CLASSICAL FIELD THEORY IN THE TIME OF COVID-19
9. LECTURE BATCH

An unavoidable starter on vector bundles

In the present lecture, we intend to discuss a natural structure on the tangent bundle over the total space of a fibre bundle that geometrises (globally) the decomposition of the tangent bundle of a local model (a trivialisation) into a (fibred) direct sum of base and fibre (*i.e.*, vertical) components. Such a structure plays an instrumental rôle in the mathematical description of the dynamical aspect of the universal gauge principle, it also has an independent physical application in the modelling of the dynamics of gauge fields (massless vector bosons, such as, *e.g.*, the photon, the $W_\pm$ and Z bosons (prior to the spontaneous breakdown of the isospin symmetry through the Higgs effect) and the gluons). In order to be able to navigate comfortably in the domain of immediate interest, we need a few more requisites from the theory of bundles with a linear structure on the (typical) fibre that we provide in this here ancillary section. We begin with

Definition 1. Adopt the hitherto notation, fix $n \in \mathbb{N}$ (arbitrarily) and consider $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with the standard (euclidean) topology and differential structure of class C^∞. A **vector bundle of rank** n **over the field** $\mathbb{K}$ of class C^∞ is a fibre bundle $(\mathbb{V}, B, \mathbb{K}^{\times n}, \pi_{\mathbb{V}})$ with the following properties:

- the fibre $\mathbb{V}_x \equiv \pi_{\mathbb{V}}^{-1}(\{x\})$ over an arbitrary point $x \in B$ is a $\mathbb{K}$-linear space;
- the restrictions of diffeomorphisms of class C^∞ (local trivialisations)

$$\mathrm{pr}_2 \circ \tau_i \restriction_{\mathbb{V}_x} \;:\; \mathbb{V}_x \xrightarrow{\cong} \mathbb{K}^{\times n}\,, \qquad x \in B$$

are isomorphisms of $\mathbb{K}$-linear spaces,

and the maps defining the $\mathbb{K}$-linear structure on fibres of $\mathbb{V}$ are of class C^∞, so that – in particular – we have a diffeomorphism

$$\mathbb{A} \;:\; \mathbb{V} \times_B \mathbb{V} \longrightarrow \mathbb{V} \tag{1}$$

modelled on the defining binary operation $A^n \;:\; \mathbb{K}^{\times n} \times \mathbb{K}^{\times n} \longrightarrow \mathbb{K}^{\times n}$ (component-wise addition) in the sense expressed by the commutative diagram

$$\begin{array}{ccc} \pi_{\mathbb{V}}^{-1}(\mathcal{O}_i) \times_B \pi_{\mathbb{V}}^{-1}(\mathcal{O}_i) & \xrightarrow{\quad\mathbb{A}\quad} & \pi_{\mathbb{V}}^{-1}(\mathcal{O}_i) \\ \big\downarrow {\scriptstyle \tau_i \times \tau_i} & & \big\downarrow {\scriptstyle \tau_i} \\ \big(\mathcal{O}_i \times \mathbb{K}^{\times n}\big) \times_{\mathcal{O}_i} \big(\mathcal{O}_i \times \mathbb{K}^{\times n}\big) & \xrightarrow[(\mathrm{pr}_1, A^n \circ \mathrm{pr}_{2,4})]{} & \mathcal{O}_i \times \mathbb{K}^{\times n} \end{array}\,, \tag{2}$$

and a family of diffeomorphisms

$$\mathbb{K}^\times \longrightarrow \mathrm{Diff}^k(\mathbb{V}) \;:\; \lambda \longmapsto \mathbb{L}_\lambda \tag{3}$$

with $\mathbb{K}$-linear restrictions to fibres, augmented with the $\mathbb{K}$-linear map $\mathbb{L}_{0_\mathbb{K}}$, that are modelled on the defining action $\ell^n \;:\; \mathbb{K} \times \mathbb{K}^{\times n} \longrightarrow \mathbb{K}^{\times n}$ (component-wise multiplication in $\mathbb{K}$) in the sense expressed by the commutative diagram

(4)
$$\begin{array}{ccc} \pi_{\mathbb{V}}^{-1}(\mathcal{O}_i) & \xrightarrow{\mathbb{L}_\lambda} & \pi_{\mathbb{V}}^{-1}(\mathcal{O}_i) \\ \downarrow \tau_i & & \downarrow \tau_i \\ \mathcal{O}_i \times \mathbb{K}^{\times n} & \xrightarrow[\mathrm{id}_{\mathcal{O}_i} \times \ell_\lambda^n]{} & \mathcal{O}_i \times \mathbb{K}^{\times n} \end{array} .$$

If $\mathbb{K} = \mathbb{R}$, we speak of a **real vector bundle**, whereas if $\mathbb{K} = \mathbb{C}$, we have a **complex vector bundle**.

The rank of the bundle is denoted as $\mathrm{rk}\,\mathbb{V}$. Whenever $\mathrm{rk}\,\mathbb{V} = 1$, the bundle is termed a **line bundle** and customarily denoted as L,

$$\begin{array}{ccc} \mathbb{K} & \longrightarrow & L \\ & & \downarrow \pi_L \\ & & B \end{array} .$$

The map (of class C^∞)

$$\mathbf{0}_{\mathbb{V}} \;:\; B \longrightarrow \mathbb{V} \;:\; x \longmapsto \tau_i^{-1}(x, \mathbf{0}^n) , \quad x \in \mathcal{O}_i ,$$

is called the **zero section** of the vector bundle $\mathbb{V}$. It is a global section of $\mathbb{V}$. The set of local sections $\Gamma_{\mathrm{loc}}(\mathbb{V})$ and the set of global sections $\Gamma(\mathbb{V})$ both carry a natural (pointwise) structure of a module over the ring $C^\infty(B, \mathbb{K})$.

A **vector subbundle of rank** m of a vector bundle $(\mathbb{V}, B, \mathbb{K}^{\times n}, \pi_{\mathbb{V}})$ is a subbundle $(\mathbb{W}, B, \mathbb{K}^{\times m}, \pi_{\mathbb{V}} \upharpoonright_{\mathbb{W}})$, $m < n$ of the latter fibre bundle with the additional property: over an arbitrary point $x \in B$ in the base, its fibre $\mathbb{W}_x \subset \mathbb{V}_x$ is a $\mathbb{K}$-linear subspace.

A **morphism of vector bundles** (**over the field** $\mathbb{K}$) $(\mathbb{V}_A, B_A, \mathbb{K}^{\times n_A}, \pi_{\mathbb{V}_A})$, $n_A \in \mathbb{N}$, $A \in \{1, 2\}$ is a fibre-bundle morphism

$$(\Phi, f) \;:\; (\mathbb{V}_1, B_1, \mathbb{K}^{\times n_1}, \pi_{\mathbb{V}_1}) \longrightarrow (\mathbb{V}_2, B_2, \mathbb{K}^{\times n_2}, \pi_{\mathbb{V}_2})$$

whose restriction to the fibre $\mathbb{V}_{1\,x}$ over an arbitrary point $x \in B_1$ in the base,

(5)
$$\Phi \upharpoonright_{\mathbb{V}_{1\,x}} \;:\; \mathbb{V}_{1\,x} \longrightarrow \mathbb{V}_{2\,f(x)} ,$$

is a $\mathbb{K}$-linear map. The **rank** of the vector-bundle morphism (Φ, f) is the map

$$\mathrm{rk}\,(\Phi, f) \;:\; B_1 \longrightarrow \mathbb{N} \;:\; x \longmapsto \mathrm{rk}\big(\Phi \upharpoonright_{\mathbb{V}_{1\,x}}\big) .$$

The canonical example of a vector bundle is the tangent bundle $\mathsf{T}M \longrightarrow M$ over a smooth manifold M. Its rank is equal to the dimension of M.

An elementary structural property of vector bundles is stated in

Theorem 1. Adopt the hitherto notation. A vector bundle $(\mathbb{V}, B, \mathbb{K}^{\times n}, \pi_{\mathbb{V}})$ of rank $n \in \mathbb{N}$ is globally trivial, *i.e.*, isomorphic with the bundle $(B \times \mathbb{K}^{\times n}, B, \mathbb{K}^{\times n}, \mathrm{pr}_1)$, iff there exist n global sections $\sigma^k \in \Gamma(\mathbb{V})$, $k \in \overline{1, n}$ which are linearly independent, *i.e.*, such that, over every point $x \in B$ of its base, the vectors $\sigma^k(x) \in \mathbb{V}_x$, $k \in \overline{1, n}$ defined by them are linearly independent.

Proof: Left to the Reader as an easy exercise. □

Staying in the context defined by the last theorem, we readily establish the following geometric counterpart of the Steinitz Exchange Lemma:

Proposition 1. Adopt the hitherto notation and consider a vector bundle $(\mathbb{V}, B, \mathbb{K}^{\times n}, \pi_{\mathbb{V}})$ of rank $n \in \mathbb{N}$. Given $m \leq n$ linearly independent local sections σ^k, $k \in \overline{1, m}$ of $\mathbb{V}$ over an open subset $\mathcal{O} \subseteq B$ of the base B, there exists an open subset $\mathcal{U} \subseteq \mathcal{O}$ and local sections $\sigma^l \in \Gamma(\mathbb{V} \upharpoonright_{\mathcal{U}})$, $l \in \overline{m+1, n}$ which compose a linearly independent set $\{\sigma^j\}^{j \in \overline{1,n}}$ of local sections over $\mathcal{U}$ together with the fomerly introduced ones and thus determine a trivialisation of $\mathbb{V} \upharpoonright_{\mathcal{U}}$.

Proof: Left to the Reader as an easy exercise. □

The last result prepares us for the proof of the following fundamental theorem which constitutes the point of departure of our subsequent physics-oriented discussion.

