CLASSICAL FIELD THEORY IN THE TIME OF COVID-19
7. LECTURE BATCH

Associated bundles

The detailed discussion of the structure of principal bundles presented in the previous lecture prepared us for the study of geometric objects of prime relevance to the modelling of the procedure of rendering global symmetries of a mechanical resp. field-theoretic system local. These, we introduce in

Definition 1. Let $(\mathsf{P}_{\rm G}, B, {\rm G}, \pi_{\mathsf{P}_{\rm G}})$ be a principal bundle, and M – a manifold with a smooth (left) action $\lambda : {\rm G} \times M \longrightarrow M$ of the Lie group ${\rm G}$. A **bundle associated with** $\mathsf{P}_{\rm G}$ **by** λ is a fibre bundle

$$\big(\mathsf{P}_{\rm G} \times_\lambda M, B, M, \pi_{\mathsf{P}_{\rm G} \times_\lambda M}\big)$$

composed of

- the total space $\mathsf{P}_{\rm G} \times_\lambda M \equiv (\mathsf{P}_{\rm G} \times M)/{\rm G}$ given by the quotient manifold determined – along the lines of Cor. 6.2 (and based on Thm. 5.2) and in the notation introduced *ibid.* – by the action of Eq. (6.2);
- the projection to the base

$$\pi_{\mathsf{P}_{\rm G} \times_\lambda M} : \mathsf{P}_{\rm G} \times_\lambda M \longrightarrow B : [(p, m)] \longmapsto \pi_{\mathsf{P}_{\rm G}}(p) .$$

Here, local trivialisations $\tau_i : \pi_{\mathsf{P}_{\rm G}}^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times {\rm G}, \ i \in I$ of the principal bundle $\mathsf{P}_{\rm G}$ associated with the open cover $\mathscr{O} = \{\mathcal{O}_i\}_{i \in I}$ of the base B induce local trivialisations

$$\widetilde{\tau}_i : \pi_{\mathsf{P}_{\rm G} \times_\lambda M}^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times M : [(p, m)] \longmapsto \big(\pi_{\mathsf{P}_{\rm G}}(p), \lambda_{{\rm pr}_2 \circ \tau_i(p)}(m)\big) ,$$

with the ensuing transition maps

$$\widetilde{\tau}_i \circ \widetilde{\tau}_j^{-1} : \mathcal{O}_{ij} \times M \circlearrowleft : (x, m) \longmapsto \big(x, \lambda_{g_{ij}(x)}(m)\big) .$$

Upon fixing (arbitrarily) a point $x \in B$, we choose (also arbitrarily) $p_* \in (\mathsf{P}_{\rm G})_x$. Diffeomorphisms

$$[p_*]_\lambda : M \xrightarrow{\cong} (\mathsf{P}_{\rm G} \times_\lambda M)_x : m \longmapsto [(p_*, m)] ,$$

with inverses

$$[p_*]_\lambda^{-1} : (\mathsf{P}_{\rm G} \times_\lambda M)_x \xrightarrow{\cong} M : [(p, m)] \longmapsto \lambda_{\phi_{\mathsf{P}_{\rm G}}(p_*, p)}(m)$$

and the obvious property

$$\forall_{g \in {\rm G}} : [p_* \triangleleft g]_\lambda = [p_*]_\lambda \circ \lambda_g , \tag{1}$$

are called **fibre-modelling isomorphisms**. These induce **fibre-transport isomorphisms**

$$\begin{aligned} [p_2, p_1]_\lambda \equiv [p_2]_\lambda \circ [p_1]_\lambda^{-1} \quad &: \quad (\mathsf{P}_{\rm G} \times_\lambda M)_{\pi_{\mathsf{P}_{\rm G}}(p_1)} \xrightarrow{\cong} (\mathsf{P}_{\rm G} \times_\lambda M)_{\pi_{\mathsf{P}_{\rm G}}(p_2)} \\ &: \quad [(p, m)] \longmapsto \big[\big(p_2, \lambda_{\phi_{\mathsf{P}_{\rm G}}(p_1, p)}(m)\big)\big] , \end{aligned}$$

defined for all pairs $(p_1, p_2) \in \mathsf{P}_{\rm G}$.

For any pair $(\mathsf{P}_{\rm G} \times_{\lambda_\alpha} M_\alpha, B, M_\alpha, \pi_{\mathsf{P}_{\rm G} \times_{\lambda_\alpha} M_\alpha}), \ \alpha \in \{1, 2\}$ of bundles associated with the same principal bundle $(\mathsf{P}_{\rm G}, B, {\rm G}, \pi_{\mathsf{P}_{\rm G}})$, we define also the **associated-bundle invariant** as the fibre-bundle morphism

$$(\Phi, {\rm id}_B) : \mathsf{P}_{\rm G} \times_{\lambda_1} M_1 \longrightarrow \mathsf{P}_{\rm G} \times_{\lambda_2} M_2$$

with the fundamental property expressed by the commutative diagram (written for any $p_1, p_2 \in \mathsf{P}_{\rm G}$)

$$\begin{array}{ccc} \big(\mathsf{P}_{\rm G} \times_{\lambda_1} M_1\big)_{\pi_{\mathsf{P}_{\rm G}}(p_1)} & \xrightarrow{[p_2,p_1]_{\lambda_1}} & \big(\mathsf{P}_{\rm G} \times_{\lambda_1} M_1\big)_{\pi_{\mathsf{P}_{\rm G}}(p_2)} \\ {\scriptstyle \Phi\upharpoonright_{(\mathsf{P}_{\rm G}\times_{\lambda_1} M_1)_{\pi_{\mathsf{P}_{\rm G}}(p_1)}}} \downarrow & & \downarrow {\scriptstyle \Phi\upharpoonright_{(\mathsf{P}_{\rm G}\times_{\lambda_1} M_1)_{\pi_{\mathsf{P}_{\rm G}}(p_2)}}} \\ \big(\mathsf{P}_{\rm G} \times_{\lambda_2} M_2\big)_{\pi_{\mathsf{P}_{\rm G}}(p_1)} & \xrightarrow[{[p_2,p_1]_{\lambda_2}}]{} & \big(\mathsf{P}_{\rm G} \times_{\lambda_2} M_2\big)_{\pi_{\mathsf{P}_{\rm G}}(p_2)} \end{array} .$$

Remark 1. Existence of the structure of a smooth manifold on the space of orbits $\mathsf{P}_{\rm G} \times_\lambda M$ of the action $\widetilde{\lambda}$ is a direct consequence of Thm. 5.2, which can be invoked in the present context in virtue of Cor. 6.2. Smoothness of the projection to the base $\pi_{\mathsf{P}_{\rm G}\times_\lambda M}$ is readily inferred from Prop. Niezb-10, once we note that the projection closes the commutative diagram

$$\begin{array}{ccc} & & B \\ & \nearrow^{\pi_{\mathsf{P}_{\rm G}}\circ{\rm pr}_1} & \uparrow {\scriptstyle \pi_{\mathsf{P}_{\rm G}\times_\lambda M}} \\ \mathsf{P}_{\rm G} \times M & \xrightarrow[\pi_{(\mathsf{P}_{\rm G}\times M)/{\rm G}}]{} & \mathsf{P}_{\rm G} \times_\lambda M \end{array} ,$$

in which $\pi_{(\mathsf{P}_{\rm G}\times M)/{\rm G}}$ is a surjective submersion (by the very same Thm. 5.2), and $\pi_{\mathsf{P}_{\rm G}} \circ {\rm pr}_1$ is manifestly smooth. JAs the latter map is also submersive, this property is inherited by $\pi_{\mathsf{P}_{\rm G}\times_\lambda M}$, a fact that can be demonstrated directly by applying the tangent functor T to the above diagram.

We shall, next, examine the local trivialisations, beginning with a check of their well-definedness. For that, we must verify that the value taken by the map $\widetilde{\tau}_i$ on the class $[(p,m)]$ does not depend on the choice of the representative thereof. Thus, we compute

$$\big(\pi_{\mathsf{P}_{\rm G}}(p \triangleleft g), \lambda\big({\rm pr}_2 \circ \tau_i(p \triangleleft g), \lambda\big(g^{-1}, m\big)\big)\big) = \big(\pi_{\mathsf{P}_{\rm G}}(p), \lambda\big({\rm pr}_2 \circ \tau_i(p) \cdot g, \lambda\big(g^{-1}, m\big)\big)\big)$$

$$= \big(\pi_{\mathsf{P}_{\rm G}}(p), \lambda\big({\rm pr}_2 \circ \tau_i(p) \cdot g \cdot g^{-1}, m\big)\big) = \big(\pi_{\mathsf{P}_{\rm G}}(p), \lambda\big({\rm pr}_2 \circ \tau_i(p), m\big)\big) .$$

Furthermore, since maps

$$\underline{\tau}_i \ : \ \pi_{\mathsf{P}_{\rm G}}^{-1}(\mathcal{O}_i) \times M \longrightarrow \mathcal{O}_i \times M \ : \ (p,m) \longmapsto \big(\pi_{\mathsf{P}_{\rm G}}(p), \lambda_{{\rm pr}_2\circ\tau_i(p)}(m)\big) , \quad i \in \{1,2\}$$

are manifestly smooth, and related to $\widetilde{\tau}_i$ through the commutative diagram

$$\begin{array}{ccc} & & \mathcal{O}_i \times M \\ & \nearrow^{\underline{\tau}_i} & \uparrow {\scriptstyle \widetilde{\tau}_i} \\ \pi_{\mathsf{P}_{\rm G}}^{-1}(\mathcal{O}_i) \times M & \xrightarrow[\pi_{(\mathsf{P}_{\rm G}\times M)/{\rm G}}]{} & \pi_{\mathsf{P}_{\rm G}\times_\lambda M}^{-1}(\mathcal{O}_i) \end{array} ,$$

in which the canonical projection $\pi_{(\mathsf{P}_{\rm G}\times M)/{\rm G}}$ is smooth by Thm. 5.2 and Cor. 6.2, we conclude that also the maps $\widetilde{\tau}_i$ are smooth in virtue of Prop. Niezb-10. There is no doubt about smoothness (also local) of their inverses

$$\widetilde{\tau}_i^{-1} \ : \ \mathcal{O}_i \times M \longrightarrow \pi_{\mathsf{P}_{\rm G}\times_\lambda M}^{-1}(\mathcal{O}_i) \ : \ (x,m) \longmapsto \big[\big(\tau_i^{-1}(x,e), m\big)\big] .$$

In all the hitherto considerations, we have *implicitly* assumed well-definedness of the definition of the maps $\widetilde{\tau}_i$ and $\widetilde{\tau}_i^{-1}$, and that calls for a separate verification – the latter justifies *a posteriori* our identification of the typical fibre

$$\pi_{\mathsf{P}_{\rm G}\times_\lambda M}^{-1}\big(\big\{\pi_{\mathsf{P}_{\rm G}\times_\lambda M}([(p,m)])\big\}\big) \cong M , \quad [(p,m)] \in \mathsf{P}_{\rm G} \times_\lambda M$$

of the fibre bundle under reconstruction. We readily demonstrate the desired properties: Thus, for any $(x,m) \in \mathcal{O}_i \times M$, we have

$$\widetilde{\tau}_i \circ \widetilde{\tau}_i^{-1}(x,m) = \widetilde{\tau}_i\big(\big[\big(\tau_i^{-1}(x,e), m\big)\big]\big)$$

$$= \big(\pi_{\mathsf{P}_{\rm G}} \circ \tau_i^{-1}(x,e), \lambda\big(\mathrm{pr}_2 \circ \tau_i \circ \tau_i^{-1}(x,e), m\big)\big) = \big(x, \lambda(e,m)\big) = (x,m)\,,$$

and for $[(p,m)] \in \mathsf{P}_{\rm G} \times_\lambda M$, $p = \tau_i^{-1}(x,g)$, we obtain

$$\widetilde{\tau}_i^{-1} \circ \widetilde{\tau}_i\big([(p,m)]\big) = \widetilde{\tau}_i^{-1}\big(\pi_{\mathsf{P}_{\rm G}}(p), \lambda\big(\mathrm{pr}_2 \circ \tau_i(p), m\big)\big)$$

$$= \big[\big(\tau_i^{-1}\big(\pi_{\mathsf{P}_{\rm G}}(p), e\big), \lambda\big(\mathrm{pr}_2 \circ \tau_i(p), m\big)\big)\big] = \big[\big(\tau_i^{-1}(x,e), \lambda(g,m)\big)\big]$$

$$= \big[\big(\tau_i^{-1}(x,e) \triangleleft g, m\big)\big] = \big[\big(\tau_i^{-1}(x,g), m\big)\big] \equiv [(p,m)]\,.$$

Finally, we calculate

$$\begin{aligned} \widetilde{\tau}_i \circ \widetilde{\tau}_j^{-1}(x,m) &\equiv \widetilde{\tau}_i \circ \widetilde{\tau}_j^{-1}\big(\pi_{\mathsf{P}_{\rm G}} \circ \tau_j^{-1}(x,e), \lambda\big(\mathrm{pr}_2 \circ \tau_j(\tau_j^{-1}(x,e)), m\big)\big) \\ &= \widetilde{\tau}_i\big(\big[\tau_j^{-1}(x,e), m\big]\big) = \big(x, \lambda\big(\mathrm{pr}_2 \circ \tau_i \circ \tau_j^{-1}(x,e), m\big)\big) \\ &= \big(x, \lambda\big(\mathrm{pr}_2(x, g_{ij}(x)), m\big)\big) \equiv \big(x, \lambda\big(g_{ij}(x), m\big)\big)\,. \end{aligned}$$

The construction of the associated bundle is, therefore, well-defined.

