CLASSICAL FIELD THEORY II IN THE TIME OF COVID-19
6. LECTURE BATCH

Principal bundles with a structure group

Our hitherto tour d'horizon in the realm of Lie groups and algebras has equipped us with intuitions and formal tools indispensable for an efficient and constructive navigation in the environment of geometric structures central to the modelling of local symmetries (or, more accurately, those rendered local, or gauged) in classical mechanics and field theory. We now return to a detailed discussion thereof.

Definition 1. Let G be a Lie group. A **principal bundle** with a **structure group** G is a fibre bundle

$$(\mathsf{P}_{\mathrm{G}}, B, \mathrm{G}, \pi_{\mathsf{P}_{\mathrm{G}}})$$

composed of

- a total space P_{G} with a free right action $r.$ of the structure group G that is transitive in the fibre over a point (arbitrary) of the base, as described by the commutative diagram

$$\begin{array}{ccc} \mathsf{P}_{\mathrm{G}} \times \mathrm{G} & \xrightarrow{\;\;r\;\;} & \mathsf{P}_{\mathrm{G}} \\ {\scriptstyle \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathsf{P}_{\mathrm{G}}}} \\ \mathsf{P}_{\mathrm{G}} & \xrightarrow[\;\;\pi_{\mathsf{P}_{\mathrm{G}}}\;\;]{} & B \end{array} \;;$$

- local trivialisations

$$\tau_i \;:\; \pi_{\mathsf{P}_{\mathrm{G}}}^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times \mathrm{G}\,, \quad i \in I$$

  associated with an open cover $\mathscr{O} = \{\mathcal{O}_i\}_{i\in I}$ of the base B and G-equivariant with respect to the right actions: r on the domain and the regular one $\wp$ on the second cartesian factor of the codomain,

$$\widetilde{\wp}^i \equiv \mathrm{id}_{\mathcal{O}_i} \times \wp \;:\; (\mathcal{O}_i \times \mathrm{G}) \times \mathrm{G} \longrightarrow \mathcal{O}_i \times \mathrm{G} \;:\; \big((x,g),h\big) \longmapsto (x, g\cdot h)\,,$$

  i.e., satisfying the conditions

$$\tau_i \circ r_g = \widetilde{\wp}^i_g \circ \tau_i\,, \qquad i \in I\,.$$

A **principal subbundle with a structure group** H of a principal bundle $(\mathsf{P}_{\mathrm{G}}, B, \mathrm{G}, \pi_{\mathsf{P}_{\mathrm{G}}})$ is a subbundle $(\mathsf{P}_{\mathrm{H}}, B, \mathrm{H}, \pi_{\mathsf{P}_{\mathrm{G}}}\restriction_{\mathsf{P}_{\mathrm{H}}})$ tof that bundle with a structure group given by a Lie subgroup $\mathrm{H} \subset \mathrm{G}$.

A **morphism of principal bundles** $(\mathsf{P}_{\mathrm{G}_\alpha}, B_\alpha, \mathrm{G}_\alpha, \pi_{\mathsf{P}_{\mathrm{G}_\alpha}})$, $\alpha \in \{1,2\}$ with the defining actions of the respective structure groups r^α is a triple (Φ, f, φ) composed of a morphism of fibre bundles

$$(\Phi, f) \;:\; (\mathsf{P}_{\mathrm{G}_1}, B_1, \mathrm{G}_1, \pi_{\mathsf{P}_{\mathrm{G}_1}}) \longrightarrow (\mathsf{P}_{\mathrm{G}_2}, B_2, \mathrm{G}_2, \pi_{\mathsf{P}_{\mathrm{G}_2}})$$

