CLASSICAL FIELD THEORY IN THE TIME OF COVID-19
8. LECTURE BATCH

Gauged symmetries in the language of principal and associated bundles – topology & geometry

We shall, now, present the basic application of principal and associated bundles in classical mechanics and field theory, to wit, the universal scheme of **gauging of symmetries** (*i.e.*, of rendering global symmetries **local**). We begin with the ancillary

Definition 1. Let (E,B,F,π_E) be a fibre bundle with local trivialisations $\tau_i\ :\ \pi_E^{-1}(\mathcal{O}_i)\xrightarrow{\cong}\mathcal{O}_i\times F$ over a trivialising open cover $\mathscr{O}=\{\mathcal{O}_i^B\}_{i\in I}$, the latter chosen[1] to support local charts $\kappa_i^B\ :\ \mathcal{O}_i^B\xrightarrow{\cong}\mathcal{U}_i\subset\mathbb{R}^{\times m},\ m=\dim B$ of an atlas $\widehat{\mathscr{A}_B}$ of the base B. Define auxilliary diffeomorphisms

$$T_v^i\ :\ \mathcal{U}_i\xrightarrow{\cong}T_v^i(\mathcal{U}_i)=:\mathcal{U}_i^0\ :\ w\longmapsto w-v\,,\qquad v\in\mathcal{U}_i$$

mapping the open sets $\mathcal{U}_i$ onto the respective neighbourhoods $\mathbf{0}_m\equiv T_v^i(v)\in\mathbb{R}^{\times m}$. Upon picking up an arbitrary atlas $\widehat{\mathscr{A}_F}=\{\kappa_\alpha^F\}_{\alpha\in J}$ of the typical fibre F, composed of charts $\kappa_\alpha^F\ :\ \mathcal{O}_\alpha^F\xrightarrow{\cong}\mathcal{V}_\alpha\subset\mathbb{R}^{\times n},\ n=\dim F,\ \alpha\in J$, we induce an atlas $\widehat{\mathscr{A}_E}=\{\kappa_{i,\alpha}^E\}_{(i,\alpha)\in I\times J}$ on E, consisting of the charts

$$\kappa_{i,\alpha}^E\equiv(\kappa_i^B\times\kappa_\alpha^F)\circ\tau_i\restriction_{\tau_i^{-1}(\mathcal{O}_i^B\times\mathcal{O}_\alpha^F)}\ :\ \tau_i^{-1}(\mathcal{O}_i^B\times\mathcal{O}_\alpha^F)\xrightarrow{\cong}\mathcal{U}_i\times\mathcal{V}_\alpha\subset\mathbb{R}^{\times m+n}\,,$$

termed **adapted charts**. On the set

$$\Gamma_x(E):=\{\ \phi\in\Gamma_{\rm loc}(E)\quad|\quad(\pi_E\circ\phi)^{-1}(\{x\})\neq\varnothing\ \}$$

of local sections of the bundle E defined near $x\in B$, we, then, establish the equivalence relation

$$\phi_1\sim_{j_x^1}\phi_2\qquad\Longleftrightarrow\qquad\big(\phi_1(x),\mathsf{T}_x\phi_1\big)=\big(\phi_2(x),\mathsf{T}_x\phi_2\big)\,.$$

The corresponding equivalence class of the section $\phi\in\Gamma_x(E)$ is denoted as

$$j_x^1\phi\equiv[\phi]_{\sim_{j_x^1}}$$

and called the **first jet of the section** ϕ. The set of such equivalence classes over a given point $x\in B$ shall be denoted as

$$\mathsf{J}_x^1E\equiv\{\ j_x^1\phi\quad|\quad\phi\in\Gamma_x(E)\ \}\,.$$

The bundle of first jets of sections of the fibre bundle E is the fibre bundle with the following components:

- the base B with the structure of a smooth manifold determined by the atlas $\widehat{\mathscr{A}_B}$;
- the total space

$$\mathsf{J}^1E:=\bigsqcup_{x\in B}\mathsf{J}_x^1E$$

  with the structure of a smooth manifold described below;
- the typical fibre $\mathsf{J}_{\mathbf{0}_m}^1(\mathbb{R}^{\times m}\times F)$ with the structure of a smooth manifold described below;
- the projection on the base

$$\pi_{\mathsf{J}^1E}\ :\ \mathsf{J}^1E\longrightarrow B\ :\ (j_x^1\phi,x)\longmapsto x\,.$$

[1] The assumption does not diminish the generality of our considerations as we may always obtain such a cover from an arbitrary trivialising cover by refining it relative to that of an arbitrary atlas of B.

Above, the maps

$$\begin{aligned} \mathsf{J}^1\kappa^E_{i,\alpha} \quad &: \quad \mathsf{J}^1 E_{i,\alpha} \equiv \{\ j^1_x\phi \quad | \quad \phi(x) \in \tau_i^{E\,-1}(\mathcal{O}^B_i \times \mathcal{O}^F_\alpha)\ \} \xrightarrow{\cong} \mathsf{J}^1\kappa^E_{i,\alpha}(\mathsf{J}^1 E_{i,\alpha}) \\ &: \quad j^1_x\phi \longmapsto \big(\kappa^B_i(x), \kappa^F_\alpha \circ \phi(x), \mathsf{D}(\kappa^F_\alpha \circ \phi \circ \kappa_i^{B\,-1})\big(\kappa^B_i(x)\big)\big)\,, \quad (i,\alpha) \in I \times J \end{aligned}$$

induce on $\mathsf{J}^1 E$ the strong pullback topology from the (subspace) product topology on the sets $\mathsf{J}^1\kappa^E_{i,\alpha}(\mathsf{J}^1 E_{i,\alpha}) \subset \mathbb{R}^{\times m+n+mn}$, *i.e.*, a subset $\mathcal{V} \subset \mathsf{J}^1 E$ is open iff the condition

$$\forall_{(i,\alpha)\in I\times J}\ :\ \mathsf{J}^1\kappa^E_{i,\alpha}(\mathcal{V} \cap \mathsf{J}^1 E_{i,\alpha}) \in \mathscr{T}\big(\mathsf{J}^1\kappa^E_{i,\alpha}(\mathsf{J}^1 E_{i,\alpha})\big)$$