Let us, next, consider the map

$$[p_*]_\lambda^{-1} \;:\; (\mathsf{P}_{\rm G} \times_\lambda M)_x \longrightarrow M \;:\; [(p,m)] \longmapsto \lambda_{\phi_{\mathsf{P}_{\rm G}}(p_*,p)}(m)\,, \quad p_* \in (\mathsf{P}_{\rm G})_x\,.$$

It is well-defined as for any representative $(\widetilde{p}, \widetilde{m}) \in [(p,m)]$ we get

$$\lambda_{\phi_{\mathsf{P}_{\rm G}}(p_*,\widetilde{p})}(\widetilde{m}) = \lambda_{\phi_{\mathsf{P}_{\rm G}}(p_*,p)} \circ \lambda_{\phi_{\mathsf{P}_{\rm G}}(p,\widetilde{p})}(\widetilde{m}) = \lambda_{\phi_{\mathsf{P}_{\rm G}}(p_*,p)}(m)\,.$$

Moreover, it is bijective because of the implication

$$[p_*]_\lambda^{-1}\big([(p_2,m_2)]\big) = [p_*]_\lambda^{-1}\big([(p_1,m_1)]\big) \qquad \Longleftrightarrow \qquad m_2 = \lambda_{\phi_{\mathsf{P}_{\rm G}}(p_2,p_1)}(m_1)$$

$$\Longrightarrow \qquad [(p_2,m_2)] = \big[\big(p_2, \lambda_{\phi_{\mathsf{P}_{\rm G}}(p_2,p_1)}(m_1)\big)\big] = \big[\big(p_2 \triangleleft \phi_{\mathsf{P}_{\rm G}}(p_2,p_1), m_1\big)\big]$$

$$= [(p_1,m_1)]\,,$$

showing injectivity of $[p_*]_\lambda^{-1}$, and any point $m \in M$ may be written as

$$m = [p_*]_\lambda^{-1}\big([(p_*,m)]\big)\,,$$

which testifies to the map's surjectivity, simultaneously indicating its inverse

$$[p_*]_\lambda \;:\; M \longrightarrow (\mathsf{P}_{\rm G} \times_\lambda M)_x \;:\; m \longmapsto [(p_*,m)]\,.$$

Indeed, the map $[p_*]_\lambda$ satisfies the identities

$$\begin{aligned} [p_*]_\lambda^{-1} \circ [p_*]_\lambda(m) &= \lambda_{\phi_{\mathsf{P}_{\rm G}}(p_*,p_*)}(m) = \lambda_e(m) = m\,, \\ [p_*]_\lambda \circ [p_*]_\lambda^{-1}\big([(p,m)]\big) &= \big[\big(p_*, \lambda_{\phi_{\mathsf{P}_{\rm G}}(p_*,p)}(m)\big)\big] \equiv \big[\big(p_* \triangleleft \phi_{\mathsf{P}_{\rm G}}(p_*,p), m\big)\big] \\ &= [(p,m)]\,. \end{aligned}$$

It is manifestly smooth as a superposition of the immersion $(p_*, \mathrm{id}_M) \;:\; M \longrightarrow \{p_*\} \times M \subset (\mathsf{P}_{\rm G})_{\pi_{\mathsf{P}_{\rm G}}(p_*)} \times M$ and the surjective submersion $\pi_{(\mathsf{P}_{\rm G} \times M)/{\rm G}} \;:\; \mathsf{P}_{\rm G} \times M \longrightarrow (\mathsf{P}_{\rm G} \times M)/{\rm G}$. Smoothness of $[p_*]_\lambda^{-1}$, on the other hand, follows from Prop. Niezb-10 referred to the commutative diagram

$$\begin{array}{ccc} & & M \\ & \nearrow^{\lambda(\phi_{\mathsf{P}_{\rm G}}(p_*,\mathrm{pr}_1),\mathrm{pr}_2)} & \uparrow{\scriptstyle [p_*]_\lambda^{-1}} \\ (\mathsf{P}_{\rm G})_x \times M & \xrightarrow[\pi_{(\mathsf{P}_{\rm G}\times{\rm G})/{\rm G}}\restriction_{(\mathsf{P}_{\rm G})_x \times M}]{} & (\mathsf{P}_{\rm G} \times_\lambda M)_x \end{array}\,,$$

with a surjective submersion on the horizontal edge. The construction of the diffeomorphism $[p_*]_\lambda^{-1}$ thus provides us with an independent proof of the identification of the typical fibre of the associated bundle advanced above.

Examples 1.

(1) A vector bundle $\mathbb{V}$ (of rank n) can be viewed/reconstructed as a bundle associated with the principal bundle of frames $\mathsf{F}_{\rm GL}\mathbb{V}$ by the defining action (evaluation),

$$\mathbb{V} \cong \mathsf{F}_{\rm GL}\mathbb{V} \times_{\rm ev} \mathbb{K}^{\times n}\,.$$

(2) The **adjoint bundle**

$$\big({\rm Ad}\,\mathsf{P}_{\rm G} \equiv \mathsf{P}_{\rm G} \times_{\rm Ad} {\rm G}, B, {\rm G}, \pi_{\mathsf{P}_{\rm G}\times_{\rm Ad}{\rm G}}\big)\,.$$

(3) A principal bundle $\mathsf{P}_{\rm G}$ can be realised as an associated bundle

$$\big(\mathsf{P}_{\rm G} \times_\ell {\rm G}, B, {\rm G}, \pi_{\mathsf{P}_{\rm G}\times_\ell{\rm G}}\big)\,,$$

where $\ell\ :\ {\rm G}\times{\rm G}\longrightarrow{\rm G}$ is the left regular action of ${\rm G}$ on itself. The relevant fibre-bundle isomorphism is given by

$$\widetilde\iota\ :\ \mathsf{P}_{\rm G}\times_\ell{\rm G}\longrightarrow\mathsf{P}_{\rm G}\ :\ [(p,g)]\longmapsto p\triangleleft g\,,$$

its smoothness following from the fact that it closes the commutative diagram

$$\begin{array}{ccc} & & B \\ & \overset{r}{\nearrow} & \Big\uparrow{\scriptstyle \pi_{\mathsf{P}_{\rm G}\times_\ell{\rm G}}} \\ \mathsf{P}_{\rm G}\times{\rm G} & \xrightarrow[\pi_{(\mathsf{P}_{\rm G}\times{\rm G})/{\rm G}}]{} & \mathsf{P}_{\rm G}\times_\ell{\rm G} \end{array}\,,$$

in which $\pi_{(\mathsf{P}_{\rm G}\times{\rm G})/{\rm G}}$ is a surjective submersion, and r – a smooth map. The inverse $\widetilde\iota$ is given, in a manifestly smooth form, by

$$\widetilde\iota^{-1}\ :\ \mathsf{P}_{\rm G}\longrightarrow\mathsf{P}_{\rm G}\times_\ell{\rm G}\ :\ p\longmapsto[(p,e)]\,.$$

On the associated bundle $\mathsf{P}_{\rm G}\times_\ell{\rm G}$, we find the right action of ${\rm G}$ given by

$$\widetilde r\ :\ (\mathsf{P}_{\rm G}\times_\ell{\rm G})\times{\rm G}\longrightarrow\mathsf{P}_{\rm G}\times_\ell{\rm G}\ :\ \big([(p,g)],h\big)\longmapsto[(p,g\cdot h)]\,.$$

Relative to it, each fibre is a torsor. The isomorphism $\widetilde\iota$ is G-equivariant,

$$\widetilde\iota\circ\widetilde r\big([(p,g)],h\big)=\widetilde\iota\big([(p,g\cdot h)]\big)=p\triangleleft(g\cdot h)=(p\triangleleft g)\triangleleft h=r\circ\widetilde\iota\big([(p,g)],h\big)\,,$$

and so we do, indeed, have a principal-bundle isomorphism.

In a search for automorphisms of the associated bundle $\mathsf{P}_{\rm G}\times_\ell{\rm G}$, we note that due to mutual commutativity of the left $\ell.$ and right $\wp.\ :\ {\rm G}\times{\rm G}\longrightarrow{\rm G}\ :\ (g,h)\longmapsto g\cdot h$ regular actions the latter induces – in virtue of Prop. 1, and for any $g\in{\rm G}$ – an associated-bundle invariant

$$\Phi[r_g]\ :\ \mathsf{P}_{\rm G}\times_\ell{\rm G}\circlearrowleft\ :\ [(p,h)]\longmapsto\Phi[r_g]^{\pi_{\mathsf{P}_{\rm G}}(p)}\big([(p,h)]\big)\,,$$

with

$$\begin{aligned}\Phi[r_g]^{\pi_{\mathsf{P}_{\rm G}}(p)}\big([(p,h)]\big) &= [p]_{\mathsf{P}_{\rm G}\times_\ell{\rm G}}\circ r_g\circ[p]^{-1}_{\mathsf{P}_{\rm G}\times_\ell{\rm G}}\big([(p,h)]\big)\\ &= [p]_{\mathsf{P}_{\rm G}\times_\ell{\rm G}}\circ r_g\circ\ell_{\phi_{\mathsf{P}_{\rm G}}(p,p)}(h)=[p]_{\mathsf{P}_{\rm G}\times_\ell{\rm G}}\circ r_g(h)\\ &= [p]_{\mathsf{P}_{\rm G}\times_\ell{\rm G}}(h\cdot g)=[(p,h\cdot g)]\equiv\widetilde r_g\big([(p,h)]\big)\,,\end{aligned}$$

whence

$$\Phi[r_g]\equiv\widetilde r_g\,,$$

and since

$$[(p,h)]=[(p\triangleleft h,e)]\equiv\widetilde\iota^{-1}(p\triangleleft h)$$

and

$$[(p, h\cdot g)] = [(p \triangleleft h\cdot g, e)] = \big[\big((p\triangleleft h)\triangleleft g, e\big)\big] = \big[\big(r_g(p\triangleleft h), e\big)\big] \equiv \widetilde{\iota}^{-1}\circ r_g(p\triangleleft h)\,,$$

we obtain

$$\widetilde{\iota}\circ\Phi[r_g]\circ\widetilde{\iota}^{-1} = r_g\,.$$

It is in this sense that automorphisms $\Phi[r_g]$ are induced by $r_\cdot$, and the latter can be regarded as a model associated-bundle invariant.

The practical (*e.g.*, physical) purpose of the construction of the associated bundle is to obtain a smooth distribution of manifolds of a predetermined (iso)type M over a give base B (*e.g.*, a spacetime), endowed with a distinguished action of a fixed Lie group G (*e.g.*, of symmetries of a physical theory), the latter being local over the base. In other words, it is to obtain a manifold locally modelled on $\mathcal{O}\times M$, $\mathcal{O}\subset B$ with an action of G locally modelled on λ. That the goal thus defined has been attained is demonstrated convincingly in the following two propositions.

Proposition 1. Bundles associated with the given principal bundle $(\mathsf{P}_{\mathrm{G}}, B, \mathrm{G}, \pi_{\mathsf{P}_{\mathrm{G}}})$ together with the attendant associated-bundle invariants form the **category of bundles associated with the principal bundle** P_{G}, to be denoted as

$$\mathbf{AssBun}(\mathsf{P}_{\mathrm{G}})\,.$$

The latter category is canonically equivalent to the category $\mathbf{Man}_{\mathrm{G}}$ of manifolds with a left action of G with G-equivariant maps as morphisms.

Proof: The first part of the statement is merely an indication of the class of morphisms to be considered, and as such, it does not require a separate proof (associated-bundle invariants can be superposed, and the identity map is – of course – an associated-bundle invariant). Also the one-to-one correspondence between objects of the category $\mathbf{AssBun}(\mathsf{P}_{\mathrm{G}})$ and G-manifolds is obvious. Thus, the only thing that needs to be checked is the relevant bijective correspondence between associated-bundle invariants and G-equivariant maps.