and of a Lie-group homomorphism related as in the commutative diagram

$$\begin{array}{ccccc} \mathsf{P}_{\mathrm{G}_1} \times \mathrm{G}_1 & \xrightarrow{\;\;r^1\;\;} & \mathsf{P}_{\mathrm{G}_1} & \xrightarrow{\;\;\pi_{\mathsf{P}_{\mathrm{G}_1}}\;\;} & B_1 \\ {\scriptstyle \Phi\times\varphi}\big\downarrow & & \big\downarrow{\scriptstyle \Phi} & & \big\downarrow{\scriptstyle f} \\ \mathsf{P}_{\mathrm{G}_2} \times \mathrm{G}_2 & \xrightarrow[\;\;r^2\;\;]{} & \mathsf{P}_{\mathrm{G}_2} & \xrightarrow[\;\;\pi_{\mathsf{P}_{\mathrm{G}_2}}\;\;]{} & B_2 \end{array} \;. \tag{1}$$

Examples 1.

(1) The trivial principal bundle over B with a structure group G, *i.e.*,

$$(B \times \mathrm{G}, B, \mathrm{G}, \mathrm{pr}_1)\,.$$

(2) The frame bundle of a vector bundle $\mathbb{V}$ modelled on $\mathbb{K}^{\times n}$, *i.e.*,

$$\big(\mathsf{F}_{\mathrm{GL}}\mathbb{V}, B, \mathrm{GL}(n;\mathbb{K}), \pi_{\mathsf{F}_{\mathrm{GL}}\mathbb{V}}\big)\,,$$

and in particular –the **frame bundle over a manifold** M of dimension n, *i.e.*,

$$\big(\mathsf{F}_{\mathrm{GL}}\mathsf{T}M, M, \mathrm{GL}(\mathsf{T}M) \cong \mathrm{GL}(n;\mathbb{R}), \pi_{\mathsf{F}_{\mathrm{GL}}\mathsf{T}M}\big)\,.$$

(3) The **Hopf fibration**

$$\big(\mathrm{SU}(2) \equiv \mathbb{S}^3, \mathbb{S}^2, \mathrm{U}(1), \pi_{\mathrm{SU}(2)/\mathrm{U}(1)}\big)\,.$$

Definition 2. Adopt the notation of Def. 1 and let

$$\mathsf{P}_{\mathrm{G}} \times_B \mathsf{P}_{\mathrm{G}} := \{\ (p_1, p_2) \in \mathsf{P}_{\mathrm{G}} \times \mathsf{P}_{\mathrm{G}} \quad | \quad \pi_{\mathsf{P}_{\mathrm{G}}}(p_1) = \pi_{\mathsf{P}_{\mathrm{G}}}(p_2)\ \}$$

be the fibred square of the total space of a principal bundle. The **quotient map** on the principal bundle $(\mathsf{P}_{\mathrm{G}}, B, \mathrm{G}, \pi_{\mathsf{P}_{\mathrm{G}}})$ is the mapping

$$\phi_{\mathsf{P}_{\mathrm{G}}}\ :\ \mathsf{P}_{\mathrm{G}} \times_B \mathsf{P}_{\mathrm{G}} \longrightarrow \mathrm{G}$$

determined by the condition

$$\forall_{(p_1,p_2) \in \mathsf{P}_{\mathrm{G}} \times_B \mathsf{P}_{\mathrm{G}}}\ :\ p_2 = p_1 \triangleleft \phi_{\mathsf{P}_{\mathrm{G}}}(p_1, p_2)\,.$$

Remark 1. The postulated smoothness of the quotient map is most readily checked with the help of local trivialisations. Indeed, let $p_1, p_2 \in (\mathsf{P}_{\mathrm{G}})_x,\ x \in \mathcal{O}_i,\ i \in I$, with $p_\alpha = \tau_i^{-1}(x, g_\alpha),\ \alpha \in \{1,2\}$ for some elements $g_\alpha \in \mathrm{G}$, and then, in virtue of the identity

$$p_2 = \tau_i^{-1}(x, g_2) \equiv \tau_i^{-1}\big(x, g_1 \cdot (g_1^{-1} \cdot g_2)\big) = \tau_i^{-1}(x, g_1) \triangleleft \big(g_1^{-1} \cdot g_2\big) \equiv p_1 \triangleleft \big(g_1^{-1} \cdot g_2\big)\,,$$