is satisfied. In this topology, the maps $\mathsf{J}^1\kappa^E_{i,\alpha}$ są are – by construction – homeomorphisms and (hence) we may use them as charts, called **induced** (or **natural**) **charts**, with the corresponding transition maps

$$\mathsf{J}^1 E_{ij,\alpha\beta} \equiv \{\ j^1_x\phi \quad | \quad \phi(x) \in \tau_i^{E\,-1}(\mathcal{O}^B_{ij} \times \mathcal{O}^F_{\alpha\beta})\ \}$$

in the form

$$\begin{aligned} \mathsf{J}^1\kappa^E_{ij,\alpha\beta} := \mathsf{J}^1\kappa^E_{i,\alpha} \circ \big(\mathsf{J}^1\kappa^E_{j,\beta}\big)^{-1}\ :\ \mathsf{J}^1\kappa^E_{j,\beta}(\mathsf{J}^1 E_{ij,\alpha\beta}) &\xrightarrow{\cong} \mathsf{J}^1\kappa^E_{i,\alpha}(\mathsf{J}^1 E_{ij,\alpha\beta}) \\ :\ \big(\kappa^B_j(x), \kappa^F_\beta \circ \phi(x), &\mathsf{D}(\kappa^F_\beta \circ \phi \circ \kappa_j^{B\,-1})\big(\kappa^B_j(x)\big)\big) \\ \longmapsto \big(\kappa^B_i(x), \kappa^F_\alpha \circ \phi(x), &\mathsf{D}(\kappa^F_\alpha \circ \phi \circ \kappa_i^{B\,-1})\big(\kappa^B_i(x)\big)\big) \\ &= \big(t^B_{ij}\big(\kappa^B_j(x)\big), t^F_{\alpha\beta}\big(\kappa^F_\alpha \circ \phi(x)\big), \\ \mathsf{D}t^F_{\alpha\beta}\big(\kappa^F_\beta \circ \phi(x)\big) \circ \mathsf{D}(\kappa^F_\beta \circ \phi \circ \kappa_j^{B\,-1})\big(\kappa^B_j(x)\big) &\circ \mathsf{D}t^B_{ij}\big(\kappa^B_j(x)\big)^{-1}\big) \end{aligned}$$

The dependence of the point in the image on the argument is of class C^∞ in the base component and in that from the fibre of E (by assumption),

$$t^B_{ij} \in \mathrm{Diff}^\infty\big(\kappa^B_j(\mathcal{O}^B_{ij}), \kappa^B_i(\mathcal{O}^B_{ij})\big)\,, \qquad t^F_{\alpha\beta} \in \mathrm{Diff}^\infty\big(\kappa^F_\beta(\mathcal{O}^F_{\alpha\beta}), \kappa^F_\alpha(\mathcal{O}^F_{\alpha\beta})\big)\,,$$

and of the same class C^∞ in the last component regarded as a function of the point in the base and that of the point in the fibre of E,

$$\mathsf{D}t^F_{\alpha\beta} \in C^\infty\big(\kappa^F_\beta(\mathcal{O}^F_{\alpha\beta}), \mathbb{R}^{\times n^2}\big)\,, \qquad \mathsf{D}t^B_{ij} \in C^\infty\big(\kappa_j(\mathcal{O}^B_{ij}), \mathbb{R}^{\times m^2}\big)\,.$$

It is, moreover, linear and so of class C^∞ in the last argument $\mathsf{D}(\kappa^F_\beta \circ \phi \circ \kappa_j^{B\,-1})\big(\kappa^B_j(x)\big)$, therefore, altogether, of class C^∞. Consequently, it defines on $\mathsf{J}^1 E$ the structure of a smooth manifold. Note also that the projection on the baseis a smooth surjection as a superposition of maps of the same class,

$$\pi_{\mathsf{J}^1 E}\restriction_{\mathsf{J}^1 E_{i,\alpha}} = \kappa_i^{B\,-1} \circ \mathrm{pr}_1 \circ \mathsf{J}^1\kappa^E_{i,\alpha}\,.$$

The structure of a smooth manifold on the set $\mathsf{J}^1_{\mathbf{0}_m}(\mathbb{R}^{\times m} \times F)$ is induced analogously, in which we employ the global chart on the base $\mathbb{R}^{\times m}$ of the trivial bundle $\mathbb{R}^{\times m} \times F$ together with local charts κ^F_α, $\alpha \in J$ on its fibre. Thus, we use, on the subsets

$$\mathsf{J}^1_{\mathbf{0}_m}\big(\mathbb{R}^{\times m} \times \mathcal{O}^F_\alpha\big) \equiv \{\ j^1_{\mathbf{0}_m}\phi \quad | \quad \phi(0) \in \tau_i^{E\,-1}\big(\{\mathbf{0}_m\} \times \mathcal{O}^F_\alpha\big)\ \}\,,$$

the maps

$$\begin{aligned} \mathsf{J}^1_{\mathbf{0}_m}\kappa^F_\alpha \quad &: \quad \mathsf{J}^1_{\mathbf{0}_m}\big(\mathbb{R}^{\times m} \times \mathcal{O}^F_\alpha\big) \xrightarrow{\cong} \mathsf{J}^1_0\tau^F_\alpha\big(\mathsf{J}^1_0\big(\mathbb{R}^{\times m} \times \mathcal{O}^F_\alpha\big)\big) \\ &: \quad j^1_{\mathbf{0}_m}\phi \longmapsto \big(\kappa^F_i \circ \phi(\mathbf{0}_m), \mathsf{D}\big(\kappa^F_i \circ \phi\big)(\mathbf{0}_m)\big) \end{aligned}$$

to induce on $\mathsf{J}^1_{\mathbf{0}_m}(\mathbb{R}^{\times m} \times F)$ the strong pullback topology from the (subspace) product topology on the sets $\mathsf{J}^1_0\tau^F_\alpha\big(\mathsf{J}^1_0\big(\mathbb{R}^{\times m} \times \mathcal{O}^F_\alpha\big)\big) \subset \mathbb{R}^{\times n(1+m)}$. These maps simultaneously play the rôle of smooth local charts in this topology.