Theorem 2. Adopt the notation of Def. 1 and let $(\Phi, f) : (\mathbb{V}_1, B_1, \mathbb{K}^{\times n_1}, \pi_{\mathbb{V}_1}) \longrightarrow (\mathbb{V}_2, B_2, \mathbb{K}^{\times n_2}, \pi_{\mathbb{V}_2})$ be a morphism of vector bundles $\mathbb{V}_A,\ A \in \{1,2\}$ of a constant rank $\mathrm{rk}\,(\Phi, f) \equiv r \in \mathbb{N}$. Its **kernel**

$$\mathrm{Ker}\,(\Phi, f) := \bigcup_{x \in B_1} \ker\big(\Phi\restriction_{\mathbb{V}_{1\,x}}\big)$$

carries a canonical structure of a vector subbundle of the domain $\mathbb{V}_1$, and satisfies

$$\mathrm{rk}\,\mathrm{Ker}\,(\Phi, f) = n_1 - r\,.$$

Proof: Clearly, at an arbitrary point $x \in B$, we obtain an isomorphism

$$\big(\pi_{\mathbb{V}_1}\restriction_{\mathrm{Ker}\,(\Phi,f)}\big)^{-1}(\{x\}) \cong \mathbb{K}^{\times n_1 - r}$$

by the purely algebraic dimension count. Furthermore, the subset $\mathrm{Ker}\,(\Phi, f) \subset \mathbb{V}_1$ carries the structure of a topological subspace of the topological space $(\mathbb{V}, \mathscr{T}(\mathbb{V}))$ on which we should, first, find a submanifold atlas. We do that with the help of Prop. 1. To this end, we choose (arbitrarily) open neighbourhoods: $\mathcal{O}_1$ containing $x_1 \in B_1$ and $\mathcal{O}_2 \supset f(\mathcal{O}_1)$ containing $f(x_1) \in B_2$ that support the respective local trivialisations

$$\tau_{\mathcal{O}_A} \ :\ \pi_{\mathbb{V}_A}^{-1}(\mathcal{O}_A) \xrightarrow{\ \cong\ } \mathcal{O}_A \times \mathbb{K}^{\times n_A}\,, \qquad A \in \{1,2\}\,.$$

These enable us to present the morphism (Φ, f) as

$$\Phi_{21} \equiv \tau_{\mathcal{O}_2} \circ \Phi \circ \tau_{\mathcal{O}_1}^{-1} \ :\ \mathcal{O}_1 \times \mathbb{K}^{\times n_1} \longrightarrow \mathcal{O}_2 \times \mathbb{K}^{\times n_2} \ :\ (x, v) \longmapsto \big(f(x), L_\Phi(x)(v)\big)$$

for some smooth map

$$L_\Phi \ :\ \mathcal{O}_1 \longrightarrow \mathbb{K}(n_2) \ :\ x \longmapsto L_\Phi(x)$$

of rank

$$\mathrm{rk}\, L_\Phi(x) = r\,.$$

Consider decompositions

$$\mathbb{K}^{\times n_1} = \mathrm{Ker}\, L_\Phi(x_1) \oplus \Delta_1\,, \qquad\qquad \mathbb{K}^{\times n_2} = \mathrm{Image}\, L_\Phi(x_1) \oplus \Delta_2\,,$$

written for some direct-sum complements $\Delta_A \subset \mathbb{K}^{\times n_A}$ of the respective dimensions

$$\dim_{\mathbb{K}} \Delta_1 = n_1 - \dim_{\mathbb{K}} \mathrm{Ker}\, L_\Phi(x_1) = \dim_{\mathbb{K}} \mathrm{Image}\, L_\Phi(x_1) = n_2 - \dim_{\mathbb{K}} \Delta_2\,.$$

In view of the obvious relation

$$\Delta_1 \cong \mathrm{Image}\, L_\Phi(x_1)\,,$$

we may, next, construct a family, indexed by $\mathcal{O}_1 \ni x$, of $\mathbb{K}$-linear maps

$$\begin{aligned}\widetilde{\Lambda}_\Phi(x) \quad &: \quad \mathbb{K}^{\times n_1} \oplus \Delta_2 \equiv \mathrm{Ker}\, L_\Phi(x_1) \oplus \Delta_1 \oplus \Delta_2 \longrightarrow \mathrm{Ker}\, L_\Phi(x_1) \oplus \mathrm{Image}\, L_\Phi(x_1) \oplus \Delta_2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \equiv \mathrm{Ker}\, L_\Phi(x_1) \oplus \mathbb{K}^{\times n_2} \\ &: \quad (k, \delta_1, \delta_2) \longmapsto (k, 0_{\mathrm{Image}\, L_\Phi(x_1)}, \delta_2) +_\oplus \big(0, L_\Phi(x)\big)(k, \delta_1)\,,\end{aligned}$$

that contains the manifestly invertible member

$$\widetilde{\Lambda}_\Phi(x_1) = \mathrm{id}_{\mathrm{Ker}\, L_\Phi(x_1)} \oplus L_\Phi(x_1)\restriction_{\Delta_1} \oplus \mathrm{id}_{\Delta_2}\,.$$

Since invertible maps compose an open subset in $\mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^{\times n_1} \oplus \Delta_2, \mathrm{Ker}\, L_\Phi(x_1) \oplus \mathbb{K}^{\times n_2})$ (to wit, the complement of the preimage of the closed set $\{0_{\mathbb{K}}\}$ along the continuous map $\det_{(n_1 + \dim_{\mathbb{K}} \Delta_2)}$), the element $\widetilde{\Lambda}_\Phi(x_1)$ belongs to this subset together with some open neighbourhood $\mathcal{U}$ whose preimage along the (continuous) map $\widetilde{\Lambda}_\Phi$ is an open neighbourhood $\mathcal{V}_1 \equiv \widetilde{\Lambda}_\Phi^{-1}(\mathcal{U}) \ni x_1$ with the property $\mathcal{V}_1 \subset \mathcal{O}_1$. Thus, alongside the smooth map

$$\Lambda_\Phi \equiv \widetilde{\Lambda}_\Phi \restriction_{\mathcal{V}_1} \; : \; \mathcal{V}_1 \longrightarrow \mathrm{Iso}_{\mathbb{K}}\big(\mathbb{K}^{\times n_1} \oplus \Delta_2, \mathrm{Ker}\, L_\Phi(x_1) \oplus \mathbb{K}^{\times n_2}\big)\,,$$

we also have the smooth (pointwise, at each $x \in \mathcal{V}_1$) inverse thereof,

$$V_\Phi \equiv \mathrm{Inv} \circ \Lambda_\Phi \; : \; \mathcal{V}_1 \longrightarrow \mathrm{Iso}_{\mathbb{K}}\big(\mathrm{Ker}\, L_\Phi(x_1) \oplus \mathbb{K}^{\times n_2}, \mathbb{K}^{\times n_1} \oplus \Delta_2\big)\,,$$

Take an arbitrary vector

$$(k, \delta_1) \in \mathrm{Ker}\, L_\Phi(x_1) \oplus \Delta_1 \equiv \mathbb{K}^{\times n_1}\,.$$

Upon fixing $x \in \mathcal{V}_1$, we conclude that

$$\begin{aligned}(k, \delta_1) \in \mathrm{Ker}\, L_\Phi(x) \qquad &\Longleftrightarrow \qquad \Lambda_\Phi(x)(k, \delta_1, 0_{\Delta_2}) = (k, 0_{\Delta_1}, 0_{\Delta_2}) \\ &\Longleftrightarrow \qquad (k, \delta_1, 0_{\Delta_2}) = V_\Phi(x)(k, 0_{\Delta_1}, 0_{\Delta_2})\,.\end{aligned}$$

Taking into account the canonical injections:

$$\jmath_{\mathrm{Ker}\, L_\Phi(x_1)} \; : \; \mathrm{Ker}\, L_\Phi(x_1) \rightarrowtail \mathrm{Ker}\, L_\Phi(x_1) \oplus \mathbb{K}^{\times n_2}$$

and

$$\jmath_{\mathbb{K}^{\times n_1}} \; : \; \mathbb{K}^{\times n_1} \rightarrowtail \mathbb{K}^{\times n_1} \oplus \Delta_2\,,$$

we may, therefore, write

$$\jmath_{\mathbb{K}^{\times n_1}}\big(\mathrm{Ker}\, L_\Phi(x)\big) \subseteq V_\Phi(x)\big(\mathrm{Image}\, \jmath_{\mathrm{Ker}\, L_\Phi(x_1)}\big)\,,$$

but also

$$\begin{aligned} &\dim_{\mathbb{K}} \jmath_{\mathbb{K}^{\times n_1}}\big(\mathrm{Ker}\, L_\Phi(x)\big) = \dim_{\mathbb{K}} \mathrm{Ker}\, L_\Phi(x) = n_1 - \dim_{\mathbb{K}} \mathrm{Image}\, L_\Phi(x) \\ = \;\; & n_1 - \dim_{\mathbb{K}} \mathrm{Image}\, L_\Phi(x_1) = \dim_{\mathbb{K}} \mathrm{Ker}\, L_\Phi(x_1) = \dim_{\mathbb{K}} \mathrm{Image}\, \jmath_{\mathrm{Ker}\, L_\Phi(x_1)} \\ \equiv \;\; & \dim_{\mathbb{K}} V_\Phi(x)\big(\mathrm{Image}\, \jmath_{\mathrm{Ker}\, L_\Phi(x_1)}\big)\,,\end{aligned}$$

the latter equality being a consequence of the invertibility of $V_\Phi(x)$. Hence,

$$\jmath_{\mathbb{K}^{\times n_1}}\big(\mathrm{Ker}\, L_\Phi(x)\big) = V_\Phi(x)\big(\mathrm{Image}\, \jmath_{\mathrm{Ker}\, L_\Phi(x_1)}\big)\,,$$

from which we infer that the $\mathbb{K}$-linear map

$$\mu_x \; : \; \mathrm{Ker}\, L_\Phi(x_1) \longrightarrow \ker\big(\Phi\restriction_{\mathbb{V}_{1\,x}}\big) \; : \; k \longmapsto \tau_{\mathcal{O}_1}^{-1}\big(x, \mathrm{pr}_{1,2} \circ V_\Phi(x)(k, 0_{\mathrm{Image}\, L_\Phi(x_1)}, 0_{\Delta_2})\big)$$

is an isomorphism, with the inverse

$$\mu_x^{-1} \; : \; \ker\big(\Phi\restriction_{\mathbb{V}_{1\,x}}\big) \longrightarrow \mathrm{Ker}\, L_\Phi(x_1) \; : \; \tau_{\mathcal{O}_1}^{-1}(x, v) \longmapsto \mathrm{pr}_1 \circ \Lambda_\Phi(x)(v, 0_{\Delta_2})\,.$$

Accordingly, for a given basis $\{k_j\}_{j \in \overline{1, n_1 - r}}$ of $\mathrm{Ker}\, L_\Phi(x_1)$, the maps

$$\sigma^j \; : \; \mathcal{V}_1 \longrightarrow \mathbb{V}_1 \; : \; x \longmapsto \mu_x(k_j)$$

compose a family of $n_1 - r$ linearly independent smooth sections of $\mathbb{V}_1\restriction_{\mathcal{V}_1}$, which – in virtue of Prop. 1 – may be extended, over some $\mathcal{W}_1 \subseteq \mathcal{V}_1$, to a local basis $\{\sigma^i\}^{i \in \overline{1, n_1}}$ of $\Gamma(\mathbb{V}_1\restriction_{\mathcal{W}_1})$. Invoking Thm. 1, we associate with the latter a (new) trivialisation of $\mathbb{V}_1$ over $\mathcal{W}_1 \ni x$,

$$\tau_{\mathcal{W}_1} \; : \; \pi_{\mathbb{V}_1}^{-1}(\mathcal{W}_1) \xrightarrow{\cong} \mathcal{W}_1 \times \mathbb{K}^{\times n_1} \; : \; \mathbb{L}_{\lambda_i}\big(\sigma^i(x)\big) \longmapsto \big(x, (\lambda_1, \lambda_2, \ldots, \lambda_{n_1})\big)\,,$$

in which

$$\begin{aligned} &\tau_{\mathcal{W}_1}\big(\mathrm{Ker}\,(\Phi, f) \cap \pi_{\mathbb{V}_1}^{-1}(\mathcal{W}_1)\big) \\ = \;\; & \big\{ \big(x, (\lambda_1, \lambda_2, \ldots, \lambda_{n_1 - r}, 0, 0, \ldots, 0)\big) \;\; | \;\; x \in \mathcal{W}_1 \quad \wedge \quad \lambda_j \in \mathbb{K}\,, \; j \in \overline{1, n_1 - r} \big\} \cong \mathcal{W}_1 \times \mathbb{K}^{\times n_1 - r}\,.\end{aligned}$$

When combined with a local atlas on the base, this yields the desired submanifold atlas on $\mathrm{Ker}\,(\Phi, f)$, which proves that the latter is, indeed, an embedded submanifold of $\mathbb{V}_1$.