Let $(\Phi, \mathrm{id}_B) : \mathsf{P}_{\mathrm{G}}\times_{\lambda_1} M_1 \longrightarrow \mathsf{P}_{\mathrm{G}}\times_{\lambda_2} M_2$ be an associated-bundle invariant. We may define – for some (arbitrary) point $p\in\mathsf{P}_{\mathrm{G}}$ – a map (manifestly smooth)

$$\chi[\Phi] \coloneqq [p]_{\lambda_2}^{-1}\circ\Phi\circ[p]_{\lambda_1} \;:\; M_1 \xrightarrow{\cong} (\mathsf{P}_{\mathrm{G}}\times_{\lambda_1} M_1)_{\pi_{\mathsf{P}_{\mathrm{G}}}(p)} \longrightarrow (\mathsf{P}_{\mathrm{G}}\times_{\lambda_2} M_2)_{\pi_{\mathsf{P}_{\mathrm{G}}}(p)} \xrightarrow{\cong} M_2\,,$$

which, owing to the defining property of Φ,

$$\Phi\circ[p_2]_{\lambda_1}\circ[p_1]_{\lambda_1}^{-1} = [p_2]_{\lambda_2}\circ[p_1]_{\lambda_2}^{-1}\circ\Phi\,,$$

does not depend on the choice of the point p used in its definition. G-equivariance of the thus determined maps

$$\chi[\Phi]\in\mathrm{Hom}_{\mathrm{G}}(M_1, M_2)$$

follows from a direct computation, invoking Eq. (1) and carried out for arbitrary $(p,g)\in\mathsf{P}_{\mathrm{G}}\times\mathrm{G}$,

$$\begin{aligned}\chi[\Phi]\circ\lambda_{1\,g} &\equiv [p]_{\lambda_2}^{-1}\circ\Phi\circ\big([p]_{\lambda_1}\circ\lambda_{1\,g}\big) = [p]_{\lambda_2}^{-1}\circ\Phi\circ[p\triangleleft g]_{\lambda_1} \equiv \big([p\triangleleft g]_{\lambda_2}\circ\lambda_{2\,g^{-1}}\big)^{-1}\circ\Phi\circ[p\triangleleft g]_{\lambda_1}\\ &= \lambda_{2\,g}\circ[p\triangleleft g]_{\lambda_2}^{-1}\circ\Phi\circ[p\triangleleft g]_{\lambda_1} = \lambda_{2\,g}\circ[p]_{\lambda_2}^{-1}\circ\Phi\circ[p]_{\lambda_1} \equiv \lambda_{2\,g}\circ\chi[\Phi]\,.\end{aligned}$$

Conversely, to every map $\chi\in\mathrm{Hom}_{\mathrm{G}}(M_1, M_2)$, we may associate a (smooth) map

$$\begin{aligned}\Phi[\chi]^{\pi_{\mathsf{P}_{\mathrm{G}}}(p)} \coloneqq [p]_{\lambda_2}\circ\chi\circ[p]_{\lambda_1}^{-1} \quad &: \quad (\mathsf{P}_{\mathrm{G}}\times_{\lambda_1} M_1)_{\pi_{\mathsf{P}_{\mathrm{G}}}(p)} \longrightarrow (\mathsf{P}_{\mathrm{G}}\times_{\lambda_1} M_2)_{\pi_{\mathsf{P}_{\mathrm{G}}}(p)}\\ &: \quad [(p,m)] \longmapsto \big[\big(p, \chi(m)\big)\big]\,,\end{aligned}$$

depending on $p\in\mathsf{P}_{\mathrm{G}}$ exclusively through its projection to the base B,

$$\begin{aligned}\Phi[\chi]^{\pi_{\mathsf{P}_{\mathrm{G}}}(p\triangleleft g)} &= [p\triangleleft g]_{\lambda_2}\circ\chi\circ[p\triangleleft g]_{\lambda_1}^{-1} = [p]_{\lambda_2}\circ\big(\lambda_{2\,g}\circ\chi\circ\lambda_{1\,g^{-1}}\big)\circ[p]_{\lambda_1}^{-1}\\ &= [p]_{\lambda_2}\circ\chi\circ\big(\lambda_{1\,g}\circ\lambda_{1\,g^{-1}}\big)\circ[p]_{\lambda_1}^{-1} = [p]_{\lambda_2}\circ\chi\circ[p]_{\lambda_1}^{-1} \equiv \Phi[\chi]^{\pi_{\mathsf{P}_{\mathrm{G}}}(p)}\,,\end{aligned}$$

and hence defining an associated-bundle invariant given by the formula

$$\Phi[\chi] \;:\; \mathsf{P}_{\mathrm{G}} \times_{\lambda_1} M_1 \longrightarrow \mathsf{P}_{\mathrm{G}} \times_{\lambda_2} M_2 \;:\; [(p,m)] \longmapsto \Phi[\chi]^{\pi_{\mathsf{P}_{\mathrm{G}}}(p)}\big([(p,m)]\big).$$

Indeed, we calculate

$$\begin{aligned}\Phi[\chi] \circ [p_2,p_1]_{\lambda_1} &\equiv \big([p_2]_{\lambda_2} \circ \chi \circ [p_2]_{\lambda_1}^{-1}\big) \circ \big([p_2]_{\lambda_1} \circ [p_1]_{\lambda_1}^{-1}\big) = [p_2]_{\lambda_2} \circ \chi \circ [p_1]_{\lambda_1}^{-1} \\ &= \big([p_2]_{\lambda_2} \circ [p_1]_{\lambda_2}^{-1}\big) \circ \big([p_1]_{\lambda_2} \circ \chi \circ [p_1]_{\lambda_1}^{-1}\big) \equiv [p_2,p_1]_{\lambda_2} \circ \Phi[\chi].\end{aligned}$$

The two assignments given above:

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{AssBun}(\mathsf{P}_{\mathrm{G}})}(\mathsf{P}_{\mathrm{G}} \times_{\lambda_1} M_1, \mathsf{P}_{\mathrm{G}} \times_{\lambda_2} M_2) \longrightarrow \mathrm{Hom}_{\mathrm{G}}(M_1,M_2) \\ &: \quad (\Phi, \mathrm{id}_B) \longmapsto \chi[\Phi]\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{G}}(M_1,M_2) \longrightarrow \mathrm{Hom}_{\mathbf{AssBun}(\mathsf{P}_{\mathrm{G}})}(\mathsf{P}_{\mathrm{G}} \times_{\lambda_1} M_1, \mathsf{P}_{\mathrm{G}} \times_{\lambda_2} M_2) \\ &: \quad \chi \longmapsto \big(\Phi[\chi], \mathrm{id}_B\big)\end{aligned}$$

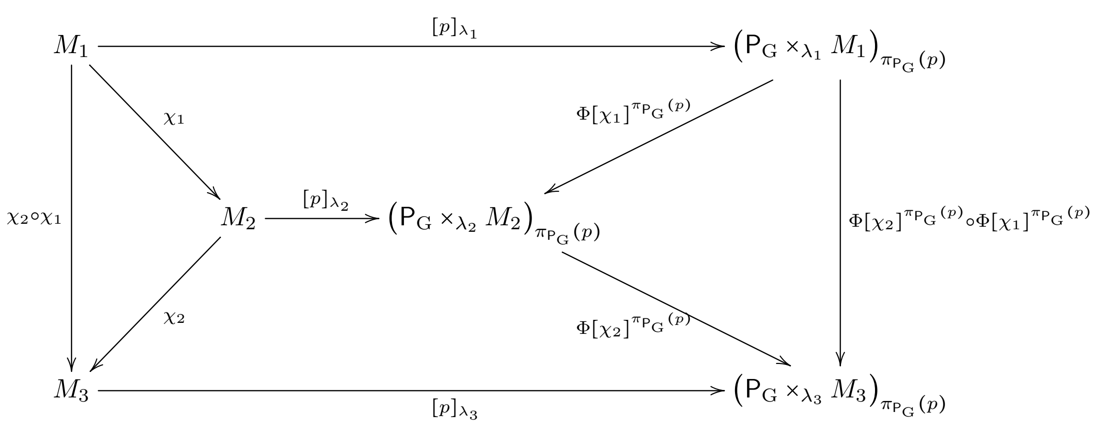
are mutually inverse, and each of them is functorial. Indeed, given a manifold M with an action $\lambda \;:\; \mathrm{G} \times M \longrightarrow M$, we obtain, over an arbitrary point $p \in \mathsf{P}_{\mathrm{G}}$, the equality

$$\Phi[\mathrm{id}_M]^{\pi_{\mathsf{P}_{\mathrm{G}}}(p)} = [p]_{\lambda_2} \circ \mathrm{id}_B \circ [p]_{\lambda_1}^{-1} = [p]_{\lambda_2} \circ [p]_{\lambda_1}^{-1} = \mathrm{id}_{(\mathsf{P}_{\mathrm{G}} \times_\lambda M)_{\pi_{\mathsf{P}_{\mathrm{G}}}(p)}},$$

and so also

$$\Phi[\mathrm{id}_M] = \mathrm{id}_{\mathsf{P}_{\mathrm{G}} \times_\lambda M}.$$

Furthermore, for any pair of G-equivariant maps $\chi_\alpha \;:\; M_\alpha \longrightarrow M_{\alpha+1},\ \alpha \in \{1,2\}$ between G-manifolds $M_\beta,\ \beta \in \{1,2,3\}$ with the respective actions $\lambda_\beta \;:\; \mathrm{G} \times M_\beta \longrightarrow M_\beta$, we arrive at the commutative diagram (for an arbitrary point $p \in \mathsf{P}_{\mathrm{G}}$)



in which commutativity of the upper (resp. lower) trapeze expresses the definition of the invariant $\Phi[\chi_1]$ (resp. $\Phi[\chi_2]$), and commutativity of the left and right triangles encodes definitions of the respective superpositions of maps, and in which the identity (by definition) between the rightmost edge and the map $\Phi[\chi_2 \circ \chi_1]^{\pi_{\mathsf{P}_{\mathrm{G}}}(p)}$ implies, in keeping with our expectations,

$$\Phi[\chi_2 \circ \chi_1] = \Phi[\chi_2] \circ \Phi[\chi_1].$$

The same diagram convinces us of the functoriality of the inverse assignment, if only we treat the associated-bundle invariants as given and the G-equivariant maps as associated with the latter. □

Remark 2. The term "adjoint bundle" is sometimes used, in the literature, with regard to the particular associated bundle

$$(\mathrm{ad}\,\mathsf{P}_{\mathrm{G}} \equiv \mathsf{P}_{\mathrm{G}} \times_{\mathsf{T}_e \mathrm{Ad}} \mathfrak{g}, B, \mathfrak{g}, \pi_{\mathsf{P}_{\mathrm{G}} \times_{\mathsf{T}_e \mathrm{Ad}} \mathfrak{g}}),$$

with the typical fibre identical with the Lie algebra $\mathfrak{g}$ of the Lie group G.

We also have the fundamental

Proposition 2. Adopt the notation of Def. 1 and Example 1 (2). There exists a canonical structure of a bundle of groups on $\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}}$, locally modelled on the Lie-group structure on the typical fibre G, *i.e.*, there are well-defined: an associative binary operation

$$[M] \;:\; \mathrm{Ad}\,\mathsf{P}_{\mathrm{G}} \times_B \mathrm{Ad}\,\mathsf{P}_{\mathrm{G}} \longrightarrow \mathrm{Ad}\,\mathsf{P}_{\mathrm{G}}$$

with a neutral element, and a unary operation

$$[\mathrm{Inv}] \;:\; \mathrm{Ad}\,\mathsf{P}_{\mathrm{G}} \circlearrowleft,$$

satisfying (fibrewise) the axioms of a group. This structure canonically induces the structure of a (Fréchet–Lie) group on the space $\Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}})$ of sections of the bundle, admitting a realisation on the space $\Gamma(\mathsf{P}_{\mathrm{G}} \times_\lambda M)$ of sections of the associated bundle $\mathsf{P}_{\mathrm{G}} \times_\lambda M$ which is induced by the mapping

$$[\lambda]. \;:\; \mathrm{Ad}\,\mathsf{P}_{\mathrm{G}} \times_B (\mathsf{P}_{\mathrm{G}} \times_\lambda M) \longrightarrow \mathsf{P}_{\mathrm{G}} \times_\lambda M$$

that satisfies (fibrewise) axioms of the action of a Lie group on a manifold and locally modelled on λ.