Local trivialisations of the bundle $\mathsf{J}^1 E$ are given by

$$J^1\tau^E_i \quad : \quad \pi^{-1}_{\mathsf{J}^1 E}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times \mathsf{J}^1_{\mathbf{0}_m}(\mathbb{R}^{\times m} \times F)$$

$$:\quad \big(j^1_x\phi, x\big) \longmapsto \Big(x, j^1_{\mathbf{0}_m}\Big(\big(T_{\kappa_i^B(x)} \circ \kappa_i^B \times \mathrm{id}_F\big) \circ \tau_i^E \circ \phi \circ \kappa_i^{B\,-1} \circ T^{-1}_{\kappa_i^B(x)}\Big)\Big)\,.$$

It is on the basis of the above definition, formalising the intuitive notion of co-tangency classes (of first order) of local sections of the bundle (in analogy with the construction of the tangent bundle as the bundle of co-tangency classes of paths in the manifold), that we now introduce the fundamental physical concept.

Definition 2. Let Σ be a smooth manifold equipped with a metric structure $\mathrm{g} \in \Gamma(\mathsf{T}^*\Sigma \otimes_{\mathbb{R},\Sigma} \mathsf{T}^*\Sigma)$ of signature $(d,1)$, $d \in \mathbb{N}$ and let F be a smooth manifold with the group of automorphisms (*i.e.*, of those auto-diffeomorphisms which preserve any potential extra structure on F, *e.g.*, a linear one or that of a torsor of a group) $\mathrm{Aut}(F)$. A **lagrangean field theory of type** F **over** (Σ, g) is the pair

$$\mathscr{F} := \big((\mathcal{F}, \Sigma, F, \pi_{\mathcal{F}}), \mathcal{A}_{\mathrm{DF}}\big)$$

composed of a fibre bundle $(\mathcal{F}, \Sigma, F, \pi_{\mathcal{F}})$, termed the **covariant configuration bundle of type** F (or simply the **field bundle of type** F), with the typical fibre F and local trivialisations

$$\tau_i \;:\; \pi_{\mathcal{F}}^{-1}(\mathcal{O}_i) \xrightarrow{\;\cong\;} \mathcal{O}_i \times F\,, \quad i \in I$$

associated with the open cover $\mathscr{O} = \{\mathcal{O}_i\}_{i \in I}$ of the base Σ, termed the **spacetime**, and of the functional

$$\mathcal{A}_{\mathrm{DF}}^{\mathscr{F}} \;:\; \Gamma(\mathcal{F}) \longrightarrow \mathrm{U}(1)\,,$$

termed the **Dirac–Feynman amplitude** whose critical points are customarily called **classical field configurations of type** F. Here, we assume existence of a functional[2]

$$S_{\mathscr{F}} \;:\; \Gamma(\mathcal{F}) \longrightarrow \mathbb{R}/2\pi\mathbb{Z}\,,$$

determining the Dirac–Feynman amplitude by the formula

$$\mathcal{A}_{\mathrm{DF}}^{\mathscr{F}} = \exp \circ \big(\mathsf{i}\, S_{\mathscr{F}}\big)\,,$$

that we call the **action functional of the field theory** $\mathscr{F}$. The functional is given by the class *modulo* 2π of an integral over Σ of the fibre-bundle morphism

$$\mathcal{L}_{\mathscr{F}} \;:\; \mathsf{J}^1\mathcal{F} \longrightarrow \bigwedge{}^{d+1}\mathsf{T}^*\Sigma$$

termed the **lagrangean density of the field theory** $\mathscr{F}$, *i.e.*, for any section $\phi \in \Gamma(\mathcal{F})$, we have

$$S_{\mathscr{F}}[\phi] = \int_\Sigma \mathcal{L}_{\mathscr{F}}(\mathrm{j}^1_\cdot\phi) + 2\pi\mathbb{Z}\,.$$

In the present context, global sections of the bundle $\mathcal{F}$ are called **fields of type** F, and their space $\Gamma(\mathcal{F})$ is called the **configuration space of the field theory** $\mathscr{F}$.

A **global symmetry** of the field theory $\mathscr{F}$ is an arbitrary automorphism $(\Phi, \mathrm{id}_\Sigma) \in \mathrm{Aut}_{\mathbf{Bun}(\Sigma)}(\mathcal{F}\,|\,\Sigma)$ (covering the identity on the base) of the field bundle of that theory, (locally) modelled on an automorphism of the typical fibre $\varphi \in \mathrm{Aut}_{\mathbf{Man}^{(2)}}(F)$ in the sense expressed by the family of commutative diagrams indexed by $I \ni i$

$$\begin{CD} \pi_{\mathcal{F}}^{-1}(\mathcal{O}_i) @>{\Phi}>> \pi_{\mathcal{F}}^{-1}(\mathcal{O}_i) \\ @V{\tau_i}VV @VV{\tau_i}V \\ \mathcal{O}_i \times F @>>{\mathrm{id}_{\mathcal{O}_i} \times \varphi}> \mathcal{O}_i \times F \end{CD}$$

and inducing an automorphism

$$\Gamma\Phi \;:\; \Gamma(\mathcal{F}) \circlearrowleft \;:\; \phi \longmapsto \Phi \circ \phi$$

[2] The choice of the functional's codomain was elucidated at an earlier stage when we discussed the quantum-mechanical interpretation of Dirac–Feynman amplitudes.

of the configuration space with the property

$$\mathcal{A}_{\rm DF}^{\mathscr{F}} \circ \Phi = \mathcal{A}_{\rm DF}^{\mathscr{F}} .$$

The group of all automorphisms of the field bundle of the above form is called the **global-symmetry group of the field theory** $\mathscr{F}$. We shall denote it as

$$\mathrm{Symm}(\mathscr{F} \nwarrow F) .$$

In the above definition, global symmetries *implicitly* acquire an ***active*** interpretation, in which Φ maps *different* field configurations into one another, and in particular – critical (that is classical) ones to other critical ones that are *not to be idetified* with the former (physically). As such, global symmetries determine a correspondence between *different* field configurations in the space of states of the field theory .

