$

(i.3) Use the Jacobi identity for $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ to induce on $\mathfrak{g}$ a natural structure of a $\mathfrak{g}$-module (the so-called ad.-module). Reinterpret the said identity in terms of the ensuing $\mathfrak{g}$-valued Lie-algebra cohomology of $\mathfrak{g}$.

(ii) **The algebraic meaning of $H^2(\mathfrak{g},\mathfrak{a})$.**

We shall now establish a natural correspondence between classes in $H^2(\mathfrak{g},\mathfrak{a})$ and equivalence classes of supercentral extensions of $\mathfrak{g}$ by a commutative Lie algebra $\mathfrak{a}$ considered as a $\mathfrak{g}$-module with the *trivial* $\mathfrak{g}$-action. We begin our discussion with

Proposition 4. *Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra, and let $(\mathfrak{a}, 0)$ be a commutative Lie algebra. An equivalence class of central extensions $(\widetilde{\mathfrak{g}}, [\cdot,\cdot]_{\widetilde{\mathfrak{g}}})$ of $\mathfrak{g}$ by $\mathfrak{a}$ canonically determines a class in $H^2(\mathfrak{g},\mathfrak{a})$. This class vanishes iff the short exact sequence determined by the extensions splits.*

Prove the Proposition by using the vector-space isomorphism (demonstrate that it is well-defined and that it is what we call it!)

$$\widetilde{\imath} \,:\, \widetilde{\mathfrak{g}} \xrightarrow{\cong} \mathfrak{a} \oplus \mathfrak{g} \,:\, \widetilde{X} \longmapsto \big(\jmath^{-1}\big(\widetilde{X} - \sigma \circ \pi(\widetilde{X})\big), \pi_{\mathfrak{g}}(\widetilde{X})\big)$$

induced by a $\mathbb{K}$-linear section σ of π, *i.e.*, of a $\mathbb{K}$-linear map $\sigma \,:\, \mathfrak{g} \longrightarrow \widetilde{\mathfrak{g}}$ with the property $\pi_{\mathfrak{g}} \circ \sigma = \mathrm{id}_{\mathfrak{g}}$. Next, use $\widetilde{\imath}$ to induce on $\mathfrak{a} \oplus \mathfrak{g}$ a Lie bracket that extends $[\cdot,\cdot]_{\mathfrak{g}}$ on $\mathfrak{g}$ in such a manner that $\widetilde{\imath}$ is promoted to the rank of a Lie-algebra isomorphism. The first part is proven by considering (linear, symmetry and cohomological) properties of the mapping

$$\Theta_\sigma \,:\, \mathfrak{g}^{\times 2} \longrightarrow \mathfrak{a} \,:\, (X_1, X_2) \longmapsto \jmath_{\mathfrak{a}}^{-1}\left([\sigma(X_1), \sigma(X_2)]_{\widetilde{\mathfrak{g}}} - \sigma([X_1,X_2]_{\mathfrak{g}})\right)\,,$$

whereupon the mapping

$$\varepsilon^{-1} \circ \sigma_2 - \sigma_1\,,$$

defined for the vertical isomorphism $\varepsilon \,:\, \widetilde{\mathfrak{g}}_1 \xrightarrow{\cong} \widetilde{\mathfrak{g}}_2$ of the equivalence and for the sections $\sigma_A \,:\, \mathfrak{g} \longrightarrow \widetilde{\mathfrak{g}}_A$, $A \in \{1,2\}$ that determine the respective extensions of $\mathfrak{g}$ by $\mathfrak{a}$, should be scrutinised in order to establish cohomological equivalence of extension 2-cocycles coming from equivalent extensions.

The second part of the Proposition is concerned with the situation in which $\Theta_\sigma = \delta_{\mathfrak{g}}^{(1)}\mu$. Thus, one should look at the linear mapping

$$\sigma_\mu := \sigma - \jmath_{\mathfrak{a}} \circ \mu \in \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g}, \widetilde{\mathfrak{g}})\,.$$

From the point of view of physical applications, it is of utmost significance that the assignment of classes in $H^2(\mathfrak{g},\mathfrak{a})$ to central extensions of $\mathfrak{g}$ by a commutative Lie algebra $\mathfrak{a}$ detailed above may, in fact, be inverted. This is stated in

Proposition 5. *Let* $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$ *be a Lie algebra, and let* $(\mathfrak{a},0)$ *be a commutative Lie algebra, regarded as a trivial* $\mathfrak{g}$*-module. A class in* $H^2(\mathfrak{g},\mathfrak{a})$ *canonically induces an equivalence class of central extensions* $(\widetilde{\mathfrak{g}},[\cdot,\cdot]_{\widetilde{\mathfrak{g}}})$ *of* $\mathfrak{g}$ *by* $\mathfrak{a}$*. The extensions split iff the former class vanishes.*

Prove the Proposition by investigating properties the bi-$\mathbb{K}$-bilinear map

$$[\cdot,\cdot\}_\Theta\ :\ \widetilde{\mathfrak{g}}^{\times 2}\longrightarrow\widetilde{\mathfrak{g}}\ :\ \big((A_1,X_1),(A_2,X_2)\big)\longmapsto\big(\Theta(X_1,X_2),[X_1,X_2\}_{\mathfrak{g}}\big)$$

determined by a given 2-cocycle $\Theta\in Z^2(\mathfrak{g},\mathfrak{a})$ on $\widetilde{\mathfrak{g}}:=\mathfrak{a}\oplus\mathfrak{g}$, alongside the natural $\mathbb{K}$-linear maps $\mathfrak{a}\longrightarrow\widetilde{\mathfrak{g}}$ (injection) and $\widetilde{\mathfrak{g}}\longrightarrow\mathfrak{g}$ (projection). In the case of cohomologous 2-cocycles, $\Theta_2=\Theta_1+\delta^{(1)}_{\mathfrak{g}}\mu,\ \mu\in C^1(\mathfrak{g},\mathfrak{a})$, take a closer look at the mapping

$$\varepsilon_\mu\ :\ \widetilde{\mathfrak{g}}\longrightarrow\widetilde{\mathfrak{g}}\ :\ (A,X)\longmapsto\big(A-\mu(X),X\big).$$

For the second part of the thesis, associate with $\Theta=\delta^{(1)}_{\mathfrak{g}}\mu,\ \mu\in C^1(\mathfrak{g},\mathfrak{a})$ the $\mathbb{K}$-linear mapping

$$\sigma_\mu\ :\ \mathfrak{g}\longrightarrow\widetilde{\mathfrak{g}}\ :\ X\longmapsto\big(-\mu(X),X\big)$$

and study its properties.

Let us conclude the purely algebraic part of our exposition with the following remark that sheds some light upon our results:

Remark 6. The existence of an extension of $\mathfrak{g}$ by $\mathfrak{a}$ determined by Θ is tantamount to a trivialisation of the pullback 2-cocycle

$$\widetilde{\Theta}:=\pi^*_{\mathfrak{g}}\Theta\ :\ \widetilde{\mathfrak{g}}^{\times 2}\longrightarrow\mathfrak{a}\ :\ \big((A_1,X_1),(A_2,X_2)\big)\longmapsto\Theta(X_1,X_2)$$

given by

$$\widetilde{\Theta}=\delta^{(1)}_{\widetilde{\mathfrak{g}}}\widetilde{\mu}\,,\qquad\widetilde{\mu}:=-\pi_{\mathfrak{a}}\ :\ \widetilde{\mathfrak{g}}\longrightarrow\mathfrak{a}\ :\ (A,X)\longmapsto -A\,.$$