<u>*Proof:*</u> Consider, first, the binary operation

$$\begin{aligned} [M] \quad &: \quad \mathrm{Ad}\,\mathsf{P}_{\mathrm{G}} \times_B \mathrm{Ad}\,\mathsf{P}_{\mathrm{G}} \longrightarrow \mathrm{Ad}\,\mathsf{P}_{\mathrm{G}} \\ &: \quad \big([(p_1, g_1)], [(p_2, g_2)]\big) \longmapsto \big[\big(p_1, g_1 \cdot \mathrm{Ad}_{\phi_{\mathsf{P}_{\mathrm{G}}}(p_1, p_2)}(g_2)\big)\big], \end{aligned}$$

alongside the fibrewise assignment

$$[\varepsilon]_{\pi_{\mathsf{P}_{\mathrm{G}}}(p)} \;:\; \{\bullet\} \longrightarrow \mathrm{Ad}\,\mathsf{P}_{\mathrm{G}} \;:\; \bullet \longmapsto [(p, e)], \quad p \in \mathsf{P}_{\mathrm{G}}$$

and the unary operation

$$[\mathrm{Inv}] \;:\; \mathrm{Ad}\,\mathsf{P}_{\mathrm{G}} \circlearrowleft \;:\; [(p, g)] \longmapsto \big[\big(p, g^{-1}\big)\big].$$

We begin by verifying that all three maps ar ewell-defined. Thus, let $(p_3, g_3) \in [(p_1, g_1)]$, so that $(p_3, g_3) = (p_1 \triangleleft g_{13}, \mathrm{Ad}_{g_{13}^{-1}}(g_1))$ and $(p_4, g_4) \in [(p_2, g_2)]$, tj. $(p_4, g_4) = (p_2 \triangleleft g_{24}, \mathrm{Ad}_{g_{24}^{-1}}(g_2))$, where we have used the notation $g_{ij} \equiv \phi_{\mathsf{P}_{\mathrm{G}}}(p_i, p_j)$, $(i, j) \in \{(1, 3), (2, 4)\}$ for the sake of brevity. In virtue of Prop. 6.1, we obtain

$$\begin{aligned} &\big[\big(p_3, g_3 \cdot \mathrm{Ad}_{g_{34}}(g_4)\big)\big] = \big[\big(p_1, \mathrm{Ad}_{g_{13}}\big(g_3 \cdot \mathrm{Ad}_{g_{34}}(g_4)\big)\big)\big] \\ = \;& \big[\big(p_1, \mathrm{Ad}_{g_{13}}\big(\mathrm{Ad}_{g_{13}^{-1}}(g_1) \cdot \mathrm{Ad}_{g_{34} \cdot g_{24}^{-1}}(g_2)\big)\big)\big] = \big[\big(p_1, g_1 \cdot \mathrm{Ad}_{g_{13} \cdot g_{34} \cdot g_{42}}(g_2)\big)\big] \\ = \;& \big[\big(p_1, g_1 \cdot \mathrm{Ad}_{g_{12}}(g_2)\big)\big] \end{aligned}$$

and

$$\big[\big(p_3, g_3^{-1}\big)\big] = \big[\big(p_1, \mathrm{Ad}_{g_{13}}\big(g_3^{-1}\big)\big)\big] = \big[\big(p_1, \mathrm{Ad}_{g_{13}}(g_3)^{-1}\big)\big] = \big[\big(p_1, g_1^{-1}\big)\big].$$

Besides, we readily establish that the value taken by the map $[\varepsilon]_{\pi_{\mathsf{P}_{\mathrm{G}}}(p)}$ does not depend on the choice of the point in the fibre over $\pi_{\mathsf{P}_{\mathrm{G}}}(p)$ as for any $\widetilde{p} = p \triangleleft \phi_{\mathsf{P}_{\mathrm{G}}}(p, \widetilde{p})$, we get

$$[(\widetilde{p}, e)] = \big[\big(p \triangleleft \phi_{\mathsf{P}_{\mathrm{G}}}(p, \widetilde{p}), e\big)\big] = \big[\big(p, \mathrm{Ad}_{\phi_{\mathsf{P}_{\mathrm{G}}}(p, \widetilde{p})}(e)\big)\big] = [(p, e)].$$

Our proof of the claim that the above structure is locally modelled on G boils down to demonstrating that the fibre-modelling isomorphisms

$$[p_*]_{\mathrm{Ad}} \;:\; \big(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}}\big)_x \longrightarrow \mathrm{G} \;:\; [(p, g)] \longmapsto \mathrm{Ad}_{\phi_{\mathsf{P}_{\mathrm{G}}}(p_*, p)}(g), \quad x \in B,$$

are group homomorphisms, which we do below (for an arbitrary pair of points $(p_1,g_1),(p_2,g_2)\in \mathsf{P}_{\rm G}\times{\rm G}$ such that $p_1,p_2\in(\mathsf{P}_{\rm G})_x$), invoking Prop. 6.1 along the way,

$$\begin{aligned}[p_*]_{\rm Ad}\circ[M]\big([(p_1,g_1)],[(p_2,g_2)]\big) &= [p_*]_{\rm Ad}\big(\big[\big(p_1,g_1\cdot{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(p_1,p_2)}(g_2)\big)\big]\big)\\ &= {\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(p_*,p_1)}\big(g_1\cdot{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(p_1,p_2)}(g_2)\big)\\ &= {\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(p_*,p_1)}(g_1)\cdot{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(p_*,p_1)\cdot\phi_{\mathsf{P}_{\rm G}}(p_1,p_2)}(g_2)\\ &= {\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(p_*,p_1)}(g_1)\cdot{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(p_*,p_2)}(g_2)\\ &\equiv M\big([p_*]_{\rm Ad}\big([p_1,g_1]\big),[p_*]_{\rm Ad}\big([p_2,g_2]\big)\big)\,.\end{aligned}$$

The first step towards a reconstruction of the fibrewise action of the group $\Gamma({\rm Ad}\,\mathsf{P}_{\rm G})$ on the space $\Gamma(\mathsf{P}_{\rm G}\times_\lambda M)$ consists in identifying the following left action of the adjoint bundle on $\mathsf{P}_{\rm G}$:

$$[r]_\cdot\ :\ {\rm Ad}\,\mathsf{P}_{\rm G}\times_B\mathsf{P}_{\rm G}\longrightarrow\mathsf{P}_{\rm G}\ :\ \big([(p,g)],\widetilde p\big)\longmapsto r_{{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(\widetilde p,p)}(g)}(\widetilde p)\,.$$

The latter is defined unequivocally since for any representative $(p_2,g_2)\in[(p_1,g_1)]$, we obtain

$$\begin{aligned}r_{{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(\widetilde p,p_2)}(g_2)}(\widetilde p) &= r_{{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(\widetilde p,p_1)\cdot\phi_{\mathsf{P}_{\rm G}}(p_1,p_2)}(g_2)}(\widetilde p)=r_{{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(\widetilde p,p_1)}({\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(p_1,p_2)}(g_2))}(\widetilde p)\\ &= r_{{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(\widetilde p,p_1)}(g_1)}(\widetilde p)\,.\end{aligned}$$

Its smoothness is ensured by Prop. Niezb-10 – indeed, $[r]_\cdot$ is the (only) smooth map induced by the (manifestly smooth) map

$$\widetilde r_\cdot\ :\ (\mathsf{P}_{\rm G}\times{\rm G})\times_B\mathsf{P}_{\rm G}\longrightarrow\mathsf{P}_{\rm G}\ :\ \big((p,g),\widetilde p\big)\longmapsto r_{{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(\widetilde p,p)}(g)}(\widetilde p)\,,$$

constant on level sets of the canonical projection $\pi_{(\mathsf{P}_{\rm G}\times{\rm G})/{\rm G}}$. We readily convince ourselves that $[r]_\cdot$ has properties analogous to the defining ones of a (left) group action: The neutral element acts trivially,

$$[r]_{[(p,e)]}(\widetilde p)=r_{{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(\widetilde p,p)}(e)}(\widetilde p)=r_e(\widetilde p)=\widetilde p\,,$$

and $[r]_\cdot$ is multiplicative in the first argument, *i.e.*, for any pair $[(p_1,g_1)],[(p_2,g_2)]\in(\mathsf{P}_{\rm G})_{\pi_{\mathsf{P}_{\rm G}}(\widetilde p)}$, the identity

$$\begin{aligned}&[r]_{[M]([(p_1,g_1)],[(p_2,g_2)])}(\widetilde p)=r_{{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(\widetilde p,p_1)}(g_1\cdot{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(p_1,p_2)}(g_2))}(\widetilde p)\\ =\ & r_{{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(\widetilde p,p_1)}(g_1)\cdot{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(\widetilde p,p_1)\cdot\phi_{\mathsf{P}_{\rm G}}(p_1,p_2)}(g_2)}(\widetilde p)=r_{{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(\widetilde p,p_1)}(g_1)\cdot{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(\widetilde p,p_2)}(g_2)}(\widetilde p)\\ =\ & r_{{\rm Ad}_{{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(\widetilde p,p_2)}(g_2^{-1})}({\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(\widetilde p,p_1)}(g_1))}\circ r_{{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(\widetilde p,p_2)}(g_2))}(\widetilde p)\\ \equiv\ & {\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(\widetilde p,p_2)\cdot g_2^{-1}\cdot\phi_{\mathsf{P}_{\rm G}}(p_2,p_1)}(g_1)\circ[r]_{[(p_2,g_2)]}(\widetilde p)\end{aligned}$$

holds true, which we may rewrite, using the equality

$$\begin{aligned}&\phi_{\mathsf{P}_{\rm G}}\big([r]_{[(p_2,g_2)]}(\widetilde p),p_1\big)=\phi_{\mathsf{P}_{\rm G}}\big(r_{g_2\cdot\phi_{\mathsf{P}_{\rm G}}(p_2,\widetilde p)}(p_2),p_1\big)\\ \equiv\ & \phi_{\mathsf{P}_{\rm G}}\Big(r_{g_2\cdot\phi_{\mathsf{P}_{\rm G}}(p_2,\widetilde p)}(p_2),r_{g_2\cdot\phi_{\mathsf{P}_{\rm G}}(p_2,\widetilde p)\cdot(g_2\cdot\phi_{\mathsf{P}_{\rm G}}(p_2,\widetilde p))^{-1}\cdot\phi_{\mathsf{P}_{\rm G}}(p_2,p_1)}(p_2)\Big)\\ =\ & \big(g_2\cdot\phi_{\mathsf{P}_{\rm G}}(p_2,\widetilde p)\big)^{-1}\cdot\phi_{\mathsf{P}_{\rm G}}(p_2,p_1)\,,\end{aligned}$$

in the desired form

$$\begin{aligned}[r]_{[M]([(p_1,g_1)],[(p_2,g_2)])}(\widetilde p) &= r_{{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}([r]_{[(p_2,g_2)]}(\widetilde p),p_1)}(g_1)}\big([r]_{[(p_2,g_2)]}(\widetilde p)\big)\\ &\equiv [r]_{[(p_1,g_1)]}\circ[r]_{[(p_2,g_2)]}(\widetilde p)\,.\end{aligned}$$

It ought to be underlined that that the action of the adjoint bundle defined above commutes with the defining (right) action $r.$ – indeed, for any $[(p,g)] \in \mathrm{Ad}\,\mathsf{P}_{\mathrm{G}}$, $h \in \mathrm{G}$ and $\widetilde{p} \in (\mathsf{P}_{\mathrm{G}})_{\pi_{\mathsf{P}_{\mathrm{G}}}(p)}$, we conclude that

$$\begin{aligned}[][r]_{[(p,g)]} \circ r_h(\widetilde{p}) &= r_{\mathrm{Ad}_{\phi_{\mathsf{P}_{\mathrm{G}}}(r_h(\widetilde{p}),p)}(g)}\big(r_h(\widetilde{p})\big) = r_{g\cdot\phi_{\mathsf{P}_{\mathrm{G}}}(p,r_h(\widetilde{p}))}(p) \\ &= r_{\phi_{\mathsf{P}_{\mathrm{G}}}(\widetilde{p},p)\cdot\phi_{\mathsf{P}_{\mathrm{G}}}(p,r_h(\widetilde{p}))}\big(r_{\mathrm{Ad}_{\phi_{\mathsf{P}_{\mathrm{G}}}(\widetilde{p},p)}(g)}(\widetilde{p})\big) \\ &\equiv r_{\phi_{\mathsf{P}_{\mathrm{G}}}(\widetilde{p},r_h(\widetilde{p}))}\big([r]_{[(p,g)]}(\widetilde{p})\big) = r_h \circ [r]_{[(p,g)]}(\widetilde{p})\,.\end{aligned}$$

We may, next, lift this action, without losing any of its desired properties verified above, from the total space of the adjoint bundle to the space of its (global) sections, according to the prescription

$$\Gamma[r]. \;:\; \Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}}) \times \mathsf{P}_{\mathrm{G}} \longrightarrow \mathsf{P}_{\mathrm{G}} \;:\; (\sigma,p) \longmapsto [r]_{\sigma\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p)}(p)\,.$$

The space $\Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}})$ (equipped with the natural structure of a Fréchet manifold) thus assumes the rôle of the support of the structure of a (Fréchet–Lie) group with group operations

$$\begin{aligned}\Gamma[M] &: \Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}}) \times \Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}}) \longrightarrow \Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}}) \;:\; (\sigma_1\sigma_2) \longmapsto [M] \circ (\sigma_1,\sigma_2)\,, \\ \Gamma[\mathrm{Inv}] &: \Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}}) \circlearrowleft \;:\; \sigma \longmapsto [\mathrm{Inv}] \circ \sigma\,, \\ \Gamma[\varepsilon] &: \{\bullet\} \longrightarrow \Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}}) \;:\; \bullet \longmapsto \big[\big(\sigma(\cdot),e\big)\big]\,,\end{aligned}$$

induced, in an obvious (pointwise) manner, from the respective operations on $\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}}$, and, at the same time, that of a subgroup of the group of automorphisms of the principal bundle P_{G} (covering the identity on the base). Here, the map $\Gamma[r]_\sigma$ is identified with the automorphism $(\Gamma[r]_\sigma, \mathrm{id}_{\mathrm{G}}, \mathrm{id}_B)$ in the notation of Def. 6.1. We may subsequently extend, in an obvious way, the thus understood action of the group of sections of the adjoint bundle on P_{G} to the bundle $\mathsf{P}_{\mathrm{G}} \times M$ over the same base B by setting

$$\begin{aligned}\Gamma[\widetilde{r}]. := \Gamma[r]. \times \mathrm{id}_M \quad &: \quad \Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}}) \times (\mathsf{P}_{\mathrm{G}} \times M) \longrightarrow \mathsf{P}_{\mathrm{G}} \times M \\ &: \quad \big(\sigma,(p,m)\big) \longmapsto \big([r]_{\sigma\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p)}(p),m\big)\,.\end{aligned}$$