Automorphisms of the typical fibre of the field bundle also admit a ***passive*** interpretation in which they feature as coordinate transformations on F, or – in other words – as redefinitions of the *description* of a single field configuration. In order to better understand this statement, let us denote by $D \equiv \dim F$ the number of independent internal degrees of freedom of the field theory $\mathscr{F}$ and consider local charts: $\kappa_1 : \mathcal{O}_x^F \xrightarrow{\cong} \mathcal{U}_x \subset \mathbb{R}^D$ on a neighbourhood of $x \in F$ and $\kappa_2 : \mathcal{O}_{\varphi(x)}^F \xrightarrow{\cong} \mathcal{U}_{\phi(x)} \subset \mathbb{R}^D$ on a neighbourhood of $\varphi(x) \in F$, in which we assume, for the sake of simplicity, that $\varphi(\mathcal{O}_x^F) \subset \mathcal{O}_{\varphi(x)}^F$. Under such circumstances, we may view the diffeomorphism φ, or, more accurately – its local coordinate presentation $\varphi_{21} \equiv \kappa_2 \circ \varphi \circ \kappa_1^{-1}$, as a (smooth) coordinate redefinition for points $y \in \mathcal{O}_x^F$ that maps the original coordinates $\kappa_1(y)$ to the new ones $\kappa_2 \circ \varphi(y)$. This makes sense as

$$\kappa_2 \circ \varphi(y) \equiv \varphi_{21}\big(\kappa_1(y)\big) .$$

If we regard symmetry transformations as such arbitrary redefinitions of the (local-)coordinate system on the space of internal degrees of freedom of the field theory, then it becomes clear that there is no reason to expect (in a spacetime in which information propagates at a finite speed) that observers decribing field-theoretic phenomena from *non-coinciding* points in the spacetime Σ should perform *simultaneous* redefinitions of their respective local descriptions. In the commonly adopted (albeit non-unique) paradigm of a smooth description of physical phenomena, we postulate, accordingly, a ***localisation***, or ***gauging*** of global symmetries. It boils down to replacing the original field bundle $\mathcal{F}$ with a new fibre bundle with the *same* typical fibre F (and so also with the very same internal degrees of freedom), on which, however, symmetries modelled by $\mathrm{G} \equiv \mathrm{Symm}(\mathscr{F} \nwarrow F)$ (under local trivialisations) are realised *locally*, *i.e.*, in whic we deal with (locally) smooth profiles of symmetry transformations in $\mathrm{Symm}(\mathscr{F} \nwarrow F)$, that is with ***local gauge transformations*** $\gamma : \mathcal{O}^B \longrightarrow \mathrm{G}$, further assumed to leave unchanged the Dirac–Feynman amplitude of a theory with the symmetry G ***gauged***. The latter is required to be structurally *akin*, in some sense[3], to the original field theory $\mathscr{F}$, by which we mean that there should exist an (essentially) algorithmic scheme of transcription of $\mathscr{F}$ into the new form with the global symmetry gauged. The point of departure for a universal formalisation of the *non-dynamical* aspect of such a transcription, *i.e.*, that which concerns a redefinition of the field bundle $\mathcal{F}$, and not any additional structures on the tangent bundle $\mathsf{T}\mathscr{F}$ over it that allow us to define those terms of the lagrangean density which depend on its sections through their *derivatives*, is provided by the construction of a bundle associated with a principal bundle with the structural group G. This conclusion is readily backed up by a moment's thought on the statement of Prop. 7.2 put in conjunction with Prop. 7.3 i 7.4. The only generalisation of the construction leading up to Def. 7.1 that is dictated by the necessity of taking into account a generic structure of a field *bundle* over the spacetime Σ is the replacement of the manifold M in that definition (corresponding to the*trivial* bundle $\Sigma \times M \longrightarrow \Sigma$) by an arbitrary fibre bundle over the base Σ of the principal bundle P_{G}. Below, we

[3]There does *not* exist a fully universal scheme of transcription of the original field theory $\mathscr{F}$ to a new form with the global symmetry gauged. There do exist, though, certain standard gauging schemes of limited applicability, among which the most commonly employed is the so-called minimal scheme, which we discuss shortly.

give analogons of definitions, propositions and theorems of Lecture 7. that accomodate this physically motivated generalisation. A careful Reader of the hitherto lectures will easily find in those lectures all the requisite formal tools for the verification of the meaningfulness and consistency of the definitions and for proving the propositions and theorems, which we, therefore, leave to Her or Him as an exercise.

We begin with

Definition 3. Let $(\mathsf{P}_{\mathrm{G}}, B, \mathrm{G}, \pi_{\mathsf{P}_{\mathrm{G}}})$ be a principal bundle with local trivialisations $\tau_i^{\mathsf{P}_{\mathrm{G}}} : \pi_{\mathsf{P}_{\mathrm{G}}}^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times \mathrm{G},\ i \in I$ associated with an open cover $\mathscr{O} = \{\mathcal{O}_i\}_{i\in I}$, and let (E, B, F, π_E) be a fibre bundle with local trivialisations $\tau_i^E : \pi_E^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times F,\ i \in I$, associated with the same cover $\mathscr{O}$, on which there exists a smooth (left) action $\Lambda_\cdot : \mathrm{G} \times E \longrightarrow E$ of a Lie group G that gives rise to a family of automorphisms $\{(\Lambda_g, \mathrm{id}_B)\}_{g\in\mathrm{G}}$ of the fibre bundle locally modelled on the automorphisms $\{\lambda_g\}_{g\in\mathrm{G}}$ of the typical fibre in the manner fixed by the commutative diagram

(1)
$$\begin{CD} \mathrm{G} \times \pi_E^{-1}(\mathcal{O}_i) @>{\Lambda_\cdot}>> \pi_E^{-1}(\mathcal{O}_i) \\ @V{\mathrm{id}_{\mathrm{G}} \times \tau_i^E}VV @VV{\tau_i^E}V \\ \mathrm{G} \times \mathcal{O}_i \times F @>>{(\mathrm{pr}_2, \lambda_\cdot \circ \mathrm{pr}_{1,3})}> \mathcal{O}_i \times F \end{CD} .$$