we recover a local presentation of the quotient map:

$$\phi_{\mathsf{P}_{\mathrm{G}}}(p_1, p_2) \equiv g_1^{-1} \cdot g_2 = \mathrm{m}\big(\mathrm{Inv} \circ \mathrm{pr}_2 \circ \tau_i(p_1), \mathrm{pr}_2 \circ \tau_i(p_2)\big)\,,$$

given as a superposition of smooth maps (m is the binary operation in G), and hence smooth.

Basic structural properties of the quotient map, to be employed presently, are described in

Proposition 1. Adopt the notation of Def. 2. The quotient map satisfies conditions expressed by the following commutative diagrams

(DM1) skew symmetry

$$\begin{array}{ccc} \mathsf{P}_{\mathrm{G}} \times_B \mathsf{P}_{\mathrm{G}} & \xrightarrow{\tau_{\mathsf{P}_{\mathrm{G}},\mathsf{P}_{\mathrm{G}}}} & \mathsf{P}_{\mathrm{G}} \times_B \mathsf{P}_{\mathrm{G}} \\ \Big\downarrow{\scriptstyle \phi_{\mathsf{P}_{\mathrm{G}}}} & & \Big\downarrow{\scriptstyle \phi_{\mathsf{P}_{\mathrm{G}}}} \\ \mathrm{G} & \xrightarrow[\mathrm{Inv}]{} & \mathrm{G} \end{array}\,,$$

where $\tau_{\mathsf{P}_{\mathrm{G}},\mathsf{P}_{\mathrm{G}}}\ :\ \mathsf{P}_{\mathrm{G}} \times_B \mathsf{P}_{\mathrm{G}} \circlearrowleft\ :\ (p_1, p_2) \longmapsto (p_2, p_1)$ is the canonical transposition (restricted to the fibred square), *i.e.*,

$$\forall_{(p_1,p_2) \in \mathsf{P}_{\mathrm{G}} \times_B \mathsf{P}_{\mathrm{G}}}\ :\ \phi_{\mathsf{P}_{\mathrm{G}}}(p_2, p_1) = \phi_{\mathsf{P}_{\mathrm{G}}}(p_1, p_2)^{-1}\,;$$

(DM2) the 1-cocycle condition

$$\begin{array}{ccc} \mathsf{P}_{\mathrm{G}} \times_B \mathsf{P}_{\mathrm{G}} \times_B \mathsf{P}_{\mathrm{G}} & \xrightarrow{(\phi_{\mathsf{P}_{\mathrm{G}}} \circ \mathrm{pr}_{1,2}, \phi_{\mathsf{P}_{\mathrm{G}}} \circ \mathrm{pr}_{2,3})} & \mathrm{G} \times \mathrm{G} \\ & \searrow{\scriptstyle \phi_{\mathsf{P}_{\mathrm{G}}} \circ \mathrm{pr}_{1,3}} & \Big\downarrow{\scriptstyle M} \\ & & \mathrm{G} \end{array}\,,$$

where $\mathrm{pr}_{i,j} : \mathsf{P}_{\mathrm{G}} \times_B \mathsf{P}_{\mathrm{G}} \times_B \mathsf{P}_{\mathrm{G}} \longrightarrow \mathsf{P}_{\mathrm{G}} \times_B \mathsf{P}_{\mathrm{G}} : (p_1, p_2, p_3) \longmapsto (p_i, p_j),\ (i,j) \in \{(1,2),(2,3),(1,3)\}$ is the canonical projection (restricted to the fibred cube), *i.e.*,

$$\forall_{(p_1,p_2,p_3)\in\mathsf{P}_{\mathrm{G}}\times_B\mathsf{P}_{\mathrm{G}}\times_B\mathsf{P}_{\mathrm{G}}} \ :\ \phi_{\mathsf{P}_{\mathrm{G}}}(p_2,p_3) \circ \phi_{\mathsf{P}_{\mathrm{G}}}(p_1,p_3)^{-1} \circ \phi_{\mathsf{P}_{\mathrm{G}}}(p_1,p_2) = e\,;$$