It now suffices to establish local trivialisations of $\mathrm{Ker}\,(\Phi, f)$ that give it a structure of a vector subbundle. This we readily achieve over $\mathcal{V}_1$ with the help of the μ_x. Indeed, we have the smooth map

$$\tau_{\mathcal{V}_1}^{-1} \;:\; \mathcal{V}_1 \times \mathrm{Ker}\, L_\Phi(x_1) \longrightarrow \mathrm{Ker}\,(\Phi, f)\restriction_{\mathcal{V}_1} \;:\; (x, k) \longmapsto \mu_x(k)$$

which is none other than the smooth inverse of the sought-after trivialisation (likewise smooth)

$$\tau_{\mathcal{V}_1} \;:\; \mathrm{Ker}\,(\Phi, f)\restriction_{\mathcal{V}_1} \longrightarrow \mathcal{V}_1 \times \mathrm{Ker}\, L_\Phi(x_1) \;:\; \tau_{\mathcal{O}_1}^{-1}(x, v) \longmapsto \big(x, \mu_x^{-1}(v)\big)\,.$$

□

We find the following important application of the above result:

Corollary 1. Adopt the hitherto notation, and in particular that of Thm. 2, and let (E, B, F, π_E) be a fibre bundle. The kernel of the vector-bundle epimorphism

$$(\mathsf{T}\pi_E, \pi_E) \;:\; \mathsf{T}E \longrightarrow \mathsf{T}B$$

is a vector subbundle

$$\big(\mathsf{V}E \equiv \mathrm{Ker}\,(\mathsf{T}\pi_E, \pi_E), E, \mathbb{K}^{\times \dim F}, \pi\big)$$

of the tangent bundlej $\mathsf{T}E$. We call it the **vertical (sub)bundle** over E. Its fibre $\mathsf{V}_pE \equiv (\mathsf{V}E)_p$ over $p \in E$, termed the **vertical (sub)space**, is spanned on **vertical vectors**.

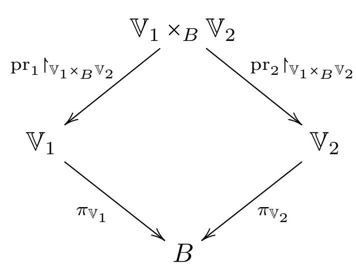
Proof: Left to the Reader as an easy exercise. □

We close the ancillary section by introducing one of the basic algebraic operations on vector spaces that lift to the category of vector bundles, to wit, the physically all-important fibred direct sum, described in

Definition 2. The **Whitney sum of vector bundles** $(\mathbb{V}_\alpha, B, \mathbb{K}^{\times n_\alpha}, \pi_{\mathbb{V}_\alpha})$, $\alpha \in \{1, 2\}$ over $\mathbb{K}$, with a common base B is the vector bundle

$$\big(\mathbb{V}_1 \oplus_{\mathbb{K},B} \mathbb{V}_2 \equiv \mathbb{V}_1 \times_B \mathbb{V}_2, B, \mathbb{K}^{\times n_1} \oplus \mathbb{K}^{\times n_2} \equiv \mathbb{K}^{\times n_1 + n_2}, \pi_{\mathbb{V}_1} \circ \mathrm{pr}_1\restriction_{\mathbb{V}_1 \times_B \mathbb{V}_2}\big)\,,$$

where $\mathbb{V}_1 \times_B \mathbb{V}_2$ is the fibred product of manifolds $\mathbb{V}_\alpha$, $\alpha \in \{1, 2\}$ described by the commutative diagram



and endowed with the structure of a submanifold smoothly embedded in the product manifold $\mathbb{V}_1 \times \mathbb{V}_2$, in conformity with the statement of Thm. Niezb-4.

Remark 1. The fibre of the Whitney sum over an arbitrary point $x \in B$ takes the form

$$\big(\mathbb{V}_1 \oplus_{\mathbb{K},B} \mathbb{V}_2\big)_x \equiv \mathbb{V}_{1\,x} \oplus \mathbb{V}_{2\,x}\,,$$

and so the Whitney sum is a natural adaptation of the construction of the direct sum of vector spaces in the geometric category of vector bundles.

Alternatively, the bundle may be described – along the lines of the Clutching Theorem – in terms of local data of its components, *i.e.*, a common trivialising cover $\mathscr{O} = \{\mathcal{O}_i\}_{i \in I}$ of the bundles $\mathbb{V}_\alpha$, $\alpha \in \{1, 2\}$ (obtained, *e.g.*, through common refinement of respective trivialising covers) together with the associated transition maps $g_{ij}^\alpha \;:\; \mathcal{O}_{ij} \longrightarrow \mathrm{GL}_{\mathbb{K}}(n_\alpha)$, $(i, j) \in I^{\times 2}$. Transition maps of the Whitney sum of the two bundles, associated with the same trivialising cover and constituting the point of departure of the reconstruction of (the equivalence class of) the bundle $\mathbb{V}_1 \oplus_{\mathbb{K},B} \mathbb{V}_2$, read

$$g_{ij}^{1\oplus 2} := g_{ij}^1 \oplus g_{ij}^2 \;:\; \mathcal{O}_{ij} \longrightarrow \mathrm{GL}_{\mathbb{K}}(n_1) \oplus \mathrm{GL}_{\mathbb{K}}(n_2) \subset \mathrm{GL}_{\mathbb{K}}(n_1 + n_2)\,.$$

Introduction to the theory of connection on fibre bundles

The study of the tangential structure and applications of fibre bundles – from investigation of their topology (differential topology, variational problems *etc.*) to physical modelling using these geometric objects (the "dynamics" of sections described by a variational principle for a distinguished action functional defined on the set of global sections, the gauging of global symmetries of the physical model, the description of the gravitational resp. electromagnetic background of the dynamics of a charged material point and of fluctuations of that background *etc.*) – often requires us to give a rigorous meaning to the differentiation of sections of a fibre bundle[1], *i.e.*, to provide a definition of the derivative which, in analogy with that of the standard differentiation (e.g., the directional derivative) of the algebra of functions on a manifold, would assign to a section of class C^k a new section of class C^{k-1}, with analogous covariance poperties with respect to the choice of local trivialisations and compatible with potential additional structure on the space of sections and on the fibre (*e.g.*, the structure of a module over the ring of smooth functions on the base or that of the torsor of the structure group). Unfortunately, the most obvious definition of a derivative of a section $\sigma \in \Gamma_{\mathrm{loc}}(E)$ of a fibre bundle E over a base B along a vector field $\mathcal{V} \in \mathfrak{X}(B)$, that is its directional derivative

$$(\mathcal{V}, \sigma) \longmapsto \mathsf{T}\sigma(\mathcal{V})\,,$$

does *not* yield objects tensorially (tangentially) covariant with respect to transition maps over intersections of elements of a trivialising cover in general, nor does it accomodate the extra information encoded in local trivialisations $E\restriction_{\mathcal{O}} \cong \mathcal{O} \times F$ whose existence entails the tangential splitting $\mathsf{T}(E\restriction_{\mathcal{O}}) \cong \mathsf{T}\mathcal{O} \times \mathsf{T}F$. Of course, the above differentiation is a local operation, and so we may always choose a distinguished local chart on the total space of the bundle adapted to a local trivialisation, and subsequently look for the desired objects in it (employing the decomposition of $\mathsf{T}(E\restriction_{\mathcal{O}})$ and, potentially, some additional structure on the tangent bundle over the typical fibre, such as, *e.g.*, the metric-tensor field), but then we leave open the question of their global differential-geometric status.

Restricting our considerations to the category of vector bundles, we may try to circumnavigate the difficulties met by noting that the tangent bundle $\mathsf{T}\mathbb{V}_x$ over the fibre $\mathbb{V}_x$ over a fixed point $x \in B$ in the base, embedded in the codomain $\mathsf{T}\mathbb{V}$ of the map $\mathsf{T}\sigma$, carries – over every point in the fibre – a linear structure *isomorphic with* $\mathbb{V}_x$, which enables us to identify vertical vector fields on $\mathbb{V}$ with sections of the bundle $\mathbb{V}$. In the light of this remark, it suffices to project $\mathsf{T}\sigma(\mathcal{V})$ onto that tangent bundle, which, however, requires existence of a decomposition of the tangent bundle over the total space $\mathbb{V}$ into a Whitney sum of the vertical subbundle and its complement. Under a local trivialisation of the vector bundle $\mathbb{V}$, the projection alluded to above can be defined naturally and yields the expected result. The original difficulty translates into the difficulty with establishing relations (transition maps) between the local objects. Thus, while the hitherto discussion indicates quite clearly the properties that the sought-after solution to the problem posed should possess, our attempts at a direct solution meet with various difficulties (*cp* also: later lectures in which we pass to render the differentiation compatible with any extra structure on the fibre). Below, we address each and every one of them separately, which leads us to several seemingly different definitions of a structural derivative of a section along a vector field on the base. Their equivalence, to be demonstrated towards the end of our discussion, will provide us with a solid *a posteriori* confirmation of the adequacy and naturality of the chosen path of formalisation of the geometric intuitions employed.

Remark 2. All considerations in the set of lectures on (compatible) connections are placed in the category of smooth manifolds (*i.e.*, those of class C^∞). In particular, the base fields $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$

[1] The necessity becomes obvious if we think of those sections as objects modelling physical fields over a spacetime.

of the vector bundles discussed implicitly carry the natural differential structur, and the structure groups of the principal bundles contemplated are Lie groups.