A **product bundle associated with** P_{G} **by** $\Lambda_\cdot$ is a fibre bundle

$$\big(\mathsf{P}_{\mathrm{G}}^{\Lambda} E, B, F, \pi_{\mathsf{P}_{\mathrm{G}}^{\Lambda} E}\big)$$

composed of

- the total space $\mathsf{P}_{\mathrm{G}}^{\Lambda} E \equiv (\mathsf{P}_{\mathrm{G}} \times_B E)/\mathrm{G}$ given by the quotient manifold determined by the action
$$\widetilde{\Lambda}_\cdot : \mathrm{G} \times (\mathsf{P}_{\mathrm{G}} \times_B E) \longrightarrow \mathsf{P}_{\mathrm{G}} \times_B E : \big(g, (p, \epsilon)\big) \longmapsto \big(r_{g^{-1}}(p), \Lambda_g(\epsilon)\big),$$
induced by $\Lambda_\cdot$ on the fibred product of the bundles P_{G} and E and by the defining action $r_\cdot$ of the structure group G on P_{G};
- the projection on the base
$$\pi_{\mathsf{P}_{\mathrm{G}}^{\Lambda} E} : \mathsf{P}_{\mathrm{G}}^{\Lambda} E \longrightarrow B : [(p, \epsilon)] \longmapsto \pi_{\mathsf{P}_{\mathrm{G}}}(p).$$

Here, the local trivialisations induced by the above trivialisations of the components take the form

$$[\tau_i] : \pi_{\mathsf{P}_{\mathrm{G}}^{\Lambda} E}^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times F : [(p, \epsilon)] \longmapsto \big(\pi_{\mathsf{P}_{\mathrm{G}}}(p), \lambda_{\mathrm{pr}_2 \circ \tau_i^{\mathsf{P}_{\mathrm{G}}}(p)}\big(\mathrm{pr}_2 \circ \tau_i^E(m)\big)\big),$$

and the ensuing transition maps are determined by the mapping

$$[\tau_i] \circ [\tau_j]^{-1} : \mathcal{O}_{ij} \times F \circlearrowleft : (x, f) \longmapsto \big(x, \lambda_{g_{ij}^{\mathsf{P}_{\mathrm{G}}}(x)} \circ g_{ij}^E(x)(f)\big)\big).$$

Physical applications of the above construction follow directly from

Proposition 1. Adopt the notation of Def. 3. The structure of a (Fréchet) group on the space $\Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}})$ of global sections of the adjoint bundle has a realisation on the space $\Gamma(\mathsf{P}_{\mathrm{G}}^{\Lambda} E)$ of global sections of the product associated bundle $\mathsf{P}_{\mathrm{G}}^{\Lambda} E$, the realisation being induced by the smooth map

$$[\Lambda]_\cdot^\times : \mathrm{Ad}\,\mathsf{P}_{\mathrm{G}} \times_B \mathsf{P}_{\mathrm{G}}^{\Lambda} E \longrightarrow \mathsf{P}_{\mathrm{G}}^{\Lambda} E$$

that satisfies (fibre-wise) axioms of a group action G on the manifold M and is locally modelled on $\Lambda_\cdot$.

We also have the following result, central to the construction of a local presentation of gauge symmetries and matter fields in the theory with the global symmetry G gauged.

Proposition 2. Adopt the notation of Def. 3 and denote the set of fibre-bundle morphisms $\mathsf{P}_{\rm G} \longrightarrow E$ covering the identity on the (common) base B as $\mathrm{Hom}_{\mathbf{Bun}(B)}(\mathsf{P}_{\rm G}, E\,|\,B)$. There exists a bijection

$$\Gamma(\mathsf{P}_{\rm G}^{\Lambda}E) \cong \mathrm{Hom}_{\rm G}(\mathsf{P}_{\rm G}, E) \cap \mathrm{Hom}_{\mathbf{Bun}(B)}(\mathsf{P}_{\rm G}, E\,|\,B)\big(=: \mathrm{Hom}_{\mathbf{Bun}(B)}(\mathsf{P}_{\rm G}, E\,|\,B)^{\rm G}\big)\,,$$

determined by the pair of mutually inverse mappings (written in terms of a pair $(\pi, \epsilon) \in \Gamma_{\rm loc}(\mathsf{P}_{\rm G}) \times \Gamma_{\rm loc}(E)$ composing a global section of $\mathsf{P}_{\rm G}^{\Lambda}E$):

$$\Phi_{\Lambda}^{\times}\ :\ \Gamma(\mathsf{P}_{\rm G}^{\Lambda}E) \longrightarrow \mathrm{Hom}_{\mathbf{Bun}(B)}(\mathsf{P}_{\rm G}, E\,|\,B)^{\rm G}\,,$$

$$\Phi_{\Lambda}^{\times}[(\pi,\epsilon)]\ :\ \mathsf{P}_{\rm G} \longrightarrow E\ :\ p \longmapsto \Lambda_{\phi_{\mathsf{P}_{\rm G}}(p,\pi\circ\pi_{\mathsf{P}_{\rm G}}(p))}\big(\epsilon\circ\pi_{\mathsf{P}_{\rm G}}(p)\big)$$

and

$$S_{\Lambda}^{\times}\ :\ \mathrm{Hom}_{\mathbf{Bun}(B)}(\mathsf{P}_{\rm G}, E\,|\,B)^{\rm G} \longrightarrow \Gamma_{\rm loc}(\mathsf{P}_{\rm G}^{\Lambda}E)\,,$$

$$S_{\Lambda}^{\times}[\Phi]\ :\ B \longrightarrow \mathsf{P}_{\rm G}^{\Lambda}E\ :\ (\mathcal{O}_i \ni)x \longmapsto \big[\big(\tau_i^{\mathsf{P}_{\rm G}\,-1}(x,e), \Phi\circ\tau_i^{\mathsf{P}_{\rm G}\,-1}(x,e)\big)\big].$$

as well as

Proposition 3. Adopt the notation of Props. 1 i 2. The bijection $\Phi_{\Lambda}^{\times}$ is (left) equivariant with respect to the following actions of the group $\Gamma(\mathrm{Ad}\,\mathsf{P}_{\rm G})$: the action