(DM3) G-equivariance

$$\begin{array}{ccccc} \mathsf{P}_{\mathrm{G}} \times_B \mathsf{P}_{\mathrm{G}} & \xleftarrow{(r\circ\mathrm{pr}_{1,3},\mathrm{pr}_2)} & (\mathsf{P}_{\mathrm{G}} \times_B \mathsf{P}_{\mathrm{G}}) \times \mathrm{G} & \xrightarrow{\mathrm{id}_{\mathsf{P}_{\mathrm{G}}}\times r} & \mathsf{P}_{\mathrm{G}} \times_B \mathsf{P}_{\mathrm{G}} \\ \downarrow{\scriptstyle \phi_{\mathsf{P}_{\mathrm{G}}}} & & \downarrow{\scriptstyle \phi_{\mathsf{P}_{\mathrm{G}}}\times\mathrm{id}_{\mathrm{G}}} & & \downarrow{\scriptstyle \phi_{\mathsf{P}_{\mathrm{G}}}} \\ \mathrm{G} & \xleftarrow[\ell\circ(\mathrm{Inv}\times\mathrm{id}_{\mathrm{G}})\circ\tau_{\mathrm{G},\mathrm{G}}]{} & \mathrm{G}\times\mathrm{G} & \xrightarrow[\wp]{} & \mathrm{G} \end{array}\ ,$$

i.e.,

$$\forall_{(p_1,p_2)\in\mathsf{P}_{\mathrm{G}}\times_B\mathsf{P}_{\mathrm{G}},\ g_1,g_2\in\mathrm{G}} \ :\ \phi_{\mathsf{P}_{\mathrm{G}}}(p_1 \triangleleft g_1, p_2 \triangleleft g_2) = g_1^{-1} \cdot \phi_{\mathsf{P}_{\mathrm{G}}}(p_1,p_2) \cdot g_2\,.$$

<u>*Proof:*</u> Obvious. □

Sometimes, we do not have the full 'package' of objects enumerated in the definition of a principal bundle, or the information is encoded differently. Therefore, it seems apposite to ponder which of its elements are of a *constitutive* nature in that they determine existence of the structure of a principal bundle on a manifold. An answer to a question thus posed is given in

Proposition 2. Let P, B be manifolds and let G be a Lie group. Assume, moreover, that there exists a surjective submersion $\pi : \mathsf{P} \longrightarrow B$ and a smooth right action $r_\cdot : \mathsf{P} \times \mathrm{G} \longrightarrow \mathsf{P}$ of G on P. If $r_\cdot$ is free, its orbits coincide with level sets of π, and the map $\phi_{\mathsf{P}} : \mathsf{P} \times_B \mathsf{P} \longrightarrow \mathrm{G}$ determined (uniquely) by the condition

$$\forall_{(p_1,p_2)\in\mathsf{P}\times_B\mathsf{P}} \ :\ p_2 = r_{\phi_{\mathsf{P}}(p_1,p_2)}(p_1)$$

is smooth, then the quadruple

$$(\mathsf{P}, B, \mathrm{G}, \pi)$$

is a principal bundle.