The property of the latter action of key significance for our later considerations is its commutativity with the action $\widetilde{\lambda}.$ defined in Eq. (6.2) that serves as the basis of the construction of the associated bundle $\mathsf{P}_{\mathrm{G}} \times_\lambda M$. Indeed, for any $\sigma \equiv [(\pi,\gamma)] \in \Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}})$, $g \in \mathrm{G}$ and $(p,m) \in \mathsf{P}_{\mathrm{G}} \times M$, we find – upon invoking relative commutativity of the actions: $[r].$ i $r.$, checked formerly – the identity

$$\begin{aligned}\Gamma[\widetilde{r}]_\sigma \circ \widetilde{\lambda}_g(p,m) &= \big([r]_{\sigma\circ\pi_{\mathsf{P}_{\mathrm{G}}}(r_g(p))}\big(r_g(p)\big), \lambda_{g^{-1}}(m)\big) \\ &\equiv \big([r]_{\sigma\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p)} \circ r_g(p), \lambda_{g^{-1}}(m)\big) = \big(r_g \circ [r]_{\sigma\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p)}(p), \ell_{g^{-1}}(m)\big) \\ &= \widetilde{\lambda}_g \circ \Gamma[\widetilde{r}]_\sigma(p,m)\,.\end{aligned}$$

As a result of the above, the induced action $\Gamma[\widetilde{r}].$ descends to the quotient manifold $(\mathsf{P}_{\mathrm{G}} \times M)/\mathrm{G} \equiv \mathsf{P}_{\mathrm{G}} \times_\lambda M$, *i.e.*, it canonically induces a left action of the group $\Gamma(P_{\mathrm{Ad.}}\mathrm{G})$ on the manifold $\mathsf{P}_{\mathrm{G}} \times_\lambda M$, given by

$$\begin{aligned}[][\Gamma[\widetilde{r}]]^\lambda_. \quad &: \quad \Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}}) \times \mathsf{P}_{\mathrm{G}} \times_\lambda M \longrightarrow \mathsf{P}_{\mathrm{G}} \times_\lambda M \\ &: \quad \big(\sigma,[(p,m)]\big) \longmapsto \big[\big([r]_{\sigma\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p)}(p),m\big)\big]\,.\end{aligned}$$

Our hitherto analysis shows that the latter map is well-defined and has all the requisite properties of a (left) group action. In the last step, we induce with its help the action, postulated in the statement of the proposition, of the group $\Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}})$ on the space of (global) sections of the assciated bundle,

$$\Gamma[\Gamma[\widetilde{r}]]^\lambda_. \quad : \quad \Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}}) \times \Gamma(\mathsf{P}_{\mathrm{G}} \times_\lambda M) \longrightarrow \Gamma(\mathsf{P}_{\mathrm{G}} \times_\lambda M)$$

$$(2) \qquad : \quad \big(\sigma,[(\pi,\mu)]\big)\longmapsto\big[\big([r]_{\sigma\circ\pi_{\mathsf{P}_{\rm G}}\circ\pi(\cdot)}\circ\pi(\cdot),\mu(\cdot)\big)\big]\equiv\big[\big([r]_{\sigma(\cdot)}\circ\pi(\cdot),\mu(\cdot)\big)\big]\,.$$

This is, self-evidently, a lift, to the space of sections, of the map

$$[\lambda]. \quad : \quad {\rm Ad}\,\mathsf{P}_{\rm G}\times_B(\mathsf{P}_{\rm G}\times_\lambda M)\longrightarrow\mathsf{P}_{\rm G}\times_\lambda M$$

$$: \quad \big([(p_1,g_1)],[(p_2,m_2)]\big)\longmapsto\big[\big(p_2,\lambda_{{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(p_2,p_1)}(g_1)}(m_2)\big)\big]$$

whose well-definedness and multiplicativit in the first argument is a direct consequence of the respective properties of the action $\Gamma[\Gamma[\widetilde{r}]].$, checked previously . That the action $[\lambda].$ is locally modelled on $\lambda.$, as claimed, is most straightforwardly proven with the help of the isomorphisms $[p_*]_{\rm Ad}$ oraz $[p_*]_\lambda$, indicated before. Thus, we carry out the following calculation:

$$\lambda_{[p_*]_{\rm Ad}([(p_1,g_1)])}\big([p_*]_\lambda\big([(p_2,m_2)]\big)\big)=\lambda_{{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(p_*,p_1)}(g_1)}\circ\lambda_{\phi_{\mathsf{P}_{\rm G}}(p_*,p_2)}(m_2)$$

$$=\lambda_{\phi_{\mathsf{P}_{\rm G}}(p_*,p_2)\cdot{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(p_2,p_1)}(g_1)}(m_2)=\lambda_{\phi_{\mathsf{P}_{\rm G}}(p_*,p_2)}\big(\lambda_{{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(p_2,p_1)}(g_1)}(m_2)\big)$$

$$\equiv [p_*]_\lambda\big(\big[\big(p_2,\lambda_{{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(p_2,p_1)}(g_1)}(m_2)\big)\big]\big)\equiv[p_*]_\lambda\circ[\lambda]_{[(p_1,g_1)]}\big([(p_2,m_2)]\big)\,.$$

□

The above proposition together with its constructive proof demonstrate convincingly that the goal set before has been attained. In so doing, they emphasise the rôle played by the space of smooth sections of the associated bundle, which prompts us to take a closer look at the latter. We do that in

Proposition 3. Adopt the notation of Def. 1 and Przykł. 1 (2). There exists a bijection

$$\Gamma(\mathsf{P}_{\rm G}\times_\lambda M)\cong{\rm Hom}_{\rm G}(\mathsf{P}_{\rm G},M)\,,$$

where ${\rm Hom}_{\rm G}(\mathsf{P}_{\rm G},M)$ is the set of G-equivariant maps of Def. 5.1.

Proof: Let $\sigma=[(\pi,\mu)]\in\Gamma(\mathsf{P}_{\rm G}\times_\lambda M)$ be a *global* section determined by (local) sections $\pi\in\Gamma_{\rm loc}(\mathsf{P}_{\rm G})$ and $\mu\in\Gamma_{\rm loc}(B\times M)$. Using the quotient map and the canonical projection on the base of the bundle $\mathsf{P}_{\rm G}$, we may define the map

$$\Phi_\lambda[\sigma]\ :\ \mathsf{P}_{\rm G}\longrightarrow M\ :\ p\longmapsto\lambda_{\phi_{\mathsf{P}_{\rm G}}(p,\pi\circ\pi_{\mathsf{P}_{\rm G}}(p))}\big(\mu\circ\pi_{\mathsf{P}_{\rm G}}(p)\big)\,.$$

We readily convince ourselves that the above definition makes sense as for any pair $(\pi',\mu')=(\pi\triangleleft{\rm Inv}\circ\gamma,\gamma\triangleright\mu)$ associated, in an obvious manner, with $\gamma\in\Gamma_{\rm loc}(B\times{\rm G})$, we find – upon invoking the axioms of an action of a group on a set – the desired equality

$$\begin{aligned}\lambda_{\phi_{\mathsf{P}_{\rm G}}(p,\pi'\circ\pi_{\mathsf{P}_{\rm G}}(p))}\big(\mu'\circ\pi_{\mathsf{P}_{\rm G}}(p)\big) &= \lambda_{\phi_{\mathsf{P}_{\rm G}}(p,\pi\circ\pi_{\mathsf{P}_{\rm G}}(p)\triangleleft\gamma\circ\pi_{\mathsf{P}_{\rm G}}(p)^{-1})}\big(\lambda_{\gamma\circ\pi_{\mathsf{P}_{\rm G}}(p)}\big(\mu\circ\pi_{\mathsf{P}_{\rm G}}(p)\big)\big)\\ &= \lambda_{\phi_{\mathsf{P}_{\rm G}}(p,\pi\circ\pi_{\mathsf{P}_{\rm G}}(p))\cdot\gamma\circ\pi_{\mathsf{P}_{\rm G}}(p)^{-1}\cdot\gamma\circ\pi_{\mathsf{P}_{\rm G}}(p)}\big(\mu\circ\pi_{\mathsf{P}_{\rm G}}(p)\big)\\ &= \lambda_{\phi_{\mathsf{P}_{\rm G}}(p,\pi\circ\pi_{\mathsf{P}_{\rm G}}(p))}\big(\mu\circ\pi_{\mathsf{P}_{\rm G}}(p)\big)\,.\end{aligned}$$

Its G-equivariance follows directly from the calculation:

$$\begin{aligned}\Phi_\lambda[\sigma]\circ r_g(p) &= \lambda_{\phi_{\mathsf{P}_{\rm G}}(p\triangleleft g,\pi\circ\pi_{\mathsf{P}_{\rm G}}(p\triangleleft g))}\big(\mu\circ\pi_{\mathsf{P}_{\rm G}}(p\triangleleft g)\big)\\ &= \lambda_{g^{-1}\cdot\phi_{\mathsf{P}_{\rm G}}(p,\pi\circ\pi_{\mathsf{P}_{\rm G}}(p))}\big(\mu\circ\pi_{\mathsf{P}_{\rm G}}(p)\big)=\lambda_{g^{-1}}\circ\Phi_{\lambda.}[\sigma](p)\,,\end{aligned}$$

carried out for arbitrary $(p,g)\in\mathsf{P}_{\rm G}\times{\rm G}$, and using Prop. 6.1 in conjunction with the aforementioned axioms.

In order to construct the inverse of the above assignment, we fix an (arbitrary) open trivialising cover $\{\mathcal{O}_i\}_{i\in I}$ for the bundle $\mathsf{P}_{\rm G}$, and subsequently assign, to an arbitrary G-equivariant map $f\ :\ \mathsf{P}_{\rm G}\longrightarrow M$, the family

$$S_\lambda[f]_i \quad : \quad \mathcal{O}_i\longrightarrow\mathsf{P}_{\rm G}\times_\lambda M\ :\ x\longmapsto\big[\big(\tau_i^{-1}(x,e),f\circ\tau_i^{-1}(x,e)\big)\big]\,,\quad i\in I$$

of local sections. Each of them is (locally) smooth as a superposition of the respective smooth maps $\big(\tau_i^{-1}(\cdot,e), f\circ\tau_i^{-1}(\cdot,e)\big) : \mathcal{O}_i \longrightarrow \mathsf{P}_{\rm G}\times M$ and the surjective submersion $\pi_{(\mathsf{P}_{\rm G}\times_\lambda M)/{\rm G}}$. We readily establish that these local sections are, in fact, restrictions (to the respective sets $\mathcal{O}_i$) of a global one upon noting that due to G-equivariance of the maps τ_i i f at any point $x\in\mathcal{O}_{ij}$, the following equality holds:

$$\begin{aligned} S_\lambda[f]_j(x) &= \big[\big(\tau_j^{-1}(x,e), f\circ\tau_j^{-1}(x,e)\big)\big] = \big[\big(\tau_i^{-1}\big(x,g_{ij}(x)\big), f\circ\tau_i^{-1}\big(x,g_{ij}(x)\big)\big)\big] \\ &= \big[\big(\tau_i^{-1}(x,e)\triangleleft g_{ij}(x), f\big(\tau_i^{-1}(x,e)\triangleleft g_{ij}(x)\big)\big] \\ &= \big[\big(\tau_i^{-1}(x,e)\triangleleft g_{ij}(x), g_{ij}(x)^{-1}\triangleright f\circ\tau_i^{-1}(x,e)\big] \\ &= \big[\big(\tau_i^{-1}(x,e), f\circ\tau_i^{-1}(x,e)\big)\big] \equiv S_\lambda[f]_i(x)\,. \end{aligned}$$

A direct calculation of both superpositions:

$$\Phi_\lambda\big[S_\lambda[f]\big] : \mathsf{P}_{\rm G}\longrightarrow M : p\longmapsto \lambda_{\phi_{\mathsf{P}_{\rm G}}(p,p)}\big(f(p)\big) = \lambda_e\big(f(p)\big) = f(p)$$

and

$$\begin{aligned} S_\lambda\big[\Phi_\lambda\big[[(\pi,\mu)]\big]\big] \quad &: \quad B\longrightarrow \mathsf{P}_{\rm G}\times_\lambda M \\ &: x\longmapsto \big[\big(\tau_i^{-1}(x,e), \lambda_{\phi_{\mathsf{P}_{\rm G}}(\tau_i^{-1}(x,e),\pi\circ\pi_{\mathsf{P}_{\rm G}}\circ\tau_i^{-1}(x,e))}\big(\mu\circ\pi_{\mathsf{P}_{\rm G}}\circ\tau_i^{-1}(x,e)\big)\big)\big] \\ &\equiv \big[\big(\tau_i^{-1}(x,e), \lambda_{\phi_{\mathsf{P}_{\rm G}}(\tau_i^{-1}(x,e),\pi(x))}\big(\mu(x)\big)\big)\big] = [(\pi,\mu)](x) \end{aligned}$$

reveals the veracity of the desired identities

$$\Phi_\lambda\circ S_\lambda = {\rm id}_{{\rm Hom}_{\rm G}(\mathsf{P}_{\rm G},M)}\,, \qquad S_\lambda\circ\Phi_\lambda = {\rm id}_{\Gamma(\mathsf{P}_{\rm G}\times_\lambda M)}\,.$$

□

A specialisation of the last result to the adjoint bundle turns out to carry further structural information, displayed in

Proposition 4. The bijection

$$\Gamma({\rm Ad}\,\mathsf{P}_{\rm G}) \cong {\rm Hom}_{\rm G}(\mathsf{P}_{\rm G},{\rm G})$$

of Prop. 3 is an isomorphism between the group of sections of the adjoint bundle, with the structure detailed in the proof of Prop. 2, and the group of maps from $\mathsf{P}_{\rm G}$ to G equivariant relative to the respective (left) actions $r_{{\rm Inv}(\cdot)}$ and Ad., with the natural pointwise group structure.