$$\begin{aligned} \Gamma[\Gamma[\widetilde{r}^{\times}]]_{\cdot}^{\Lambda}\quad &:\quad \Gamma(\mathrm{Ad}\,\mathsf{P}_{\rm G}) \times \Gamma(\mathsf{P}_{\rm G}^{\Lambda}E) \longrightarrow \Gamma(\mathsf{P}_{\rm G}^{\Lambda}E) \\ &:\quad \big(\sigma,[(\pi,\epsilon)]\big) \longmapsto \big[\big([r]_{\sigma\circ\pi_P\circ\pi(\cdot)}\circ\pi(\cdot),\epsilon(\cdot)\big)\big] \equiv \big[\big([r]_{\sigma(\cdot)}\circ\pi(\cdot),\mu(\cdot)\big)\big] \end{aligned} \tag{2}$$

on the space $\Gamma(\mathsf{P}_{\rm G}^{\Lambda}E)$ and the natural action

$$\begin{aligned} [\Phi_{\rm Ad}^{\times}\Lambda]\quad &:\quad \Gamma(\mathrm{Ad}\,\mathsf{P}_{\rm G}) \times \mathrm{Hom}_{\mathbf{Bun}(B)}(\mathsf{P}_{\rm G}, E\,|\,B)^{\rm G} \longrightarrow \mathrm{Hom}_{\mathbf{Bun}(B)}(\mathsf{P}_{\rm G}, E\,|\,B)^{\rm G} \\ &:\quad (\gamma,\Phi) \longmapsto \Lambda_{\Phi_{\rm Ad}[\gamma](\cdot)}\big(\Phi(\cdot)\big) \end{aligned}$$

on the space of G-equivariant maps $\mathrm{Hom}_{\rm G}(\mathsf{P}_{\rm G}, M)$, that is the action

$$\Phi_{\rm Ad}^{\times}\Lambda_{\cdot} \equiv [\Phi_{\rm Ad}^{\times}\Lambda]_{\cdot}\circ\big(\Phi_{\rm Ad}^{-1}\times\mathrm{id}_{\mathrm{Hom}_{\mathbf{Bun}(B)}(\mathsf{P}_{\rm G},E\,|\,B)^{\rm G}}\big)$$

of the group $\mathrm{Hom}_{\mathbf{Bun}(B)}(\mathsf{P}_{\rm G}, {\rm G}\,|\,B)^{\rm G}$ renders commutative the diagram

$$\begin{array}{ccc} \Gamma(\mathrm{Ad}\,\mathsf{P}_{\rm G}) \times \Gamma(\mathsf{P}_{\rm G}^{\Lambda}E) & \xrightarrow{\ \Gamma[\Gamma[\widetilde{r}^{\times}]]_{\cdot}^{\Lambda}\ } & \Gamma(\mathsf{P}_{\rm G}^{\Lambda}E) \\ \big\downarrow{\scriptstyle \Phi_{\rm Ad}\times\Phi_{\Lambda}^{\times}} & & \big\downarrow{\scriptstyle \Phi_{\Lambda}^{\times}} \\ \mathrm{Hom}_{\mathbf{Bun}(B)}(\mathsf{P}_{\rm G}, {\rm G}\,|\,B)^{\rm G} \times \mathrm{Hom}_{\mathbf{Bun}(B)}(\mathsf{P}_{\rm G}, E\,|\,B)^{\rm G} & \xrightarrow{\ \Phi_{\rm Ad}^{\times}\Lambda\ } & \mathrm{Hom}_{\mathbf{Bun}(B)}(\mathsf{P}_{\rm G}, E\,|\,B)^{\rm G} \end{array}\,.$$

Furthermore, we have the obvious yet useful

Proposition 4. Adopt the notation of Def. 3. Let $\mathcal{P}_{\rm G}^{\alpha} \equiv (\mathsf{P}_{\rm G}^{\alpha}, B, {\rm G}, \pi_{\mathsf{P}_{\rm G}^{\alpha}})$, $\alpha \in \{1,2\}$ be principal bundles with the structure group ${\rm G}$ over the common base B, and let (E, B, F, π_E) be fibre bundles with an action $\Lambda\ :\ {\rm G}\times E \longrightarrow E$ described *ibid.* An arbitrary principal-bundle (iso)morphism $(\Phi, \mathrm{id}_B, \mathrm{id}_{\rm G})\ :\ \mathcal{P}_{\rm G}^{1} \longrightarrow \mathcal{P}_{\rm G}^{2}$ icanonically induces a fibre-bundle (iso)morphism

$$\widetilde{\Phi}\ :\ \mathsf{P}_{\rm G}^{1\,\Lambda}E \longrightarrow \mathsf{P}_{\rm G}^{2\,\Lambda}E\ :\ [(p,\varepsilon)] \longmapsto \big[\big(\Phi(p),\varepsilon\big)\big]\,,$$

and the latter determines a bijection between the respective spaces of global sections

$$\Gamma\widetilde{\Phi}\ :\ \Gamma(\mathsf{P}_{\rm G}^{1\,\Lambda}E) \longrightarrow \Gamma(\mathsf{P}_{\rm G}^{2\,\Lambda}E)\ :\ \phi \longmapsto \widetilde{\Phi}\circ\phi\,.$$

Remark 1. With view to subsequent applications of the above proposition, it makes sense to take a closer look at the map $\Gamma\widetilde{\Phi}$ in the image of local trivialisations $\tau_i^\alpha : \pi_{\mathsf{P}^\alpha_{\mathrm{G}}}^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times \mathrm{G},\ \alpha \in \{1,2\}$ over an open cover $\{\mathcal{O}_i\}_{i\in I}$ of the base B trivialising for both principal bundles – here, the (iso)morphism Φ is represented by a family $\{h_i : \mathcal{O}_i \longrightarrow \mathrm{G}\}_{i\in I}$ of locally smooth maps that satisfy the conditions of Eqn. (6.3), *cp* Thm. 6.1. Consider a global section $\phi \in \Gamma(\mathsf{P}^{1\Lambda}_{\mathrm{G}} E)$ with restrictions

$$\phi\restriction_{\mathcal{O}_i} : \mathcal{O}_i \longrightarrow \mathsf{P}^{1\Lambda}_{\mathrm{G}} E : x \longmapsto \big[\big(\tau_i^{1\,-1}(x,e), \varphi_i(x)\big)\big]$$