We commence our discussion with

Definition 3. Let (E, B, F, π_E) be a fibre bundle. Consider a path of class C^1

$$\gamma \,:\,]-\varepsilon, \varepsilon[\longrightarrow B\,, \quad \varepsilon > 0\,.$$

A **parallel transport (of class C^∞) in E along γ** is a family of smooth diffeomorphisms

$$\mathsf{P}^\gamma_{t_1,t_2} \,:\, E_{\gamma(t_1)} \xrightarrow{\cong} E_{\gamma(t_2)}\,, \quad t_1, t_2 \in]-\varepsilon, \varepsilon[$$

with the following properties:

(PT1) the mapping (in whose definition the fibred product is associated with the pair $(\gamma \circ \mathrm{pr}_1, \pi_E)$)

$$\mathsf{P}^\gamma_{\cdot,\cdot} \,:\,]-\varepsilon, \varepsilon[^{\times 2} \times_B E \longrightarrow E \,:\, ((t_1, t_2), x) \longmapsto \mathsf{P}^\gamma_{t_1,t_2}(x)$$

is of class C^∞;

(PT2) $\mathsf{P}^\gamma_{t_1,t_1} = \mathrm{id}_{E_{\gamma(t_1)}}$;

(PT3) $\forall_{t_1,t_2,t_3 \in]-\varepsilon,\varepsilon[} \,:\, \mathsf{P}^\gamma_{t_2,t_3} \circ \mathsf{P}^\gamma_{t_1,t_2} = \mathsf{P}^\gamma_{t_1,t_3}$;

(PT4) given any section $\sigma \,:\, \mathcal{O}_x \longrightarrow E$ defined on some open neighbourhood $\mathcal{O}_x$ of the point $x \in B$, its **covariant derivative** at x along an arbitrary vector field $\mathcal{V} \in \mathfrak{X}(\mathcal{O}_x)$, given by the formula

$$\nabla_{\mathcal{V}}\sigma(x) := \tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0}\big(\mathsf{P}^\gamma_{0,t}\big)^{-1} \circ \sigma \circ \gamma(t)\,,$$

does not depend on the choice of the representatives of the co-tangency class of paths through x determined by the conditions

$$\gamma(0) = x \qquad \wedge \qquad \dot{\gamma}(0) = \mathcal{V}(x)\,;$$

(PT5) the map

$$\nabla_{\cdot}\sigma \,:\, \mathsf{T}\mathcal{O}_x \longrightarrow \mathsf{V}E \subset \mathsf{T}E$$

is $C^\infty(\mathcal{O}_x, \mathbb{R})$-linear.

Whenever there exists a parallel transport for any path γ in a neighbourhood of an arbitrary point $x \in B$, we say that a **connection on fibres (of class C^∞) of the fibre bundle E** has been established.

An elementary consequence of the existence of a parallel transport is stated in

Proposition 2. Adopt the notation of Def. 3. If there exists a parallel transport for any path in B, then at any point $p \in E$ in the fibre E_x over an arbitrary point $x \in B$, there exists a monomorphism of class C^∞

$$\mathrm{Hor}_p \,:\, \mathsf{T}_x B \rightarrowtail \mathsf{T}_p E \tag{6}$$

uniquely defined by its property

$$\mathrm{Hor}_p\big(\dot{\gamma}(0)\big) = \tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0}\mathsf{P}^\gamma_{0,t}(p)\,, \tag{7}$$

written for any path γ satisfying the condition $\gamma(0) = x$, the latter implying the identity

$$\mathsf{T}_p\pi \circ \mathrm{Hor}_p = \mathrm{id}_{\mathsf{T}_{\pi_E(p)}B}\,. \tag{8}$$

Furthermore, the tangent space $\mathsf{T}_p E$ decomposes as

$$\mathsf{T}_p E = \mathsf{V}_p E \oplus \mathrm{Image}\,\mathrm{Hor}_p\,.$$

The map Hor_p is termed the **horizontal lift of vectors** from the base to the fibre.

Proof: We begin by stressing that formula (7) does, indeed, determine Hor_p uniquely in view of arbitrariness of the vector tangent to a path at a given point in the base. Hence, it suffices to check the desired properties of the expression on the right-hand side of the equality. With that in mind, we note that the family $\mathsf{P}^\gamma_{t_1,t_2},\ t_1,t_2 \in]-\varepsilon,\varepsilon[$ of diffeomorphisms defined for a path with a nowhere vanishing tangent vector $\dot\gamma$ (for a sufficiently small ε) induces a (locally) smooth vector field $\mathcal{Y}$ over $\gamma(]-\varepsilon,\varepsilon[)$ with the flow (or, equivalently, with the local group of local diffeomorphisms)

$$\begin{aligned}\Phi_{\mathcal{Y}} \quad &: \quad]-\varepsilon,\varepsilon[\times\pi_E^{-1}\big(\gamma(]-\varepsilon,\varepsilon[)\big) \longrightarrow \pi_E^{-1}\big(\gamma(]-\varepsilon,\varepsilon[)\big)\\ &: \quad (t,p) \longmapsto \mathsf{P}^\gamma_{\gamma^{-1}\circ\pi_E(p),t}(p) \equiv \Phi_{\mathcal{Y}}(t,p)\,,\end{aligned}$$

satisfying the obvious identity

$$\Phi_{\mathcal{Y}}\big(\gamma^{-1}\circ\pi_E(p),p\big) = p\,,$$

and the vector field $\mathcal{Y}$ itself obeys

$$\mathcal{Y}(p) = \tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=\gamma^{-1}\circ\pi_E(p)}\Phi_{\mathcal{Y}}(t,p)\,.$$

Consider, next, an arbitrary local section

$$\sigma\ :\ \mathcal{O}_x \longrightarrow E\,, \qquad\qquad \pi_E\circ\sigma = \mathrm{id}_{\mathcal{O}_x}$$

with the property

$$\sigma\circ\gamma(0) \equiv \sigma(x) = p\,,$$

which entails

$$\Phi_{\mathcal{Y}}\big(0,p\big) \equiv \Phi_{\mathcal{Y}}\big(\gamma^{-1}\circ\pi_E(p),p\big) = p \equiv \mathrm{id}_E(p)\,.$$

Its covariant derivative at x along the vector field tangent to γ,

$$\nabla_{\dot\gamma}\sigma(x) \equiv \tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0}\mathsf{P}^{\gamma^{-1}}_{0,t}\big(\sigma\circ\gamma(t)\big)\,,$$

can be computed with the help of the following manipulation

$$\begin{aligned}\mathsf{T}_x\sigma(\dot\gamma(0)) &\equiv \tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0}\mathsf{P}^\gamma_{0,t}\big(\mathsf{P}^{\gamma^{-1}}_{0,t}\big(\sigma\circ\gamma(t)\big)\big) = \tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0}\Phi_{\mathcal{Y}}\big(t,\mathsf{P}^{\gamma^{-1}}_{0,t}\big(\sigma\circ\gamma(t)\big)\big)\\ &= \mathsf{D}_1\Phi_{\mathcal{Y}}\big(0,\mathsf{P}^{\gamma^{-1}}_{0,0}\big(\sigma\circ\gamma(0)\big)\big) + \mathsf{D}_2\Phi_{\mathcal{Y}}\big(0,\mathsf{P}^{\gamma^{-1}}_{0,0}\big(\sigma\circ\gamma(0)\big)\big)\big(\nabla_{\dot\gamma}\sigma(x)\big)\\ &= \mathcal{Y}\big(\mathsf{P}^{\gamma^{-1}}_{0,0}\big(\sigma\circ\gamma(0)\big)\big) + \mathsf{T}_{\mathsf{P}^{\gamma^{-1}}_{0,0}(\sigma\circ\gamma(0))}\mathrm{id}_E\big(\nabla_{\dot\gamma}\sigma(x)\big)\\ &\equiv \mathcal{Y}\big(\sigma(x)\big) + \mathrm{id}_{\mathsf{T}_{\sigma(x)}E}\big(\nabla_{\dot\gamma}\sigma(x)\big) = \mathcal{Y}(p) + \nabla_{\dot\gamma}\sigma(x)\end{aligned}$$

which yields

$$\mathcal{Y}(p) = \mathsf{T}_x\sigma(\dot\gamma(0)) - \nabla_{\dot\gamma}\sigma(x)\,,$$

and so also

$$\mathrm{Hor}_p\big(\dot\gamma(0)\big) = \tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0}\mathsf{P}^\gamma_{0,t}(p) \equiv \tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0}\Phi_{\mathcal{Y}}(t,p) = \mathcal{Y}\big(p\big) = \mathsf{T}_x\sigma(\dot\gamma(0)) - \nabla_{\dot\gamma}\sigma(x)\,.$$

Thus, we see that directly by the definition of the covariant derivative (and that of the tangent mapping) the map Hor_p is $\mathbb{R}$-linear and depends on the choice of the path through $\dot\gamma(0)$ (oraz $\gamma(0) = x$) exclusively. It is also injective as, on the one hand,

$$\nabla_{\dot\gamma}\sigma(x) \in \mathsf{V}_pE\,,$$

and, on the other hand,

$$\mathsf{T}_x\sigma(\dot\gamma(0)) \in \mathsf{V}_pE \qquad\Longleftrightarrow\qquad \dot\gamma(0) \equiv \mathsf{T}_{\sigma(x)}\pi\circ\mathsf{T}_x\sigma\big(\dot\gamma(0)\big) = 0_{\mathsf{T}_{\sigma(x)}E}\,,$$

and therefore

$$\begin{aligned}\mathsf{T}_x\sigma(\dot\gamma(0)) - \nabla_{\dot\gamma}\sigma(x) \in \mathsf{V}_pE \qquad &\Longleftrightarrow \qquad \mathsf{T}_x\sigma(\dot\gamma(0)) \in \mathsf{V}_pE\\ &\Longleftrightarrow \qquad \dot\gamma(0) = 0_{\mathsf{T}_xB}\,,\end{aligned}$$

or

$$\mathrm{Image\,Hor}_p \cap \mathsf{V}_p E = \{0_{\mathsf{T}_p E}\}\,.$$

The above implies a chain of relations between $\mathbb{R}$-linear spaces (we are dealing with the internal direct sum here)

$$\mathrm{Image\,Hor}_p \oplus \mathsf{V}_p E = \mathrm{Image\,Hor}_p +_{\mathsf{T}_p E} \mathsf{V}_p E \subset \mathsf{T}_p E\,,$$

and since – in virtue of injectivity of Hor_p, which makes it an isomorphism onto its image – the equality

$$\begin{aligned}\dim_{\mathbb{R}}\big(\mathrm{Image\,Hor}_p \oplus \mathsf{V}_p E\big) &= \dim_{\mathbb{R}} \mathrm{Image\,Hor}_p + \dim_{\mathbb{R}} \mathsf{V}_p E = \dim_{\mathbb{R}} \mathsf{T}_x B + \dim F \\ &= \dim B + \dim F = \dim E \equiv \dim_{\mathbb{R}} \mathsf{T}_p E\end{aligned}$$

holds true, we obtain the relation

$$\mathrm{Image\,Hor}_p \oplus \mathsf{V}_p E = \mathsf{T}_p E\,.$$

The identity (8) follows directly from the expression for $\mathrm{Hor}_p\big(\dot\gamma(0)\big)$ derived above. □

An alternative take on a connection is presented in

Definition 4. An **Ehresmann connection** on a fibre bundle (E, B, F, π_E) is a choice of a vector subbundle $\mathsf{H}E \subset \mathsf{T}E$ of the tangent bundle over the total space E that complements the vertical bundle $\mathsf{V}E$ in the tangent bundle $\mathsf{T}E$ as

$$\mathsf{T}E = \mathsf{V}E \oplus_B \mathsf{H}E\,.$$

The subbundle $\mathsf{H}E$ is called a **horizontal (sub)bundle** over E. Its fibre $\mathsf{H}_p E \equiv (\mathsf{H}E)_p$ over $p \in E$, termed the **horizontal (sub)space**, is spanned on **horizontal vectors**.