Proof: Borrowing the notation from the proofs of both propositions mentioned in the above statement, we check – for any pair of sections $\sigma_\alpha = [(\pi_\alpha,\gamma_\alpha)]\in\Gamma({\rm Ad}\,\mathsf{P}_{\rm G})$, $\alpha\in\{1,2\}$ and a point $p\in\mathsf{P}_{\rm G}$ –

$$\begin{aligned} &\Phi_{\rm Ad}\big[\Gamma[M](\sigma_1,\sigma_2)\big](p) = {\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(p,\pi_1\circ\pi_{\mathsf{P}_{\rm G}}(p))}\big(\gamma_1\circ\pi_{\mathsf{P}_{\rm G}}(p)\cdot{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(\pi_1\circ\pi_{\mathsf{P}_{\rm G}}(p),\pi_2\circ\pi_{\mathsf{P}_{\rm G}}(p))}\big(\gamma_2\circ\pi_{\mathsf{P}_{\rm G}}(p)\big)\big) \\ &= {\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(p,\pi_1\circ\pi_{\mathsf{P}_{\rm G}}(p))}\big(\gamma_1\circ\pi_{\mathsf{P}_{\rm G}}(p)\big)\cdot{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(p,\pi_1\circ\pi_{\mathsf{P}_{\rm G}}(p))\cdot\phi_{\mathsf{P}_{\rm G}}(\pi_1\circ\pi_{\mathsf{P}_{\rm G}}(p),\pi_2\circ\pi_{\mathsf{P}_{\rm G}}(p))}\big(\gamma_2\circ\pi_{\mathsf{P}_{\rm G}}(p)\big) \\ &= {\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(p,\pi_1\circ\pi_{\mathsf{P}_{\rm G}}(p))}\big(\gamma_1\circ\pi_{\mathsf{P}_{\rm G}}(p)\big)\cdot{\rm Ad}_{\phi_{\mathsf{P}_{\rm G}}(p,\pi_2\circ\pi_{\mathsf{P}_{\rm G}}(p))}\big(\gamma_2\circ\pi_{\mathsf{P}_{\rm G}}(p)\big) \\ &= M\circ\big(\Phi_{\rm Ad}(\sigma_1),\Phi_{\rm Ad}(\sigma_2)\big)(p)\,. \end{aligned}$$

□

The structural character of the bijection referred to in Props. 2 and 3 is best illustrated in

Proposition 5. Adopt the notation of Props. 2 and 3 and their proofs. The bijection Φ_λ is (left) equivariant relative to the following actions of the group $\Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}})$: the action $\Gamma[\Gamma[\widetilde{r}]]^\lambda_\cdot$ on the space $\Gamma(\mathsf{P}_{\mathrm{G}}\times_\lambda M)$, defined in Eq. (2), and the natural action

$$\begin{aligned}[\Phi_{\mathrm{Ad}}\lambda]_\cdot \quad &: \quad \Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}})\times\mathrm{Hom}_{\mathrm{G}}(\mathsf{P}_{\mathrm{G}},M)\longrightarrow\mathrm{Hom}_{\mathrm{G}}(\mathsf{P}_{\mathrm{G}},M)\\ &: \quad (\gamma,\mu)\longmapsto\lambda_{\Phi_{\mathrm{Ad}}[\gamma](\cdot)}\big(\mu(\cdot)\big)\end{aligned}$$

on the space of G-equivariant maps $\mathrm{Hom}_{\mathrm{G}}(\mathsf{P}_{\mathrm{G}},M)$, which means that the action

$$\Phi_{\mathrm{Ad}}\lambda_\cdot\equiv[\Phi_{\mathrm{Ad}}\lambda]_\cdot\circ\big(\Phi_{\mathrm{Ad}}^{-1}\times\mathrm{id}_{\mathrm{Hom}_{\mathrm{G}}(\mathsf{P}_{\mathrm{G}},M)}\big)$$

of the group $\mathrm{Hom}_{\mathrm{G}}(\mathsf{P}_{\mathrm{G}},\mathrm{G})$ renders the diagram

$$\begin{array}{ccc}\Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}})\times\Gamma(\mathsf{P}_{\mathrm{G}}\times_\lambda M) & \xrightarrow{\ \Gamma[\Gamma[\widetilde{r}]]^\lambda_\cdot\ } & \Gamma(\mathsf{P}_{\mathrm{G}}\times_\lambda M)\\ \Big\downarrow{\scriptstyle\Phi_{\mathrm{Ad}}\times\Phi_\lambda} & & \Big\downarrow{\scriptstyle\Phi_\lambda}\\ \mathrm{Hom}_{\mathrm{G}}(\mathsf{P}_{\mathrm{G}},\mathrm{G})\times\mathrm{Hom}_{\mathrm{G}}(\mathsf{P}_{\mathrm{G}},M) & \xrightarrow[\ \Phi_{\mathrm{Ad}}\lambda_\cdot\ ]{} & \mathrm{Hom}_{\mathrm{G}}(\mathsf{P}_{\mathrm{G}},M)\end{array}$$

commutative.

Proof: Before all else, we convince ourselves that the map $\Phi_{\mathrm{Ad}}\lambda_\cdot$ is well-defined. To this end, we pick up an arbitrary pair $(\gamma,\mu)\in\mathrm{Hom}_{\mathrm{G}}(\mathsf{P}_{\mathrm{G}},\mathrm{G})\times\mathrm{Hom}_{\mathrm{G}}(\mathsf{P}_{\mathrm{G}},M)$ and consider the result of the evaluation $\Phi_{\mathrm{Ad}}\lambda_\gamma(\mu)$ – we must prove that the latter is G-equivariant, which we do in a direct computation, carried out for arbitrary $(p,g)\in\mathsf{P}_{\mathrm{G}}\times\mathrm{G}$,

$$\begin{aligned}\Phi_{\mathrm{Ad}}\lambda_\gamma\circ r_g^*(\mu)(p) &= \lambda_{\gamma\circ r_g(p)}\big(\mu\circ r_g(p)\big)=\lambda_{\mathrm{Ad}_{g^{-1}}(\gamma(p))}\circ\lambda_{g^{-1}}\big(\mu(p)\big)\\ &= \lambda_{g^{-1}}\big(\lambda_{\gamma(p)}\big(\mu(p)\big)\big)\equiv\lambda_{g^{-1}}\circ\Phi_{\mathrm{Ad}}\lambda_\gamma(\mu)(p)\,.\end{aligned}$$

It is obvious that the map $\Phi_{\mathrm{Ad}}\lambda_\cdot$ satisfies the axioms defining a group action. Therefore, it remains to verify its equivariance. For arbitrary $\widetilde{\sigma}=[(\widetilde{\pi},\widetilde{\gamma})]\in\Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}})$ and $\sigma=[(\pi,\mu)]\in\Gamma(\mathsf{P}_{\mathrm{G}}\times_\lambda M)$ as well as $p\in\mathsf{P}_{\mathrm{G}}$, we calculate

$$\begin{aligned}&\Phi_\lambda\big[\Gamma[\Gamma[\widetilde{r}]]^\lambda_{\widetilde{\sigma}}(\sigma)\big](p)=\lambda_{\phi_{\mathsf{P}_{\mathrm{G}}}(p,\lambda_{\widetilde{\sigma}\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p)}(\pi\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p)))}\big(\mu\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p)\big)\\ &= \lambda_{\phi_{\mathsf{P}_{\mathrm{G}}}(p,r_{\mathrm{Ad}_{\phi_{\mathsf{P}_{\mathrm{G}}}(\pi\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p),\widetilde{\pi}\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p))}(\widetilde{\gamma}\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p))}(\pi\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p)))}\big(\mu\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p)\big)\\ &= \lambda_{\phi_{\mathsf{P}_{\mathrm{G}}}(p,r_{\widetilde{\gamma}\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p)\cdot\phi_{\mathsf{P}_{\mathrm{G}}}(\widetilde{\pi}\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p),\pi\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p))}(\widetilde{\pi}\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p)))}\big(\mu\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p)\big)\\ &= \lambda_{\phi_{\mathsf{P}_{\mathrm{G}}}(p,\widetilde{\pi}\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p))\cdot\widetilde{\gamma}\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p)\cdot\phi_{\mathsf{P}_{\mathrm{G}}}(\widetilde{\pi}\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p),\pi\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p))}\big(\mu\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p)\big)\\ &= \lambda_{\phi_{\mathsf{P}_{\mathrm{G}}}(p,\widetilde{\pi}\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p))\cdot\widetilde{\gamma}\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p)\cdot\phi_{\mathsf{P}_{\mathrm{G}}}(\widetilde{\pi}\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p),p)}\circ\ell_{\phi_{\mathsf{P}_{\mathrm{G}}}(p,\pi\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p))}\big(\mu\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p)\big)\\ &= \lambda_{\mathrm{Ad}_{\phi_{\mathsf{P}_{\mathrm{G}}}(p,\widetilde{\pi}\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p))}(\widetilde{\gamma}\circ\pi_{\mathsf{P}_{\mathrm{G}}}(p))}\big(\Phi_\lambda[\sigma](p)\big)\\ &= \Phi_{\mathrm{Ad}}\lambda_{\mathrm{Ad}_{\phi_{\mathsf{P}_{\mathrm{G}}}(\cdot,\widetilde{\pi}\circ\pi_{\mathsf{P}_{\mathrm{G}}}(\cdot))}(\widetilde{\gamma}\circ\pi_{\mathsf{P}_{\mathrm{G}}}(\cdot))}\big(\Phi_\lambda[\sigma]\big)(p)\\ &\equiv \Phi_{\mathrm{Ad}}\lambda_{\Phi_{\mathrm{Ad}}[\widetilde{\sigma}]}\big(\Phi_\lambda[\sigma]\big)(p)\,,\end{aligned}$$

which is the anticipated result. □

Our hitherto considerations present $\Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}})$ as a bundle of groups acting on a bundle of manifolds M in a natural manner modelled on λ. The statement that we give below deepens our observations substantially and, simultaneously, opens a path towards a natural physical interpretation of the group $\Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}})$ as the gauge group of field theory.