<u>*Proof:*</u> In virtue of the proposition of lecture I (p. 11) on existence of local sections of a surjective submersion, we infer existece of an open cover $\mathscr{O} = \{\mathcal{O}_i\}_{i\in I}$ of the manifold B over whose elements there are smooth local sections $\sigma_i : \mathcal{O}_i \longrightarrow \mathsf{P}$ of the submersion π, and so we may define the manifestly smooth mappings

$$\tau_i^{-1} \ :\ \mathcal{O}_i \times \mathrm{G} \longrightarrow \pi^{-1}(\mathcal{O}_i) \ :\ (x,g) \longmapsto r_g\big(\sigma_i(x)\big)\,.$$

Using ϕ_{P}, we readily find their (smooth) inverses

$$\tau_i \ :\ \pi^{-1}(\mathcal{O}_i) \longrightarrow \mathcal{O}_i \times \mathrm{G} \ :\ p \longmapsto \big(\pi(p), \phi_{\mathsf{P}}\big(\sigma_i \circ \pi(p), p\big)\big)\,.$$

These are well-defined as

$$\pi\big(\sigma_i \circ \pi(p)\big) = (\pi \circ \sigma_i) \circ \pi(p) = \mathrm{id}_{\mathcal{O}_i} \circ \pi(p) = \pi(p)\,,$$

and so – indeed – satisfy the postulated identities

$$\tau_i^{-1} \circ \tau_i(p) = \tau_i^{-1}\big(\pi(p), \phi_{\mathsf{P}}\big(\sigma_i \circ \pi(p), p\big)\big) = r_{\phi_{\mathsf{P}}(\sigma_i\circ\pi(p),p)}\big(\sigma_i \circ \pi(p)\big) \equiv p\,,$$

$$\begin{aligned} \tau_i \circ \tau_i^{-1}(x,g) &= \tau_i\big(r_g\big(\sigma_i(x)\big)\big) = \big(\pi \circ r_g \circ \sigma_i(x), \phi_{\mathsf{P}}\big(\sigma_i \circ \pi \circ r_g \circ \sigma_i(x), r_g \circ \sigma_i(x)\big)\big) \\ &= \big(\pi \circ \sigma_i(x), \phi_{\mathsf{P}}\big(\sigma_i \circ \pi \circ \sigma_i(x), r_g \circ \sigma_i(x)\big)\big) \end{aligned}$$

$$= \big(x, \phi_{\mathsf{P}}\big(\sigma_i(x), r_g \circ \sigma_i(x)\big)\big) = (x,g)\,,$$

of which the second follows from the fact, that the action of G maps level sets of π into themselves, and

$$\forall_{(p_1,p_2,g)\in\mathsf{P}^{\times 2}\times\mathrm{G}}\ :\ \phi_{\mathsf{P}}\big(p_1, r_g(p_2)\big) = \phi_{\mathsf{P}}(p_1,p_2)\cdot g\,.$$

The local trivialisations constructed above satisfy, at points $x \in \mathcal{O}_{ij}$, $i,j \in I$, the conditions

$$\begin{aligned}\tau_i \circ \tau_j^{-1}(x,g) &= \tau_i\big(r_g \circ \sigma_j(x)\big) = \big(\pi \circ r_g \circ \sigma_j(x), \phi_{\mathsf{P}}\big(\sigma_i \circ \pi \circ r_g \circ \sigma_j(x), r_g \circ \sigma_j(x)\big)\big)\\ &= \big(x, \phi_{\mathsf{P}}\big(\sigma_i(x), r_g \circ \sigma_j(x)\big)\big) = \big(x, \phi_{\mathsf{P}}\big(\sigma_i(x), \sigma_j(x)\big)\cdot g\big)\,,\end{aligned}$$

from which we read off the form of the transition maps

$$g_{ij}\ :\ \mathcal{O}_{ij} \longrightarrow \mathrm{G}\ :\ x \longmapsto \phi_{\mathsf{P}}\big(\sigma_i(x), \sigma_j(x)\big)\,.$$

This concludes our identification of the postulated structure of a principal bundle with the structure group G. □

An example of a principal bundle of prime relevance to the modelling of an effective field theory in the vicinity of a vacuum of a field theory with a continuous symmetry (global or local), and so, in particular, in the description of the Higgs effect, is given in

Corollary 1. Let G be a Lie group, and $\mathrm{H} \subseteq \mathrm{G}$ – its arbitrary closed subgroup. The quadruple

$$(\mathrm{G}, \mathrm{G/H}, \mathrm{H}, \pi_{\mathrm{G/H}})$$

is a principal bundle over the smooth homogeneous space $\mathrm{G/H}$.