Equivalence between the two definitions of a connection is stated in

Theorem 3. A connection on fibres of a fibre bundle determines an Ehresmann connection on it, and *vice versa*.

Proof: The first claim follows directly from Prop. 2 the dependence of Hor_p on the point $p \in E$ is – by construction – smooth (as the dependence of the smooth vector field $\mathcal{Y}$ on the point in its support), and so we have an unabmiguous smooth splitting of an arbitrary vector field $\mathcal{V}$ on E into components

$$\mathcal{V} = (\mathrm{id}_{\mathsf{T}E} - \mathrm{Hor.} \circ \mathsf{T}\pi)(\mathcal{V}) + \mathrm{Hor.} \circ \mathsf{T}\pi(\mathcal{V})\,,$$

of which the former is vertical by Eqn. (8),

$$\mathsf{T}\pi\big((\mathrm{id}_{\mathsf{T}E} - \mathrm{Hor.} \circ \mathsf{T}\pi)(\mathcal{V})\big) = \mathsf{T}\pi(\mathcal{V}) - (\mathsf{T}\pi \circ \mathrm{Hor.}) \circ \mathsf{T}\pi(\mathcal{V}) = \mathsf{T}\pi(\mathcal{V}) - \mathsf{T}\pi(\mathcal{V}) = 0_{\mathsf{T}E}\,.$$

Clearly, the pair $(\mathrm{id}_{\mathsf{T}E} - \mathrm{Hor.} \circ \mathsf{T}\pi, \mathrm{Hor.} \circ \mathsf{T}\pi)$ is a complete family of complementary projections, and as such it defines the sought-after Whitney-sum decomposition of $\mathsf{T}E$.

Conversely, an Ehresmann connection $\mathsf{T}E = \mathsf{V}E \oplus_B \mathsf{H}E$ uniquely defines a smooth lift

$$\mathrm{Hor.} := (\mathsf{T}\pi\restriction_{\mathsf{H}E})^{-1} \;:\; \mathsf{T}B \xrightarrow{\cong} \mathsf{H}E \subset \mathsf{T}E\,, \tag{9}$$

and the latter permits us – in virtue of the theorem on uniqueness of the integral curve of a vector field through a given point in its domain – to associate, with an arbitrary smooth path $\gamma \;:\;]-\varepsilon, \varepsilon[\longrightarrow B$ though $x \equiv \gamma(0) \in B$, the **horizontal lift** of that **path** $\widetilde\gamma_p \;:\;]-\varepsilon, \varepsilon[\longrightarrow E$ through an arbitrary point $p \in E_x$, *i.e.*, an integral curve of the vector field $\mathrm{Hor}_\gamma(\dot\gamma)$ that solves the initial-value problem

$$\dot{\widetilde\gamma}_p(t) = \mathrm{Hor}_{\widetilde\gamma_p(t)}\big(\dot\gamma(t)\big)\,, \qquad \widetilde\gamma_p(0) = p\,. \tag{10}$$

Upon lifting γ to all points in the fibre E_x, we obtain a family of smooth diffeomorphisms between that fibre and fibres in some neighbourhood thereof,

$$\mathsf{P}^{\gamma}_{0,t}\ :\ E_x \xrightarrow{\cong} E_{\gamma(t)}\ :\ p \longmapsto \widetilde{\gamma}_p(t)\,. \tag{11}$$

Their smoothness is ensured by the smooth dependence of the flow of the vector field $\mathrm{Hor}_{\widetilde{\gamma}_p}(\dot{\gamma})$ on the initial condition p (following, in turn, from the smooth dependence of $\mathrm{Hor}_{\widetilde{\gamma}_p}$ on p), and the local uniqueness of the solution to the above initial-value problem guarantees bijectivity of the maps $\mathsf{P}^{\gamma}_{0,t}$. Finally, the superposition law (PT3) of Def. 3, as well as the initial condition (PT2), emerge directly from the construction of the flow of the smooth vector field (on the basis of the correspondence with local groups of local diffeomorphisms). Thus, it remains to consider the covariant derivative defined by the thus established connection on fibres of E. Reasoning as in the proof of Prop. 2, we compute

$$\begin{aligned}\nabla_{\dot{\gamma}}\sigma(x) &\equiv \tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0}\mathsf{P}^{\gamma\,-1}_{0,t}\big(\sigma\circ\gamma(t)\big) = -\mathrm{Hor}_{\sigma(x)}\big(\dot{\gamma}(0)\big) + \mathsf{T}_{\sigma(x)}\mathsf{P}^{\gamma\,-1}_{0,0}\big(\tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0}\sigma\circ\gamma(t)\big)\\ &= -\mathrm{Hor}_{\sigma(x)}\big(\dot{\gamma}(0)\big) + \mathsf{T}_x\sigma\big(\dot{\gamma}(0)\big)\,,\end{aligned} \tag{12}$$

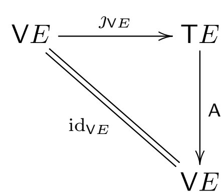
and hence conclude that the derivative depends on the path γ employed in its definition solely through $\dot{\gamma}(0)$ (and $\gamma(0) = x$), and on the field $\dot{\gamma}$ – in a manifestly $C^\infty(B,\mathbb{R})$-linear manner, in conformity with the axiom (PT4) of Def. 3. □

On the basis of the interpretation of the Whitney sum as a geometrisation of the direct sum of vector spaces, and with the help of the equivalent description of that structure in terms of a complete family of complementary projections, we infer that the description of a connection on a fibre bundle that uses the decomposition of the tangent bundle over the total space of that bundle into a (Whitneya) sum of subbundles: the vertical one and the horizontal one carries in itself a hint as to another natural reformulation of the definition of a connection. Thus, we arrive at

Definition 5. A **connection form** on a fibre bundle (E,B,F,π_E) is a $C^\infty(B,\mathbb{R})$-linear morphism of vector bundles

$$(\mathsf{A},\mathrm{id}_E)\ :\ \mathsf{T}E \longrightarrow \mathsf{V}E$$

with the property expressed by the commutative diagram



in which $\jmath_{\mathsf{VE}}$ is the canonical injection.

Also this time, we readily verify equivalence of the definitions.

Theorem 4. An Ehresmann connection on a fibre bundle canonically determines a connection form and *vice versa.*

<u>*Proof:*</u> An Ehresmann connection $\mathsf{T}E = \mathsf{V}E \oplus_B \mathsf{H}E$ induces a ($C^\infty(B,\mathbb{R})$-linear) vector-bundle epimorphism

$$\mathsf{A} := \mathrm{id}_{\mathsf{T}E} - \big(\mathsf{T}\pi_E\restriction_{\mathsf{H}E}\big)^{-1}\circ\mathsf{T}\pi_E\ :\ \mathsf{T}E \twoheadrightarrow \mathsf{V}E\,,$$

giving a smooth distribution of projections onto the fibre of the first component of the Whitney sum. For any vertical vector field $\mathcal{V}$, we obtain

$$\mathsf{A}(\mathcal{V}) \equiv \mathrm{id}_{\mathsf{T}E}(\mathcal{V}) - \big(\mathsf{T}\pi_E\restriction_{\mathsf{H}E}\big)^{-1}\circ\mathsf{T}\pi_E(\mathcal{V}) = \mathcal{V} +_{\mathsf{T}E} \mathbf{0}_{\mathsf{T}E} = \mathcal{V}\,.$$

Conversely, to any vector-bundle morphism $\mathsf{A} : \mathsf{T}E \longrightarrow \mathsf{V}E$ with the property $\mathsf{A}\restriction_{\mathsf{V}E} = \mathrm{id}_{\mathsf{V}E}$, we may – in virtue of Thm. 2 – associate the vector subbundle

$$\mathsf{H}E := \mathrm{Ker}(\mathsf{A}, \mathrm{id}_B) \subset \mathsf{T}E\,.$$

For any $v \in \mathsf{H}_pE \cap \mathsf{V}_pE,\ p \in E$, we obtain the result

$$v = \mathrm{id}_{\mathsf{V}E}(v) = \mathsf{A}(v) = 0_{\mathsf{T}_pE}\,,$$

and so

$$\mathsf{H}E \cap \mathsf{V}E = \{\mathbf{0}_{\mathsf{T}E}\}\,.$$

At the same time, any vector field $\mathcal{V}$ on E splits as

$$\mathcal{V} = (\mathrm{id}_{\mathsf{T}E} - \mathsf{A})(\mathcal{V}) + \mathsf{A}(\mathcal{V})$$

with the smooth components taking values in the respective subbundles:

$$\mathsf{A}(\mathcal{V}) \ :\ E \longrightarrow \mathsf{V}E$$

and

$$\mathcal{V} - \mathsf{A}(\mathcal{V}) \ :\ E \longrightarrow \mathsf{H}E\,,$$

which follows from the identity

$$\mathsf{A}\big(\mathcal{V} - \mathsf{A}(\mathcal{V})\big) = \mathsf{A}(\mathcal{V}) - \mathsf{A}\restriction_{\mathsf{V}E}\big(\mathsf{A}(\mathcal{V})\big) = \mathsf{A}(\mathcal{V}) - \mathrm{id}_{\mathsf{V}E}\big(\mathsf{A}(\mathcal{V})\big) = \mathsf{A}(\mathcal{V}) - \mathsf{A}(\mathcal{V}) = \mathbf{0}_{\mathsf{T}E}\,.$$