Proposition 6. Adopt the notation of Prop. 2 and its proof. There exists a canonical group isomorphism

$$\Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}}) \cong \{ (\Phi, \mathrm{id}_{\mathrm{G}}, f) \in \mathrm{Aut}_{\mathbf{GrpBun}_{\mathrm{G}}(B)}(\mathsf{P}_{\mathrm{G}}) \mid f = \mathrm{id}_B \}$$

$$=: \mathrm{Aut}_{\mathbf{GrpBun}_{\mathrm{G}}(B)}(\mathsf{P}_{\mathrm{G}} \,|\, B).$$

Proof: We begin by establishing a bijection between the sets $\mathrm{Hom}_{\mathrm{G}}(\mathsf{P}_{\mathrm{G}}, \mathrm{G})$ and $\mathrm{Aut}_{\mathbf{GrpBun}_{\mathrm{G}}(B)}(\mathsf{P}_{\mathrm{G}} \,|\, B)$. For that, we pick up (arbitrarily) $\gamma \in \mathrm{Hom}_{\mathrm{G}}(\mathsf{P}_{\mathrm{G}}, \mathrm{G})$ and define the map

$$\Psi[\gamma] : \mathsf{P}_{\mathrm{G}} \circlearrowleft : p \longmapsto r_{\gamma(p)}(p).$$

The latter is manifestly G-equivariant,

$$\forall_{(p,g)\in \mathsf{P}_{\mathrm{G}}\times \mathrm{G}} : \Psi[\gamma] \circ r_g(p) \equiv r_{\gamma\circ r_g(p)}\big(r_g(p)\big) = r_g \circ r_{\mathrm{Ad}_{g^{-1}}(\gamma(p))}(p) = r_{\gamma(p)\cdot g}(p)$$

$$= r_g \circ \Psi[\gamma](p),$$

and preserves fibres, and so it defines an automorphism

$$\big(\Psi[\gamma], \mathrm{id}_{\mathrm{G}}, \mathrm{id}_B\big) \in \mathrm{Aut}_{\mathbf{GrpBun}_{\mathrm{G}}(B)}(\mathsf{P}_{\mathrm{G}} \,|\, B).$$

Furthermore, it is a group homomorphism, a fact readily inferred from a direct computation

$$\Psi\big[\widetilde{M}(\gamma_1, \gamma_2)\big](p) = r_{\gamma_1(p)\cdot\gamma_2(p)}(p) \equiv r_{\gamma_2(p)\cdot \mathrm{Ad}_{\gamma_2(p)^{-1}}(\gamma_1(p))}(p)$$

$$= r_{\mathrm{Ad}_{\gamma_2(p)^{-1}}(\gamma_1(p))} \circ r_{\gamma_2(p)}(p) = r_{\gamma_1(p\triangleleft\gamma_2(p))} \circ r_{\gamma_2(p)}(p)$$

$$\equiv \Psi[\gamma_1] \circ \Psi[\gamma_2](p),$$

carried out for arbitrary $\gamma_1, \gamma_2 \in \mathrm{Hom}_{\mathrm{G}}(\mathsf{P}_{\mathrm{G}}, \mathrm{G})$. At this stage, it suffices to invoke Prop. 3, to obtain the sought-after group homomorphism

$$\alpha_{\cdot} \equiv \big(\Psi[\cdot], \mathrm{id}_{\mathrm{G}}, \mathrm{id}_B\big) \circ \Phi_{\mathrm{Ad}.} : \Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}}) \longrightarrow \mathrm{Aut}_{\mathbf{GrpBun}_{\mathrm{G}}(B)}(\mathsf{P}_{\mathrm{G}} \,|\, B).$$

Going in the opposite direction, we associate to an arbitrary automorphism $(\Phi, \mathrm{id}_{\mathrm{G}}, \mathrm{id}_B) \in \mathrm{Aut}_{\mathbf{GrpBun}_{\mathrm{G}}(B)}(\mathsf{P}_{\mathrm{G}} \,|\, B)$ the map

$$\chi[(\Phi, \mathrm{id}_{\mathrm{G}}, \mathrm{id}_B)] : \mathsf{P}_{\mathrm{G}} \longrightarrow \mathrm{G} : p \longmapsto \phi_{\mathsf{P}_{\mathrm{G}}}\big(p, \Phi(p)\big)$$

whose G-equivariance is proven through reference to Prop. 5.1, and for arbitrary $(p, g) \in \mathsf{P}_{\mathrm{G}} \times \mathrm{G}$,

$$\chi[(\Phi, \mathrm{id}_{\mathrm{G}}, \mathrm{id}_B)] \circ r_g(p) \equiv \phi_{\mathsf{P}_{\mathrm{G}}}\big(r_g(p), \Phi \circ r_g(p)\big) = \phi_{\mathsf{P}_{\mathrm{G}}}\big(r_g(p), r_g \circ \Phi(p)\big)$$

$$= \mathrm{Ad}_{g^{-1}}\big(\phi_{\mathsf{P}_{\mathrm{G}}}\big(p, \Phi(p)\big)\big) \equiv \mathrm{Ad}_{g^{-1}} \circ \chi[(\Phi, \mathrm{id}_{\mathrm{G}}, \mathrm{id}_B)](p).$$

It is easy to see that the map

$$\chi : \mathrm{Aut}_{\mathbf{GrpBun}_{\mathrm{G}}(B)}(\mathsf{P}_{\mathrm{G}} \,|\, B) \longrightarrow \mathrm{Hom}_{\mathrm{G}}(\mathsf{P}_{\mathrm{G}}, \mathrm{G})$$

thus obtained is a group homomorphism – indeed, for any pair of automorphisms $(\Phi_\alpha, \mathrm{id}_{\mathrm{G}}, \mathrm{id}_B) \in \mathrm{Aut}_{\mathbf{GrpBun}_{\mathrm{G}}(B)}(\mathsf{P}_{\mathrm{G}} \,|\, B)$, $\alpha \in \{1, 2\}$, we calculate

$$\chi\big[(\Phi_1, \mathrm{id}_{\mathrm{G}}, \mathrm{id}_B) \circ (\Phi_2, \mathrm{id}_{\mathrm{G}}, \mathrm{id}_B)\big](p) = \phi_{\mathsf{P}_{\mathrm{G}}}\big(p, \Phi_1 \circ \Phi_2(p)\big)$$

$$= \phi_{\mathsf{P}_{\mathrm{G}}}\big(p, \Phi_1(p)\big) \cdot \phi_{\mathsf{P}_{\mathrm{G}}}\big(\Phi_1(p), \Phi_1 \circ \Phi_2(p)\big),$$

but also

$$\phi_{\mathsf{P}_{\mathrm{G}}}\big(\Phi_1(p), \Phi_1 \circ \Phi_2(p)\big) = \phi_{\mathsf{P}_{\mathrm{G}}}\Big(\Phi_1(p), \Phi_1\big(p \triangleleft \Phi_P\big(p, \Phi_2(p)\big)\big)\Big)$$

$$= \phi_{\mathsf{P}_{\mathrm{G}}}\Big(\Phi_1(p), \Phi_1(p) \triangleleft \Phi_P\big(p, \Phi_2(p)\big)\Big) = \Phi_P\big(p, \Phi_2(p)\big),$$

and hence

$$\chi\big[(\Phi_1, \mathrm{id}_{\mathrm{G}}, \mathrm{id}_B) \circ (\Phi_2, \mathrm{id}_{\mathrm{G}}, \mathrm{id}_B)\big](p) = \phi_{\mathsf{P}_{\mathrm{G}}}\big(p, \Phi_1(p)\big) \cdot \phi_{\mathsf{P}_{\mathrm{G}}}\big(p, \Phi_2(p)\big)$$

$$\equiv \widetilde{M}\big(\chi\big[(\Phi_1,\mathrm{id}_{\mathrm{G}},\mathrm{id}_B)\big],\chi\big[(\Phi_2,\mathrm{id}_{\mathrm{G}},\mathrm{id}_B)\big]\big)(p)\,,$$

in conformity with our expectations. In the end, we arrive at the group homomorphism

$$S_{\mathrm{Ad.}}\circ\chi\ :\ \mathrm{Aut}_{\mathbf{GrpBun}_{\mathrm{G}}(B)}(\mathsf{P}_{\mathrm{G}}\,|\,B)\longrightarrow\Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}})\,.$$

In order to verify that the latter is the inverse of the previously considered homomorphism $\Psi\circ\Phi_{\mathrm{Ad}}$, it is enough to check that χ is the inverse of the automorphism $\big(\Psi[\cdot],\mathrm{id}_{\mathrm{G}},\mathrm{id}_B\big)$, which we do directly by computing – for arbitrary $(p,g,x)\in\mathsf{P}_{\mathrm{G}}\times\mathrm{G}\times B$ –

$$\begin{aligned}\big(\Psi[\cdot],\mathrm{id}_{\mathrm{G}},\mathrm{id}_B\big)\circ\chi[(\Phi,\mathrm{id}_{\mathrm{G}},\mathrm{id}_B)](p,g,x) &= \big(r_{\phi_{\mathsf{P}_{\mathrm{G}}}(p,\Phi(p))}(p),g,x\big)=\big(\Phi(p),g,x\big)\\ &\equiv (\Phi,\mathrm{id}_{\mathrm{G}},\mathrm{id}_B)(p,g,x)\end{aligned}$$

and

$$\chi\circ\big(\Psi[\cdot],\mathrm{id}_{\mathrm{G}},\mathrm{id}_B\big)[\gamma](p)=\phi_{\mathsf{P}_{\mathrm{G}}}\big(p,r_{\gamma(p)}(p)\big)=\gamma(p)\,.$$

□

Extra constructions

In this closing section, we present two specialised results that form the basis of applications of the theory of associated bundles in the modelling of physical phenomena, and in particular – of the so-called Higgs effect.

Proposition 7. Adopt the notation of Thm. 5.3 and Prop. 1 and let G be a Lie group, $\mathrm{H}\subseteq\mathrm{G}$ – its arbitrary closed subgroup, and $(\mathsf{P}_{\mathrm{G}},B,\mathrm{G},\pi_{\mathsf{P}_{\mathrm{G}}})$ – a principal bundle. The canonical projection $\pi_{\mathrm{G/H}}\ :\ \mathrm{G}\longrightarrow\mathrm{G/H}$ induces an associated-bundle invariant

$$\Phi[\pi_{\mathrm{G/H}}]\ :\ \mathsf{P}_{\mathrm{G}}\times_\ell\mathrm{G}\longrightarrow\mathsf{P}_{\mathrm{G}}\times_{[\ell]}\mathrm{G/H}$$

which, in conjunction with the canonical isomorphism $\widetilde{\iota}$ from Example 1 (3), define a fibre-bundle morphism

$$\phi_{\pi_{\mathrm{G/H}}}:=\Phi[\pi_{\mathrm{G/H}}]\circ\widetilde{\iota}^{-1}\ :\ \mathsf{P}_{\mathrm{G}}\longrightarrow\mathsf{P}_{\mathrm{G}}\times_{[\ell]}\mathrm{G/H}$$

that induces the structure of a principal bundle on

$$\big(\mathsf{P}_{\mathrm{G}},\mathsf{P}_{\mathrm{G}}\times_{[\ell]}\mathrm{G/H},\mathrm{H},\phi_{\pi_{\mathrm{G/H}}}\big)$$

and a bundle isomorphism

$$[\widetilde{\iota}]^{-1}\ :\ \mathsf{P}_{\mathrm{G}}/\mathrm{H}\xrightarrow{\cong}\mathsf{P}_{\mathrm{G}}\times_{[\ell]}\mathrm{G/H}\,.$$

<u>*Proof:*</u> The canonical projection $\pi_{\mathrm{G/H}}$ is a G-equivariant map,

$$\forall_{g,\widetilde{g}\in\mathrm{G}}\ :\ \pi_{\mathrm{G/H}}\circ\ell_{\widetilde{g}}(g)=\pi_{\mathrm{G/H}}(\widetilde{g}\cdot g)=(\widetilde{g}\cdot g)\mathrm{H}\equiv[\ell]_{\widetilde{g}}(gH)\equiv[\ell]_{\widetilde{g}}\circ\pi_{\mathrm{G/H}}(g)\,,$$

and so, in the light of Prop. 1, it induces an associated-bundle invariant as in the statement of the proposition under consideration, which – in its turn – provides us with a bundle morphism

$$\phi_{\pi_{\mathrm{G/H}}}\ :\ \mathsf{P}_{\mathrm{G}}\longrightarrow\mathsf{P}_{\mathrm{G}}\times_{[\ell]}\mathrm{G/H}\ :\ p\longmapsto\Phi[\pi_{\mathrm{G/H}}]\big([(p,e)]\big)\,,$$

that we may rewrite as

$$\begin{aligned}\phi_{\pi_{\mathrm{G/H}}}(p) &\equiv [p]_{[\ell]}\circ\pi_{\mathrm{G/H}}\circ[p]_{[\ell]}^{-1}\big([(p,e)]\big)\\ &= [p]_{[\ell]}\circ\pi_{\mathrm{G/H}}\big(\phi_{\mathsf{P}_{\mathrm{G}}}(p,p)\cdot e\big)=[p]_{[\ell]}\circ\pi_{\mathrm{G/H}}(e)=[p]_{[\ell]}(\mathrm{H})=[(p,\mathrm{H})]\,,\end{aligned}$$

and subsequently examine in detail. The morphism is a superposition of surjective submersions,

$$\phi_{\pi_{\mathrm{G/H}}}=[\cdot]_{[\ell]}\circ\pi_{\mathrm{G/H}}\circ[\cdot]_{[\ell]}^{-1}\circ\widetilde{\iota}^{-1}\,,$$

and as such is itself a surjective submersion. Its level sets are orbits orbitami of the action of the subgroup H. Indeed, for any $p_1,p_2\in\mathsf{P}_{\mathrm{G}}$, we find the equivalences

$$\phi_{\pi_{\mathrm{G/H}}}(p_2)=\phi_{\pi_{\mathrm{G/H}}}(p_1)\qquad\Longleftrightarrow\qquad[(p_2,\mathrm{H})]=[(p_1,\mathrm{H})]$$

$$\Longleftrightarrow \qquad \exists_{g\in \mathrm{G}} \ : \ (p_2, \mathrm{H}) = \big(p_1 \triangleleft g^{-1}, g\mathrm{H}\big) \qquad \Longleftrightarrow \qquad \exists_{g\in \mathrm{H}} \ : \ p_2 = p_1 \triangleleft g^{-1}$$