Proof: It suffices to note that the projection $\pi_{\mathrm{G/H}}$ is – in conformity with Thm. 3 of lecture V (p. 43) – a surjective submersion with level sets that coincide with orbits of the natural right (regular) action H, and that, furthermore, there exists a manifestly smooth mapping

$$\phi_{\mathrm{G}}\ :\ \mathrm{G} \times_{\mathrm{G/H}} \mathrm{G} \longrightarrow \mathrm{H}\ :\ (g_1, g_2) \longmapsto g_1^{-1}\cdot g_2\,.$$

□

In the previous lecture, we distinguished proper actions as the ones which enable us to descend the structure of a manifold to the space of orbits of the group action. The first example of such an action arises in

Proposition 3. The defining action of the structure group on the total space of a principal bundle is proper.

Proof: Consider sequences $p_\cdot\ :\ \mathbb{N} \longrightarrow \mathsf{P}_{\mathrm{G}}$ and $g_\cdot\ :\ \mathbb{N} \longrightarrow \mathrm{G}$ with properties

$$\lim_{n\to\infty} p_n = p\,, \qquad \lim_{n\to\infty} (p_n \triangleleft g_n) = \widetilde{p}\,.$$

In consequence of continuity of the canonical projection on the base and of the character of the action of the structure group (on the fibre), we obtain the identity

$$\begin{aligned}\pi_{\mathsf{P}_{\mathrm{G}}}(\widetilde{p}) &\equiv \pi_{\mathsf{P}_{\mathrm{G}}}\big(\lim_{n\to\infty}(p_n \triangleleft g_n)\big) = \lim_{n\to\infty} \pi_{\mathsf{P}_{\mathrm{G}}}(p_n \triangleleft g_n) = \lim_{n\to\infty} \pi_{\mathsf{P}_{\mathrm{G}}}(p_n)\\ &= \pi_{\mathsf{P}_{\mathrm{G}}}\big(\lim_{n\to\infty} p_n\big) = \pi_{\mathsf{P}_{\mathrm{G}}}(p)\,,\end{aligned}$$

which enables us – in virtue of Def. 2 – to write

$$\widetilde{p} = p \triangleleft \phi_{\mathsf{P}_{\mathrm{G}}}(p, \widetilde{p})\,.$$

Let $\pi_{\mathsf{P}_{\mathrm{G}}}(p) \in \mathcal{O}_i$, where $\mathcal{O}_i$ is an element of the trivialising cover for P_{G}. There exists an index $N \in \mathbb{N}$ such that

$$\forall_{n\geq N}\ :\ p_n, p_n \triangleleft g_n \in \pi_{\mathsf{P}_{\mathrm{G}}}^{-1}(\mathcal{O}_i)\,,$$

and so we may consider subsequences $p_{N+\cdot}$ and $p_{N+\cdot} \triangleleft g_{N+\cdot}$ in the image of the trivialisation $\tau_i : \pi_{\mathsf{P}_G}^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times \mathrm{G}$, in which

$$\tau_i(p_n) =: (x_n, \gamma_n), \qquad \tau_i(p) =: (x, \gamma),$$

whence

$$\lim_{n\to\infty} (x_n, \gamma_n) = (x, \gamma),$$

and so also

$$\tau_i(p_n \triangleleft g_n) = \tau_i(p_n) \triangleleft g_n = (x_n, \gamma_n) \triangleleft g_n = (x_n, \gamma_n \cdot g_n),$$

as well as

$$\tau_i(\widetilde{p}) = \tau_i\big(p \triangleleft \phi_{\mathsf{P}_G}(p, \widetilde{p})\big) = \tau_i(p) \triangleleft \phi_{\mathsf{P}_G}(p, \widetilde{p}) = (x, \gamma) \triangleleft \phi_{\mathsf{P}_G}(p, \widetilde{p}) = \big(x, \gamma \cdot \phi_{\mathsf{P}_G}(p, \widetilde{p})\big),$$

whence

$$\lim_{n\to\infty} (\gamma_n \triangleleft g_n) = \gamma \cdot \phi_{\mathsf{P}_G}(p, \widetilde{p}).$$