□

Remark 3. We conclude the general discussion of the construction of a connection on a fibre bundle with a formulation of its local description using local trivialisations $\tau_i \ :\ \pi_E^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times F,\ i \in I$ of the bundle, associated with an open cover $\mathscr{O} = \{\mathcal{O}_i\}_{i\in I}$, and with transition maps $g_{ij} \ :\ \mathcal{O}_{ij} \longrightarrow \mathrm{Aut}(F)$. Our detailed analysis shall be carried out in local coordinates: $(x^\mu, \xi^A),\ \mu \in \overline{1, \dim B},\ A \in \overline{1, \dim F}$ on a neighbourhood of $(x, f) \equiv (\pi_{\mathsf{T}\mathcal{O}_{ij}}(X), \pi_{\mathsf{T}F}(V)) \in \mathcal{O}_{ij} \times F$ and $(y^\mu \equiv x^\mu, \zeta^A)$ on a neighbourhood of $(x,\, g_{ij}(x)(f)) \in \mathcal{O}_{ij} \times F$. The trivialisations of E induce the local tangential trivialisations

$$\mathsf{T}\tau_i \ :\ \mathsf{T}\pi_E^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mathsf{T}(\mathcal{O}_i \times F) \cong \mathrm{pr}_1^*\mathsf{T}\mathcal{O}_i \oplus_{\mathcal{O}_i\times F,\mathbb{R}} \mathrm{pr}_2^*\mathsf{T}F\,.$$

Introducing coordinate bases in the relevant tangent spaces, let us denote

$$\mathsf{T}t_{ij} \equiv \mathsf{T}\tau_i \circ \big(\mathsf{T}\tau_j\big)^{-1} \ : \mathsf{T}(\mathcal{O}_{ij} \times F) \circlearrowleft \ :\ \big((x,f), X + V\big) \longmapsto \big(\big(x, g_{ij}(x)(f)\big), \widetilde{X} + \widetilde{V}\big)\,,$$

$$X \equiv X^\mu \triangleright \tfrac{\partial}{\partial x^\mu}(x)\,, \qquad V \equiv V^A \triangleright \tfrac{\partial}{\partial \xi^A}(f)\,,$$

$$\widetilde{X} \equiv \widetilde{X}^\mu \triangleright \tfrac{\partial}{\partial y^\mu}(x)\,, \qquad \widetilde{V} \equiv \widetilde{V}^A \triangleright \tfrac{\partial}{\partial \zeta^A}\big(g_{ij}(x)(f)\big)\,,$$

and use the identitites

$$t_{ij}^*\mathsf{d}y^\mu(x,f) = \tfrac{\partial y^\mu}{\partial x^\nu}(x) \triangleright \mathsf{d}x^\nu(x) = \delta^\mu_\nu \triangleright \mathsf{d}x^\nu(x) = \mathsf{d}x^\mu(x)$$

and

$$t_{ij}^*\mathsf{d}\zeta^A(x,f) = \tfrac{\partial \zeta^A}{\partial x^\mu}\big(\big(g_{ij}(x)(f)\big) \triangleright \mathsf{d}x^\mu(x) + \tfrac{\partial \zeta^A}{\partial \xi^B}\big(\big(g_{ij}(x)(f)\big) \triangleright \mathsf{d}\xi^B(f)$$

to derive

$$\begin{aligned}\widetilde{X} \ &\equiv\ \mathsf{T}_{(x,f)}t_{ij}(X+V) \lrcorner\, \mathsf{d}y^\mu(x) \triangleright \tfrac{\partial}{\partial y^\mu}(x) = (X+V) \lrcorner\, t_{ij}^*\mathsf{d}y^\mu(x,f) \triangleright \tfrac{\partial}{\partial x^\mu}(x)\\ &=\ (X+V) \lrcorner\, \mathsf{d}x^\mu(x) \triangleright \tfrac{\partial}{\partial x^\mu}(x) = X \lrcorner\, \mathsf{d}x^\mu(x) \triangleright \tfrac{\partial}{\partial x^\mu}(x) \equiv X\end{aligned}$$

and

$$\widetilde{V} \ \equiv\ \mathsf{T}_{(x,f)}t_{ij}(X+V) \lrcorner\, \mathsf{d}\zeta^A\big(g_{ij}(x)(f)\big) \triangleright \tfrac{\partial}{\partial \zeta^A}\big(g_{ij}(x)(f)\big)$$

$$= \quad (X+V) \lrcorner t_{ij}^* \mathsf{d}\zeta^A (g_{ij}(x,f) \triangleright \tfrac{\partial}{\partial \zeta^A}\big(g_{ij}(x)(f)\big)$$

$$= \quad \big(X^\mu \tfrac{\partial \zeta^A}{\partial x^\mu}\big(\big(g_{ij}(x)(f)\big) + V^B \tfrac{\partial \zeta^A}{\partial \xi^B}\big(\big(g_{ij}(x)(f)\big)\big) \triangleright \tfrac{\partial}{\partial \zeta^A}\big(g_{ij}(x)(f)\big).$$

In what follows, we use the symbol ϖ_i (resp. ϖ_{ij}) for the projection onto the second direct summand, $\mathsf{T}F$, in the Whitney-sum decomposition of the tangent bundle $\mathsf{T}(\mathcal{O}_i \times F) \cong \mathrm{pr}_1^* \mathsf{T}\mathcal{O}_i \oplus_{\mathcal{O}_i \times F, \mathbb{R}} \mathrm{pr}_2^* \mathsf{T}F$ (resp. of $\mathsf{T}(\mathcal{O}_{ij} \times F) \cong \mathrm{pr}_1^* \mathsf{T}\mathcal{O}_{ij} \oplus_{\mathcal{O}_i \times F, \mathbb{R}} \mathrm{pr}_2^* \mathsf{T}F$).

In order to proceed with our computation, we need to make additional assumptions with regard to the group $\mathrm{Aut}(F)$ in which the transition maps g_{ij} take values. From now onwards, we presuppose that the ditinguished elements $g_{ij}(x)$ of $\mathrm{Aut}(F)$ belong to some (finite-dimensional) Lie (sub)group $\mathrm{G} \subset \mathrm{Aut}(F)$, which happens to be true in the cases of physical interest: in that of a vector bundle n over the base field $\mathbb{K}$, we are dealing with the general linear group $\mathrm{GL}(n;\mathbb{K})$, whereas in those of a principal bundle and bundles associated with it – with the structure Lie group. The assumption permits us to employ the detailed knowledge, gathered during Lectures 4. and 5., on the differential calculus on a group manifold and on a manifold with a Lie-group action, compatible with the natural group action.

Consider a local section

$$\sigma \ : \ \mathcal{O} \longrightarrow E$$

and pick up a point $x \in \mathcal{O} \cap \mathcal{O}_{ij}$. In the local picture, we define a map $\sigma_i \ : \ \mathcal{O}_i \longrightarrow F$ of class C^∞ as follows:

$$\tau_i \circ \sigma(x) =: \big(x, \sigma_i(x)\big),$$

noting the identity

$$\big(x, \sigma_j(x)\big) \equiv \tau_j \circ \sigma(x) = \tau_j \circ \tau_i^{-1}\big(\tau_i \circ \sigma(x)\big) = \tau_j \circ \tau_i^{-1}\big(x, \sigma_i(x)\big) = \big(x, g_{ji}(x)\big(\sigma_i(x)\big)\big)$$

from which we derive the transformation rule for the mappings σ_i on $\mathcal{O} \cap \mathcal{O}_{ij}$,

$$\sigma_j(x) = g_{ji}(x)\big(\sigma_i(x)\big) \equiv \delta_{g_{ji}(x)}\big(\sigma_i(x)\big).$$

Above, we have used the symbol $\delta_\cdot \ : \ \mathrm{G} \times F \longrightarrow F$ to denote the (defining) action of the Lie group $\mathrm{G} \subset \mathrm{Aut}(F)$ on F. The objects intriduced heretofore enable us to quantify, in the local picture, the correction to the natural vertical differentiation $\mathsf{T}\sigma_i$ of the section σ sourced by the covariant derivative. Thus, for an arbitrary vector field $\mathcal{V} \in \mathfrak{X}^0(\mathcal{O})$ attaining the value $V \equiv \mathcal{V}(x)$ at the point $x \in \mathcal{O} \cap \mathcal{O}_i$, we define

$$V \lrcorner \alpha_i\big(x, \sigma(x)\big) \coloneqq \varpi_i \circ \mathsf{T}\tau_i(\nabla_{\mathcal{V}} \sigma)(x) - \mathsf{T}_x \sigma_i(V), \tag{13}$$

where

$$\alpha_i\big(\cdot, \sigma(\cdot)\big) \in \mathsf{T}_\cdot^* \mathcal{O}_i \otimes_{\mathbb{R}} \mathsf{T}_\cdot F \subset \mathsf{T}_\cdot^* \mathcal{O}_i \otimes_{\mathbb{R}} \mathsf{T}_{(\cdot, \sigma(\cdot))}(\mathcal{O}_i \times F)$$

is a measure of the discrepancy between the covariant derivative and $\mathsf{T}\sigma_i$. The latter object may also be regarded equivalently as an element of the space $\Omega^1(\mathcal{O}_i) \otimes_{\mathbb{R}} \mathsf{T}F$, and whenever we do so, we denote it suggestively as $\mathsf{d}\sigma_i$. In this convention, we write

$$\varpi_i \circ \mathsf{T}\tau_i(\nabla_{\mathcal{V}} \sigma)(x) = V \lrcorner \big(\mathsf{d}\sigma_i(x) + \alpha_i\big(x, \sigma(x)\big)\big).$$

Clearly, the criterion, implicitly invoked above, of naturality of the choice $\mathsf{T}\sigma_i$ of the reference derivation has limited strength. A decent justification for the adopted decomposition of the covariant derivative into parts that depend on σ_i "tangentially" resp. "functionally" shall be provided only by a detailed discussion of a connection compatible with an extra structure on the fibre and of its physical applications, to be launched during future lectures. Meanwhile, let us investigate transformation properties of the local objects α_i displayed in transition between different local trivialisations over intersections of the respective domains. Thus, at an arbitrary point $x \in \mathcal{O} \cap \mathcal{O}_{ij}$, we obtain the identity

$$V \lrcorner \big(\mathsf{d}\sigma_i(x) + \alpha_i\big(x, \sigma(x)\big)\big) \quad \equiv \quad \varpi_{ij} \circ \mathsf{T}\tau_i(\nabla_{\mathcal{V}} \sigma)(x) = \varpi_{ij} \circ \mathsf{T}t_{ij} \circ \mathsf{T}\tau_j(\nabla_{\mathcal{V}} \sigma)(x)$$

$$= \quad V \lrcorner \big(\mathrm{id}_{\mathsf{T}^* B} \otimes \mathsf{T}_{\sigma_j(x)} \delta_{g_{ij}(x)}\big)\big(\mathsf{d}\sigma_j(x) + \alpha_j\big(x, \sigma(x)\big)\big),$$

that is – in view of the arbitrariness of V –

$$\begin{aligned}&\big(\mathrm{id}_{\mathsf{T}^*B}\otimes\mathsf{T}_{\sigma_j(x)}\delta_{g_{ij}(x)}\big)\alpha_j\big(x,\sigma(x)\big)-\alpha_i\big(x,\sigma(x)\big)\\ =\ &\mathsf{d}\sigma_i(x)-\big(\mathrm{id}_{\mathsf{T}^*B}\otimes\mathsf{T}_{\sigma_j(x)}\delta_{g_{ij}(x)}\big)\mathsf{d}\sigma_j(x)\\ =\ &\mathsf{d}\sigma_i(x)-\big(\mathrm{id}_{\mathsf{T}^*B}\otimes\mathsf{T}_{\sigma_j(x)}\delta_{g_{ij}(x)}\big)\mathsf{d}\big(\delta_{g_{ji}(x)}\big(\sigma_i(x)\big)\big)\,.\end{aligned}$$