$$\Longleftrightarrow \qquad p_2 \in p_1 \triangleleft \mathrm{H} .$$

We also have, for arbitrary $(p_1, p_2) \in \mathsf{P}_{\mathrm{G}} \times_{\mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H}} \mathsf{P}_{\mathrm{G}}$, the relation $p_2 = p_1 \triangleleft \phi_{\mathsf{P}_{\mathrm{G}}}(p_1, p_2)$, and so also the map

$$\widetilde{\phi}_{\mathsf{P}_{\mathrm{G}}} \ : \ \mathsf{P}_{\mathrm{G}} \times_{\mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H}} \mathsf{P}_{\mathrm{G}} \longrightarrow \mathrm{H} \ : \ (p_1, p_2) \longmapsto \phi_{\mathsf{P}_{\mathrm{G}}}(p_1, p_2) \, ,$$

determined uniquely by the condition of belonging to a common level set of $\phi_{\pi_{\mathrm{G/H}}}$, is manifestly smooth. Accordingly, we may invoke Prop. 6.2 to conclude that

$$\begin{array}{ccc} \mathrm{H} & \longrightarrow & \mathsf{P}_{\mathrm{G}} \\ & & \Big\downarrow{\scriptstyle \phi_{\pi_{\mathrm{G/H}}}} \\ & & \mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H} \end{array}$$

is, in fact, a principal bundle with the structure group H. On the basis of Prop. Niezb-10, we note, next, that in view of surjective submersivity of the projection $\pi_{\mathsf{P}_{\mathrm{G}}/\mathrm{H}}$ and smoothness of $\phi_{\pi_{\mathrm{G/H}}}$, there exists a unique map

$$[\widetilde{\iota}]^{-1} \ : \ \mathsf{P}_{\mathrm{G}}/\mathrm{H} \longrightarrow \mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H}$$

that closes the commutative diagram

$$\begin{array}{ccc} & & \mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H} \\ & {\scriptstyle \phi_{\pi_{\mathrm{G/H}}}} \nearrow & \Big\uparrow{\scriptstyle [\widetilde{\iota}]^{-1}} \\ \mathsf{P}_{\mathrm{G}} & \xrightarrow[\pi_{\mathsf{P}_{\mathrm{G}}/\mathrm{H}}]{} & \mathsf{P}_{\mathrm{G}}/\mathrm{H} \end{array} .$$

Swapping the rôles of the maps $\pi_{\mathsf{P}_{\mathrm{G}}/\mathrm{H}}$ and $\phi_{\pi_{\mathrm{G/H}}}$ (*i.e.*, in particular, using surjective submersivity of the latter), we arrive at the commutative diagram

$$\begin{array}{ccc} & & \mathsf{P}_{\mathrm{G}}/\mathrm{H} \\ & {\scriptstyle \pi_{\mathsf{P}_{\mathrm{G}}/\mathrm{H}}} \nearrow & \Big\uparrow{\scriptstyle [\widetilde{\iota}]} \\ \mathsf{P}_{\mathrm{G}} & \xrightarrow[\phi_{\pi_{\mathrm{G/H}}}]{} & \mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H} \end{array}$$

whose existence ensures a diffeomorphic character of $[\widetilde{\iota}]$. This diffeomorphism preserves level sets of the respective urjective submersions ($\pi_{\mathsf{P}_{\mathrm{G}}/\mathrm{H}}$ and $\phi_{\pi_{\mathrm{G/H}}}$) since for any point $p \in \mathsf{P}_{\mathrm{G}}$, we obtain equalities

$$\begin{aligned} \pi_{\mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H}} \circ [\widetilde{\iota}]^{-1}(\mathsf{p} \triangleleft \mathrm{H}) \quad &\equiv \quad \pi_{\mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H}} \circ [\widetilde{\iota}]^{-1} \circ \pi_{\mathsf{P}_{\mathrm{G}}/\mathrm{H}}(\mathsf{p}) = \pi_{\mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H}} \circ \phi_{\pi_{\mathsf{P}_{\mathrm{G}}/\mathrm{H}}}(\mathsf{p}) \\ &= \quad \pi_{\mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H}} \circ \Phi[\pi_{\mathsf{P}_{\mathrm{G}}/\mathrm{H}}] \circ \widetilde{\iota}^{-1}(\mathsf{p}) = \pi_{\mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H}}\big([(p, \mathrm{H})]\big) \\ &\equiv \quad \pi_{\mathsf{P}_{\mathrm{G}}}(p) \, . \end{aligned}$$

Consequently, we may employ it to induce on $\mathsf{P}_{\mathrm{G}}/\mathrm{H}$ the structure of a fibre bundle with respect to which $[\widetilde{\iota}]$ is (tautologically) a fibre-bundle isomorphism. □

Proposition 8. Adopt the notation of Def. 1 and Thm. 5.3 and let $\mathrm{H}_\alpha,\ \alpha \in \{1,2\}$ be closed subgroups of the Lie group G that are mutually **conjugate**, i.e., such that there exists an element $g_{21} \in \mathrm{G}$ with the property

$$\mathrm{H}_2 = \mathrm{Ad}_{g_{21}}(\mathrm{H}_1)\,,$$

and let $(\mathsf{P}_\mathrm{G}, B, \mathrm{G}, \pi_{\mathsf{P}_\mathrm{G}})$ be a principal bundle. The G-equivariant diffeomorphism

$$[\wp_{21}]\ :\ \mathrm{G}/\mathrm{H}_1 \xrightarrow{\cong} \mathrm{G}/\mathrm{H}_2\ :\ g\mathrm{H}_1 \longmapsto \big(g \cdot g_{21}^{-1}\big)\mathrm{H}_2$$

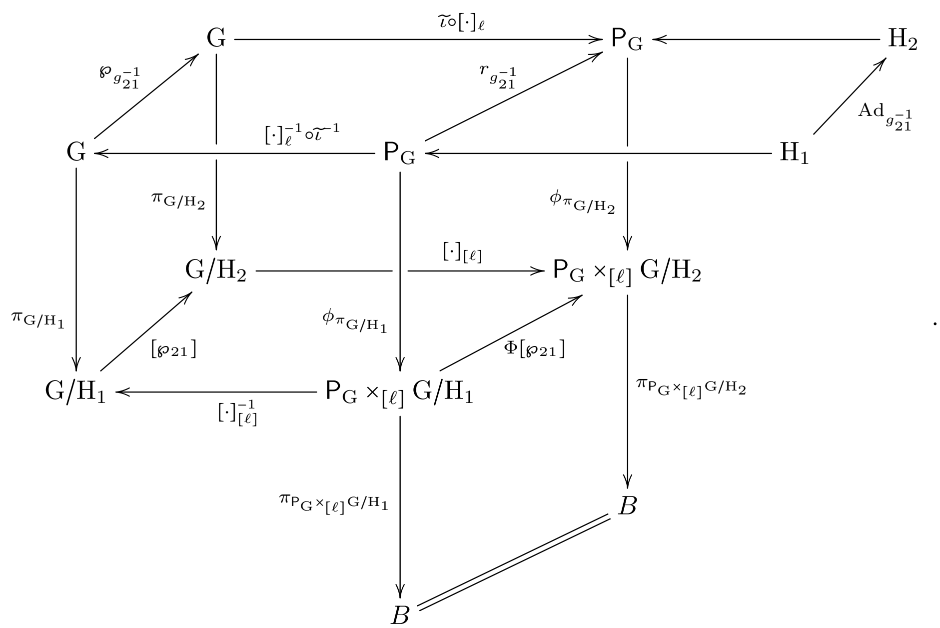
induces an associated-bundle isomorphism

$$\Phi[\wp_{21}]\ :\ \mathsf{P}_\mathrm{G} \times_{[\ell]} \mathrm{G}/\mathrm{H}_1 \xrightarrow{\cong} \mathsf{P}_\mathrm{G} \times_{[\ell]} \mathrm{G}/\mathrm{H}_2$$

which extends to an isomorphism of principal bundles

$$\big(r_{g_{21}^{-1}}, \Phi[\wp_{21}], \mathrm{Ad}_{g_{21}}\big)\ :\ \big(\mathsf{P}_\mathrm{G}, \mathsf{P}_\mathrm{G} \times_{[\ell]} \mathrm{G}/\mathrm{H}_1, \mathrm{H}_1, \phi_{\pi_{\mathsf{P}_\mathrm{G}/\mathrm{H}_1}}\big)$$
$$\xrightarrow{\cong} \big(\mathsf{P}_\mathrm{G}, \mathsf{P}_\mathrm{G} \times_{[\ell]} \mathrm{G}/\mathrm{H}_2, \mathrm{H}_2, \phi_{\pi_{\mathsf{P}_\mathrm{G}/\mathrm{H}_2}}\big)\,,$$

and, in so doing, closes the commutative diagram



.

<u>*Proof:*</u> The diffeomorphism $[\wp_{21}]$ is well-defined since for any $h \in \mathrm{H}_1$, we have

$$\big(g \cdot h \cdot g_{21}^{-1}\big)\mathrm{H}_2 = \big(g \cdot g_{21} \cdot \mathrm{Ad}_{g_{21}}(h)\big)\mathrm{H}_2 = \big(g \cdot g_{21}^{-1}\big)\mathrm{H}_2\,,$$

and manifestly G-invariant. Therefore, it determines an invertible associated-bundle invariant, or an isomorphism between the associated bundles

$$\Phi[\wp_{21}]\ :\ \mathsf{P}_\mathrm{G} \times_{[\ell]} \mathrm{G}/\mathrm{H}_1 \xrightarrow{\cong} \mathsf{P}_\mathrm{G} \times_{[\ell]} \mathrm{G}/\mathrm{H}_2\ :\ [(p, g\mathrm{H}_1)] \longmapsto \big[\big(p, \big(g \cdot g_{21}^{-1}\big)\mathrm{H}_2\big)\big]\,,$$

which – clearly – is covered by the map $r_{g_{21}^{-1}}$. Indeed, for any $p \in \mathsf{P}_\mathrm{G}$, we obtain the equality

$$\begin{aligned} \phi_{\pi_{\mathrm{G}/\mathrm{H}_2}} \circ r_{g_{21}^{-1}}(p) &\equiv \phi_{\pi_{\mathrm{G}/\mathrm{H}_2}}\big(p \triangleleft g_{21}^{-1}\big) = \big[\big(p \triangleleft g_{21}^{-1}, \mathrm{H}_2\big)\big] = \big[\big(p, g_{21}^{-1}\mathrm{H}_2\big)\big] \\ &\equiv \Phi[\wp_{21}]\big([(p, \mathrm{H}_1)]\big) = \Phi[\wp_{21}] \circ \phi_{\pi_{\mathrm{G}/\mathrm{H}_1}}(p)\,. \end{aligned}$$

Equivariance of $r_{g_{21}^{-1}}$ relative to the actions of the subgroups $\mathrm{H}_1 \ni h_1$ and H_2,

$$r_{g_{21}^{-1}}(p \triangleleft h_1) = p \triangleleft \big(h_1 \cdot g_{21}^{-1}\big) \equiv p \triangleleft \big(g_{21}^{-1} \cdot \mathrm{Ad}_{g_{21}}(h_1)\big) \equiv r_{g_{21}^{-1}}(p) \triangleleft \mathrm{Ad}_{g_{21}}(h_1)\,,$$

enables us to identify $\left(r_{g_{21}^{-1}}, \Phi[\wp_{21}], \mathrm{Ad}_{g_{21}}\right)$ as an isomorphism of principal bundles. Finally, we check commutativity of the subdiagrams containing both vertices labelled by G. Thus, over any point $p \in \mathsf{P}_{\mathrm{G}}$,

$$\begin{aligned}\widetilde{\iota} \circ [p]_\ell \circ \wp_{g_{21}^{-1}} \circ [p]_\ell^{-1} \circ \widetilde{\iota}^{-1}(p) &= \widetilde{\iota} \circ [p]_\ell \circ \wp_{g_{21}^{-1}} \circ [p]_\ell^{-1}\big([(p,e)]\big) = \widetilde{\iota} \circ [p]_\ell \circ \wp_{g_{21}^{-1}}(e) \\ &= \widetilde{\iota} \circ [p]_\ell\big(g_{21}^{-1}\big) = \widetilde{\iota}\big(\big[\big(p, g_{21}^{-1}\big)\big]\big) = p \triangleleft g_{21}^{-1} \equiv r_{g_{21}^{-1}}(p)\,,\end{aligned}$$

and, moreover, for any element $g \in \mathrm{G}$,

$$\pi_{\mathrm{G}/\mathrm{H}_2} \circ \wp_{g_{21}^{-1}}(g) = \pi_{\mathrm{G}/\mathrm{H}_2}\big(g \cdot g_{21}^{-1}\big) = \big(g \cdot g_{21}^{-1}\big)\mathrm{H}_2 \equiv [\wp_{g_{21}^{-1}}](g\mathrm{H}_1) \equiv [\wp_{g_{21}^{-1}}] \circ \pi_{\mathrm{G}/\mathrm{H}_1}(g)\,.$$

□