given in terms of local sections $\varphi_i \in \Gamma(E\restriction_{\mathcal{O}_i})$ and obeying, at an arbitrary point $y \in \mathcal{O}_{ij}$, the gluing condition

$$\begin{aligned}\big[\big(\tau_i^{1\,-1}(y,e), \varphi_i(y)\big)\big] &= \big[\big(\tau_j^{1\,-1}(y,e), \varphi_j(y)\big)\big] = \big[\big(\tau_i^{1\,-1}(y,e) \triangleleft g^1_{ij}(y), \varphi_j(y)\big)\big] \\ &= \big[\big(\tau_i^{1\,-1}(y,e), g^1_{ij}(y) \triangleright \varphi_j(y)\big)\big]\,,\end{aligned}$$

so that

$$\forall_{y\in\mathcal{O}_{ij}} : \varphi_i(y) = g^1_{ij}(y) \triangleright \varphi_j(y)\,,$$

and the information on the existence of the global section $\phi \in \Gamma(\mathsf{P}^{1\Lambda}_{\mathrm{G}} E)$ is encoded in a family of local sections $\{\varphi_i\}_{i\in I}$. The section $\Gamma\widetilde{\Phi}[\phi]$ is locally represented by a family of smooth maps (being given by a superposition of manifestly smooth maps with a surjective submersion $\pi_{(\mathsf{P}^2_{\mathrm{G}} \times_B E)/\mathrm{G}})$

$$\Gamma\widetilde{\Phi}[\phi]_i : \mathcal{O}_i \longrightarrow \mathsf{P}^{2\Lambda}_{\mathrm{G}} E : x \longmapsto \big[\big(\tau_i^{2\,-1}(x,e), h_i(x) \triangleright \varphi_i(x)\big)\big]\,.$$

We readily check that the local sections of $\mathsf{P}^{2\Lambda}_{\mathrm{G}} E$ thus defined are restrictions of a global section

$$\Gamma\widetilde{\Phi}[\phi] \in \Gamma(\mathsf{P}^{2\Lambda}_{\mathrm{G}} E)\,, \qquad \Gamma\widetilde{\Phi}[\phi]\restriction_{\mathcal{O}_i} \equiv \Gamma\widetilde{\Phi}[\phi]_i\,.$$

Indeed, at points $y \in \mathcal{O}_{ij}$, we obtain the equality

$$\begin{aligned}\big[\big(\tau_j^{2\,-1}(y,e), h_j(y) \triangleright \varphi_j(y)\big)\big] &= \big[\big(\tau_i^{2\,-1} \circ \tau^2_{ij}(y,e), \big(h_j(y) \cdot g^1_{ji}(y)\big) \triangleright \varphi_i(y)\big)\big] \\ &= \big[\big(\tau_i^{2\,-1}(y,e) \triangleleft g^2_{ij}(y), \big(h_j(y) \cdot g^1_{ji}(y)\big) \triangleright \varphi_i(y)\big)\big] \\ &= \big[\big(\tau_i^{2\,-1}(y,e), \big(g^2_{ij}(y) \cdot h_j(y) \cdot g^1_{ji}(y)\big) \triangleright \varphi_i(y)\big)\big] \\ &= \big[\big(\tau_i^{2\,-1}(y,e), h_i(y) \triangleright \varphi_i(y)\big)\big]\,.\end{aligned}$$

We may now, at long last, return to the physical structures of immediate interest. It is straightforward to identify natural candidates to the rôle of the field bundle of the theory with the symmetry gauged and of the bundle of groups acting upon it.

Definition 4. Adopt the notation of Def. 2 and Prop. 1. Let $\mathrm{G} \subseteq \mathrm{Symm}(\mathscr{F} \nwarrow F)$ be a subgroup of the group of global symmetries of the field theory $\mathscr{F}$ endowed with the structure of a finite-dimensional Lie group and let $(\mathsf{P}_{\mathrm{G}}, \Sigma, \mathrm{G}, \pi_{\mathsf{P}_{\mathrm{G}}})$ be an arbitrary principal bundle with the structure group G over the spacetime Σ. The **field bundle of type F with the gauge symmetry of type P_{G}** is the fibre bundle

$$\big(\mathsf{P}^{\mathrm{ev}}_{\mathrm{G}} \mathcal{F}, \Sigma, F, \pi_{\mathsf{P}^{\mathrm{ev}}_{\mathrm{G}} \mathcal{F}}\big)$$

associated with P_{G} by the evaluation mapping

$$\mathrm{ev.} : \mathrm{G} \times \mathcal{F} \longrightarrow \mathcal{F} : (\alpha, \varphi) \longmapsto \alpha(\varphi) \equiv \mathrm{ev}_\alpha(\varphi)\,.$$

Its global sections are termd (**matter**) **fields of type F with the gauge symmetry of type** P_{G}. In this context, the (Fréchet) group $\Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}}) \cong \mathrm{Aut}_{\mathbf{GrpBun}_{\mathrm{G}}(\Sigma)}(\mathsf{P}_{\mathrm{G}}\,|\,\Sigma)$ is called the **gauge group of type** P_{G}. The maps

$$\Gamma\big[\Gamma[\widetilde{r}^{\times}]\big]^\Lambda_\chi : \Gamma(\mathsf{P}^{\mathrm{ev}}_{\mathrm{G}} \mathcal{F}) \circlearrowleft\,, \qquad \chi \in \Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}})$$

defined in Prop. 3 (*cp* Eqn. (2)) are called **gauge transformations**, and the bundle P_{G} acquires the name of the **gauge bundle**.