In order to avoid confusion at later stages of our analysis, we emphasise: $\mathsf{T}_{\sigma_j(x)}\delta_{g_{ij}(x)}$ is the tangent of the diffeomorphism $\delta_{g_{ij}(x)}\ :\ F\circlearrowleft$ at the point $\sigma_j(x)$ in the latter's domain, whereas $\mathsf{d}\big(\delta_{g_{ji}(x)}\big(\sigma_i(x)\big)\big)$ is (a presentation of) the tangent of $\delta_{g_{ji}(\cdot)}\big(\sigma_i(\cdot)\big)\ :\ \mathcal{O}_{ij}\longrightarrow F$ at the point x. Hence, upon invoking the statement of Prop. 4.2 in conjunction with the natural presentation of the (de Rhama) exterior derivative on the Lie group G (written for an arbitrary $f\in C^1(\mathrm{G},\mathbb{R})$ and at the point $g\in\mathrm{G}$):

$$\mathsf{d}f(g)=L_A(f)(g)\triangleright\theta_{\mathrm{L}}^A(g)\equiv\tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0}\big(f\circ\mathcal{L}_t^{t_A}\big)(g)\triangleright\theta_{\mathrm{L}}^A(g)',\,,$$

we may write

$$\begin{aligned}\mathsf{d}\big(\delta_{g_{ji}(x)}\big(\sigma_i(x)\big)\big)\ &=\ \big(\mathrm{id}_{\mathsf{T}^*B}\otimes\mathsf{T}_{\sigma_i(x)}\delta_{g_{ji}(x)}\big)\mathsf{d}\sigma_i(x)+\theta_{\mathrm{L}}^A\big(g_{ji}(x)\big)\otimes_{\mathbb{R}}\tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0}\big(\delta_{\mathcal{L}_t^{t_A}(g_{ji}(x))}\big(\sigma_i(x)\big)\big)\\ &=\ \big(\mathrm{id}_{\mathsf{T}^*B}\otimes\mathsf{T}_{\sigma_i(x)}\delta_{g_{ji}(x)}\big)\big(\mathsf{d}\sigma_i(x)+g_{ji}^*\theta_{\mathrm{L}}^A(x)\otimes_{\mathbb{R}}\tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0}\big(\delta_{g_{ij}(x)\cdot\mathcal{L}_t^{t_A}(g_{ji}(x))}\big(\sigma_i(x)\big)\big)\big)\\ &=\ \big(\mathrm{id}_{\mathsf{T}^*B}\otimes\mathsf{T}_{\sigma_i(x)}\delta_{g_{ji}(x)}\big)\big(\mathsf{d}\sigma_i(x)+g_{ji}^*\theta_{\mathrm{L}}^A(x)\otimes_{\mathbb{R}}\tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0}\big(\delta_{g_{ij}(x)\cdot g_{ji}(x)\cdot\mathcal{L}_t^{t_A}(e)}\big(\sigma_i(x)\big)\big)\big)\\ &=\ \big(\mathrm{id}_{\mathsf{T}^*B}\otimes\mathsf{T}_{\sigma_i(x)}\delta_{g_{ji}(x)}\big)\big(\mathsf{d}\sigma_i(x)+g_{ji}^*\theta_{\mathrm{L}}^A(x)\otimes_{\mathbb{R}}\tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0}\big(\delta_{\mathcal{L}_t^{t_A}(e)}\big(\sigma_i(x)\big)\big)\big)\,,\end{aligned}$$

which – in the light of the explicit formula for the fundamental vector field from p. 5.7 – yields

$$\mathsf{d}\big(\delta_{g_{ji}(x)}\big(\sigma_i(x)\big)\big)=\big(\mathrm{id}_{\mathsf{T}^*B}\otimes\mathsf{T}_{\sigma_i(x)}\delta_{g_{ji}(x)}\big)\mathsf{d}\sigma_i(x)+g_{ji}^*\theta_{\mathrm{L}}^A(x)\otimes_{\mathbb{R}}\mathsf{T}_{\sigma_i(x)}\delta_{g_{ji}(x)}\big(\mathcal{K}_{t_A}\big(\sigma_i(x)\big)\big)\,.$$

Based on Props. 4.9 and 5.1, we may cast the latter in the above in the form

$$\begin{aligned}\mathsf{d}\big(\delta_{g_{ji}(x)}\big(\sigma_i(x)\big)\big)\ &=\ \big(\mathrm{id}_{\mathsf{T}^*B}\otimes\mathsf{T}_{\sigma_i(x)}\delta_{g_{ji}(x)}\big)\mathsf{d}\sigma_i(x)+g_{ji}^*\theta_{\mathrm{L}}^A(x)\otimes_{\mathbb{R}}\mathcal{K}_{\mathsf{T}_e\mathrm{Ad}_{g_{ji}(x)}(t_A)}\big(\sigma_j(x)\big)\\ &=\ \big(\mathrm{id}_{\mathsf{T}^*B}\otimes\mathsf{T}_{\sigma_i(x)}\delta_{g_{ji}(x)}\big)\mathsf{d}\sigma_i(x)+\big(\mathsf{T}_e\mathrm{Ad}_{g_{ji}(x)}\big)_A^{\ \ B}\triangleright g_{ji}^*\theta_{\mathrm{L}}^A(x)\otimes_{\mathbb{R}}\mathcal{K}_{t_B}\big(\sigma_j(x)\big)\\ &=\ \big(\mathrm{id}_{\mathsf{T}^*B}\otimes\mathsf{T}_{\sigma_i(x)}\delta_{g_{ji}(x)}\big)\mathsf{d}\sigma_i(x)+g_{ji}^*\theta_{\mathrm{R}}^B(x)\otimes_{\mathbb{R}}\mathcal{K}_{t_B}\big(\sigma_j(x)\big)\,,\end{aligned}$$

or – upon invoking the statement of Props. 4.9 once more –

$$\mathsf{d}\big(\delta_{g_{ji}(x)}\big(\sigma_i(x)\big)\big)=\big(\mathrm{id}_{\mathsf{T}^*B}\otimes\mathsf{T}_{\sigma_i(x)}\delta_{g_{ji}(x)}\big)\mathsf{d}\sigma_i(x)-g_{ij}^*\theta_{\mathrm{L}}^B(x)\otimes_{\mathbb{R}}\mathcal{K}_{t_B}\big(\sigma_j(x)\big)\,.\tag{14}$$

By the end of the day, we arrive at the sought-after transformation formula

$$\begin{aligned}\alpha_j\big(x,\sigma(x)\big)\ &=\ \big(\mathrm{id}_{\mathsf{T}^*B}\otimes\mathsf{T}_{\sigma_i(x)}\delta_{g_{ji}(x)}\big)\alpha_i\big(x,\sigma(x)\big)-g_{ji}^*\theta_{\mathrm{R}}^A(x)\otimes_{\mathbb{R}}\mathcal{K}_{t_A}\big(\sigma_j(x)\big)\\ &=\ \big(\mathrm{id}_{\mathsf{T}^*B}\otimes\mathsf{T}_{\sigma_i(x)}\delta_{g_{ij}(x)^{-1}}\big)\alpha_i\big(x,\sigma(x)\big)+g_{ij}^*\theta_{\mathrm{L}}^A(x)\otimes_{\mathbb{R}}\mathcal{K}_{t_A}\big(\sigma_j(x)\big)\,,\end{aligned}$$

of a manifestly affine nature. The formula is the point of departure for subsequent analysis taking into account any additional algebraic structure on the fibre, which we take up in the next lecture.

Our considerations are crowned with a specialisation of the notion of a fibre-bundle morphism in the presence of a connection, which we undertake below.

Definition 6. Adopt the notation of Defs. 3, 4 and 5 and let $(E_\alpha,B_\alpha,\ F_\alpha,\pi_{E_\alpha}),\ \alpha\in\{1,2\}$ be fibre bundles with connection (in any one of the equivalent formulations). A **morphism of fibre bundles with connection** (**covering a diffeomorphism between the bases**) between E_1

and E_2 is a fibre-bundle morphism described by the commutative diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \Big\downarrow{\scriptstyle \pi_{E_1}} & & \Big\downarrow{\scriptstyle \pi_{E_2}} \\ B_1 & \xrightarrow[f]{} & B_2 \end{array}$$

with the base component[2] $f \in \mathrm{Diff}^\infty(B_1, B_2)$, subject to the following conditions:

(FCM1) for arbitrary: path $\gamma \; : \;]-\varepsilon, \varepsilon[\longrightarrow B_1$, $\varepsilon > 0$ and $t \in]-\varepsilon, \varepsilon[$, the identity

$$\Phi \circ \mathsf{P}^{(1)\,\gamma}_{0,t} = \mathsf{P}^{(2)\,f\circ\gamma}_{0,t} \circ \Phi$$

holds true, and then for any: (local) section $\sigma \; : \; \mathcal{O}_x \longrightarrow E_1$ defined on some open neighbourhood $\mathcal{O}_x$ of the point $x \in B_1$ and vector field $\mathcal{V} \in \mathfrak{X}(\mathcal{O}_x)$, the **covariance condition**

$$\mathsf{T}_{\sigma(x)}\Phi\big(\nabla^{(1)}_{\mathcal{V}}\sigma(x)\big) = \nabla^{(2)}_{\mathsf{T}f(\mathcal{V})}\big(\Phi \circ \sigma \circ f^{-1}\big)\big(f(x)\big)$$

is satisfied;

(FCM2) the pair $(\mathsf{T}\Phi, \mathsf{T}f)$ of tangent maps restricts to the horizontal subbundles $\mathsf{H}E_\alpha$, $\alpha \in \{1,2\}$ and, in so doing, engenders a vector-bundle morphism described by the commutative diagram

$$\begin{array}{ccc} \mathsf{H}E_1 & \xrightarrow{\mathsf{T}\Phi\restriction_{\mathsf{H}E_1}} & \mathsf{H}E_2 \\ \Big\downarrow{\scriptstyle \mathsf{T}\pi_{E_1}\restriction_{\mathsf{H}E_1}} & & \Big\downarrow{\scriptstyle \mathsf{T}\pi_{E_2}\restriction_{\mathsf{H}E_1}} \\ \mathsf{T}B_1 & \xrightarrow[\mathsf{T}f]{} & \mathsf{T}B_2 \end{array} \;;$$

(FCM3) the tangent map $\mathsf{T}\Phi$ preserves the connection form, as expressed by the equality

$$\mathsf{T}\Phi \circ \mathsf{A}_1 = \mathsf{A}_2 \circ \mathsf{T}\Phi\,.$$

Remark 4. The covariance condition of condition (FCM1) is checked in a direct computation (carried out using an arbitrary path γ in B_1 through $x = \gamma(0)$, with the tangent vector $\dot{\gamma}(0) = \mathcal{V}(x)$),

$$\begin{aligned} \mathsf{T}_{\sigma(x)}\Phi\big(\nabla_{\mathcal{V}}\sigma(x)\big) &\equiv \mathsf{T}_{\sigma(x)}\Phi\big(\tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0} \mathsf{P}^{\gamma\,-1}_{0,t}\big(\sigma\circ\gamma(t)\big)\big) = \tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0} \Phi \circ \mathsf{P}^{\gamma\,-1}_{0,t}\big(\sigma\circ\gamma(t)\big) \\ &= \tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0} \mathsf{P}^{f\circ\gamma\,-1}_{0,t}\big(\big(\Phi\circ\sigma\circ f^{-1}\big)\circ(f\circ\gamma)(t)\big) \\ &= \nabla_{\mathsf{T}f(\mathcal{V})}\big(\Phi\circ\sigma\circ f^{-1}\big)\big(f(x)\big)\,, \end{aligned}$$

the identification of the vector field along which the section $\Phi \circ \sigma$ is differentiated at the end of the sequence of equalities following directly from the identity

$$\tfrac{\mathsf{d}}{\mathsf{d}t}\,(f\circ\gamma)(t) = \mathsf{T}_{\gamma(t)}f\big(\dot{\gamma}(t)\big)\,.$$

It is the condition verified above that explains the name given to the object $\nabla_{\mathcal{V}}\sigma$.

It is to be noted, furthermore, with regard to condition (FCM2) that the pair $(\mathsf{T}\Phi, \mathsf{T}f)$ is always a vector-bundle morphism in virtue of the functoriality of T and only the postulate that the horizontal subbundles should be preserved is a nontrivial condition additionally constraining the fibre-bundle morphism (Φ, f).

[2]The reason to restrict our choice of the base component of the morphism is obvious: Such a restriction ensures the existence of a natiral transport of vector fields between the bases, and so also between the horizontal subbundles. It is possible to generalise the definition given, but we shall not pursue it here.

Theorem 5. Adopt the notation of Def. 6. Conditions (FCM1), (FCM2) and (FCM3) are mutually equivalent.

Proof:

$(FCM1) \Rightarrow (FCM2)$ Using the isomorphisms

$$\mathrm{Hor}_p^{(1)} \ :\ \mathsf{T}_{\pi_{E_1}(p)} B_1 \xrightarrow{\cong} \mathsf{H}_p E_1 \,, \qquad p \in E_1 \,,$$

we check – for any $V \equiv \dot\gamma(0) \in \mathsf{T}_{\pi_{E_1}(p)} B_1$ (and at an arbitrary point $p \in E_1$), and adducing the arguments from the second part of Remark 4 –

$$\begin{aligned} \mathsf{T}_p \Phi\big(\mathrm{Hor}_p^{(1)}(V)\big) &= \mathsf{T}_p \Phi\big(\tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0} \mathsf{P}^{\gamma}_{0,t}(p)\big) = \tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0} \Phi \circ \mathsf{P}^{\gamma}_{0,t}(p) = \tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0} \mathsf{P}^{f\circ\gamma}_{0,t}\big(\Phi(p)\big) \\ &\equiv \mathrm{Hor}^{(2)}_{\Phi(p)}\big(\mathsf{T}_{\pi_{E_1}(p)} f(V)\big) \,. \end{aligned}$$

Linearity (fibrewise) of the restricted tangent morphism is a direct consequence of its construction.

$(FCM2) \Rightarrow (FCM1)$ Commutativity of the diagram of condition (FCM2), with the restriction to the point $p \in E_{1\,x}$, $x \in B_1$ given by

$$(15) \qquad \begin{array}{ccc} \mathsf{H}_p E_1 & \xrightarrow{\ \mathsf{T}_p \Phi\restriction_{\mathsf{H}_p E_1}\ } & \mathsf{H}_{\Phi(p)} E_2 \\ \uparrow {\scriptstyle \mathrm{Hor}_p^{(1)}} & & \uparrow {\scriptstyle \mathrm{Hor}^{(2)}_{\Phi(p)}} \\ \mathsf{T}_x B_1 & \xrightarrow[\ \mathsf{T}_x f\ ]{} & \mathsf{T}_{f(x)} B_2 \end{array} \,,$$

enables us to compute, for arbitrary paths $\widetilde{\gamma}_p$, $p \in E_{1\,x}$ lifting the path γ in B_1 through $x \equiv \gamma(0)$, *i.e.*, solving the initial-value problem (10) and defining, through that, a connection on fibres by formula (11),

$$\begin{aligned} \tfrac{\mathsf{d}}{\mathsf{d}t}(\Phi \circ \widetilde{\gamma}_p)(t) &= \mathsf{T}_{\widetilde{\gamma}_p(t)} \Phi\big(\tfrac{\mathsf{d}}{\mathsf{d}t}\widetilde{\gamma}_p(t)\big) = \mathsf{T}_{\widetilde{\gamma}_p(t)} \Phi \circ \mathrm{Hor}^{(1)}_{\widetilde{\gamma}_p(t)}\big(\dot\gamma(t)\big) \\ &= \mathrm{Hor}^{(2)}_{\Phi\circ\widetilde{\gamma}_p(t)} \circ \mathsf{T}_{\gamma(t)} f\big(\dot\gamma(t)\big) = \mathrm{Hor}^{(2)}_{\Phi\circ\widetilde{\gamma}_p(t)}\big(\tfrac{\mathsf{d}}{\mathsf{d}t}(f\circ\gamma)(t)\big) \,. \end{aligned}$$

On the other hand, directly by the definition of the horizontal lift of the path ($f \circ \gamma$ in B_2 through $f(x) \equiv f \circ \gamma(0)$ to $\Phi(p)$), we obtain the equality

$$\mathrm{Hor}^{(2)}_{\Phi(\widetilde{\gamma}_p(t))}\big(\tfrac{\mathsf{d}}{\mathsf{d}t}(f\circ\gamma)(t)\big) = \tfrac{\mathsf{d}}{\mathsf{d}t}\widetilde{(f\circ\gamma)}_{\Phi(\widetilde{\gamma}_p(0))}(t) \equiv \tfrac{\mathsf{d}}{\mathsf{d}t}\widetilde{(f\circ\gamma)}_{\Phi(p)}(t) \,,$$

and so clearly – due to the identity of the initial points, $\widetilde{(f\circ\gamma)}_{\Phi(p)}(0) = \Phi(p) = \Phi \circ \widetilde{\gamma}_p(0)$, and of the tangent vectors, and in virtue of the theorem on uniqueness of the integral curve of a vector field through a given point in its domain – we have the equality

$$\widetilde{(f\circ\gamma)}_{\Phi(p)} = \Phi \circ \widetilde{\gamma}_p \,,$$

which implies, at an arbitrary point $p \in E_1$, the desired relation

$$\big(\Phi \circ \mathsf{P}^{(1)\,\gamma}_{0,t}\big)(p) \equiv \Phi \circ \widetilde{\gamma}_p(t) = \widetilde{(f\circ\gamma)}_{\Phi(p)}(t) \equiv \mathsf{P}^{(2)\,f\circ\gamma}_{0,t}\big(\Phi(p)\big) = \big(\mathsf{P}^{(2)\,f\circ\gamma}_{0,t} \circ \Phi\big)(p) \,.$$

$(FCM2) \Rightarrow (FCM3)$ Upon noting that A_α is a projection onto $\mathsf{V}E_\alpha$ along $\mathsf{H}E_\alpha$,

$$\mathsf{A}_\alpha = \mathrm{id}_{\mathsf{T}E_\alpha} - \mathrm{Hor}^{(\alpha)}_{\cdot} \circ \mathsf{T}\pi_{E_\alpha} \,, \qquad \alpha \in \{1,2\} \,,$$

we calculate – with reference to Diag. (15) –

$$\begin{aligned} \mathsf{T}\Phi \circ \mathsf{A}_1 &= \mathsf{T}\Phi \circ \mathrm{id}_{\mathsf{T}E_1} - \big(\mathsf{T}\Phi \circ \mathrm{Hor}^{(1)}_{\cdot}\big) \circ \mathsf{T}\pi_{E_1} = \mathsf{T}\Phi - \big(\mathrm{Hor}^{(2)}_{\Phi(\cdot)} \circ \mathsf{T}f\big) \circ \mathsf{T}\pi_{E_1} \\ &= \mathsf{T}\Phi - \mathrm{Hor}^{(2)}_{\Phi(\cdot)} \circ \big(\mathsf{T}f \circ \mathsf{T}\pi_{E_1}\big) = \mathrm{id}_{\mathsf{T}E_2} \circ \mathsf{T}\Phi - \mathrm{Hor}^{(2)}_{\Phi(\cdot)} \circ \big(\mathsf{T}\pi_{E_2} \circ \mathsf{T}\Phi\big) \\ &\equiv \mathsf{A}_2 \circ \mathsf{T}\Phi \,. \end{aligned}$$

$(FCM3) \Rightarrow (FCM2)$ As $\mathsf{H}E_\alpha \equiv \mathrm{Ker}\,\mathsf{A}_\alpha,\ \alpha \in \{1,2\}$, it suffices to show that

$$\mathsf{T}\Phi(\mathrm{Ker}\,\mathsf{A}_1) \subset \mathrm{Ker}\,\mathsf{A}_2\,,$$

but this follows directly from the sequence of relations

$$\mathsf{A}_2\big(\mathsf{T}\Phi(\mathrm{Ker}\,\mathsf{A}_1)\big) = \mathsf{T}\Phi\big(\mathsf{A}_1(\mathrm{Ker}\,\mathsf{A}_1)\big) = \mathsf{T}\Phi\big(\{\mathbf{0}_{\mathsf{T}E_1}\}\big) = \{\mathbf{0}_{\mathsf{T}E_2}\}\,.$$

□