The transcription of the original field theory, constructed out of invariants of the action of the global-symmetry group $\mathrm{Symm}(\mathscr{F} \nwarrow F)$, in terms of global sections of bundles $\mathsf{P}_{\mathrm{G}}^{\Lambda}\mathcal{F}$ associated with a(n arbitrary) gauge bundle P_{G} in conformity with the postulate – largely imprecise, even heuristic – of *minimality* of the alterations of the structure of the action functional in the procedure of localisation (or gauging) of symmetries from the group $\mathrm{G} \subset \mathrm{Symm}(\mathscr{F} \nwarrow F)$, seems a highly non-obvious task, and the sections themselves – most cumbersome from the point of view of any potential formal manipulations. What comes to our rescue is the correspondence, detailed in Prop. 2, between the said sections and G-equivariant morphisms that map the gauge bundle P_{G} to the original field bundle $\mathcal{F}$ taken in conjunction with an elementary structural property of the gauge bundle, to wit, its local triviality, the latter being – in the light of Prop. 6.5 – *equivalent* to the existence of local sections. These afford a local imitation of the structure present in the original field-theoretic model *via*

Definition 5. Adopt the notation of Prop. 2 and let $\sigma_* : \mathcal{O} \longrightarrow \mathsf{P}_{\mathrm{G}}$ be an arbitrary (local) section of the principal bundle P_{G} over an open set $\mathcal{O} \subset B$. A **local presentation of the field (of type F with the gauge symmetry of type P_{G}) $\phi \in \Gamma(\mathsf{P}_{\mathrm{G}}^{\Lambda}\mathcal{F})$ in the gauge σ_*** is the local section

$$\phi_{\sigma_*} := \Phi_{\Lambda}^{\times}[\phi] \circ \sigma_* \in \Gamma_{\mathrm{loc}}(\mathcal{F}) .$$

Similarly, a **local presentation of the gauge transformation $\gamma \in \Gamma(\mathrm{Ad}\mathsf{P}_{\mathrm{G}})$ in the gauge σ_*** is the (locally) smooth map

$$\gamma_{\sigma_*} := \Phi_{\mathrm{Ad}}[\gamma] \circ \sigma_* \;:\; \mathcal{O} \longrightarrow \mathrm{G} .$$

In the present context, the choice of the section σ_* acquires the name of (the **choice of**) the **local gauge**.

Remark 2. That the map $\phi_{\sigma_*} : \mathcal{O} \longrightarrow \mathcal{F}$ is, indeed, a local section of the field bundle $\mathcal{F}$ over $\mathcal{O}$ is implied directly by the commutativity of the diagram

$$\begin{array}{ccc} \mathsf{P}_{\mathrm{G}} & \xrightarrow{\;\Phi_{\Lambda}^{\times}[\phi]\;} & \mathcal{F} \\ {\scriptstyle \pi_{\mathsf{P}_{\mathrm{G}}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{F}}} \\ \Sigma & \underset{\mathrm{id}_{\Sigma}}{=\!=\!=\!=} & \Sigma \end{array}$$

as the latter gives us

$$\pi_{\mathcal{F}} \circ \phi_{\sigma_*} \equiv \big(\pi_{\mathcal{F}} \circ \Phi_{\Lambda}^{\times}[\phi]\big) \circ \sigma_* = \pi_{\mathsf{P}_{\mathrm{G}}} \circ \sigma_* = \mathrm{id}_{\mathcal{O}} .$$

Proposition 5. In the notation of Def. 5 and Prop. 3, and for

$$^{\gamma}\phi := \Gamma[\Gamma[\widetilde{r}^{\times}]]_{\gamma}^{\Lambda}(\phi) , \qquad {}^{\mathrm{Inv}\circ\gamma}\sigma_* := r_{\mathrm{Inv}\circ\Phi_{\mathrm{Ad}}[\gamma]\circ\sigma_*(\cdot)}\big(\sigma_*(\cdot)\big) ,$$

the following identities hold true

$$\big({}^{\gamma}\phi\big)_{\sigma_*}(\cdot) = \Lambda_{\gamma_{\sigma_*}(\cdot)}\phi_{\sigma_*}(\cdot) = \phi_{{}^{\mathrm{Inv}\circ\gamma}\sigma_*}(\cdot) .$$

<u>*Proof:*</u> In the light of Prop. 3 and of the G-equivariance of $\Phi_{\Lambda}^{\times}[\phi]$, we obtain the following sequence of equalities:

$$\begin{aligned} \big({}^{\gamma}\phi\big)_{\sigma_*}(\cdot) &\equiv \Phi_{\Lambda}^{\times}\big[{}^{\gamma}\phi\big] \circ \sigma_*(\cdot) = \Phi_{\mathrm{Ad}}^{\times}\Lambda_{\Phi_{\mathrm{Ad}}[\gamma]}\big(\Phi_{\Lambda}^{\times}[\phi]\big) \circ \sigma_*(\cdot) \\ &\equiv \Lambda_{\Phi_{\mathrm{Ad}}[\gamma]\circ\sigma_*(\cdot)}\big(\Phi_{\Lambda}^{\times}[\phi] \circ \sigma_*(\cdot)\big) \equiv \Lambda_{\gamma_{\sigma_*}(\cdot)}\phi_{\sigma_*}(\cdot) \end{aligned}$$

and

$$\Lambda_{\Phi_{\mathrm{Ad}}[\gamma]\circ\sigma_*(\cdot)}\big(\Phi_{\Lambda}^{\times}[\phi] \circ \sigma_*\big)(\cdot) = \Phi_{\Lambda}^{\times}[\phi] \circ \Big(r_{\mathrm{Inv}\circ\Phi_{\mathrm{Ad}}[\gamma]\circ\sigma_*(\cdot)}\big(\sigma_*(\cdot)\big)\Big) \equiv \Phi_{\Lambda}^{\times}[\phi] \circ {}^{\mathrm{Inv}\circ\gamma}\sigma_*(\cdot) \equiv \phi_{{}^{\mathrm{Inv}\circ\gamma}\sigma_*}(\cdot) .$$

□

Remark 3. The above proposition demonstrates convincingly the meaning of the name given to the reference section σ_*. Indeed, an arbitrary change of that choice,

$$\sigma_* \longmapsto {}^{\mathrm{Inv}\circ\gamma}\sigma_* \,,$$

effects a gauge transformation of the physical field,

$$\phi \longmapsto {}^{\gamma}\phi \,.$$

The hitherto discussion does not resolve the natural (and interrelated) issues: Which of the (non-isomorphic) gauge bundles should one *choose* for the gauging of a given global symmetry in a field theory? What exactly is quantified by the potential freedom of choice of the gauge bundle? We shall return to these questions shortly.