Due to continuity of group operations, we conclude

$$\lim_{n\to\infty} g_n \equiv \lim_{n\to\infty} \big(\gamma_n^{-1} \cdot (\gamma_n \cdot g_n)\big) = \gamma^{-1} \cdot \big(\gamma \cdot \phi_{\mathsf{P}_G}(p, \widetilde{p})\big) = \phi_{\mathsf{P}_G}(p, \widetilde{p}),$$

and so we establish the veracity of the claim of the proposition in virtue of Prop. 3 of lecture V (p. 16). □

Following these lines towards physical applications, we formulate

Corollary 2. Adopt the notation of Def. 1 and let $(\mathsf{P}_G, B, \mathrm{G}, \pi_{\mathsf{P}_G})$ be a principal bundle and M – a manifold with a left action $\lambda : \mathrm{G} \times M \longrightarrow M$ of its structure group G. Consider the product manifold $\mathsf{P}_G \times M$. The action of G on it given by the formula

$$\widetilde{\lambda}_\cdot : \mathrm{G} \times (\mathsf{P}_G \times M) \longrightarrow \mathsf{P}_G \times M : \big(g, (p, x)\big) \longmapsto \big(r(p, g^{-1}), \lambda(g, m)\big) \tag{2}$$

is free and proper.

Proof: Obvious. □

We also have

Proposition 4. Adopt the notation of Def. 1 and let G be a Lie group and $\mathrm{H} \subseteq \mathrm{G}$ – its arbitrary closed subgroup. The action of the subgroup H on the total space P_G of a principal bundle $(\mathsf{P}_G, B, \mathrm{G}, \pi_{\mathsf{P}_G})$ given as the restriction (to H) of the defining action $r_\cdot$ is free and proper, and as such, it determines on P_G the structure of a principal bundle

$$(\mathsf{P}_G, \pi_{\mathsf{P}_G/\mathrm{H}}, \mathsf{P}_G/\mathrm{H}, \mathrm{H}).$$

Proof: The free character of the action of H on P_G is a consequence of the same character of the action of the structure group G on P_G, and so it remains to verify its propreness. Suppose sequences $p_\cdot \in \mathsf{P}_G^{\mathbb{N}}$ and $h_\cdot \in \mathrm{H}^{\mathbb{N}}$ satisfy conditions

$$\lim_{n\to\infty} p_n = p, \qquad \lim_{n\to\infty} (p_n \triangleleft h_n) = \widetilde{p}.$$

Then, owing to properness of the action of $\mathrm{G} \supset \mathrm{H} \ni h_n$, there exists a subsequence $h_{n_\cdot} \in \mathrm{H}^{\mathbb{N}}$ such that

$$\lim_{k\to\infty} h_{n_k} = h \in \mathrm{G}.$$

However, H is closed, and so necessarily $h \in \mathrm{H}$, which implies properness of the action of H in the light of Prop. 3 of lecture V (p. 16). This, in turn, implies that the set P_G/H of orbits carries – in virtue of the Quotient Manifold Thm. 2 of lecture V (p. 20) – the structure of a smooth manifold for which the canonical projection $\pi_{\mathsf{P}_{\rm G}/{\rm H}}$ is a surjective submersion, and since level sets of this map are orbits of the action of H and we have a well-defined mapping

$$\widetilde{\phi}_{\mathsf{P}_{\rm G}}\ :\ \mathsf{P}_{\rm G}\times_{\mathsf{P}_{\rm G}/{\rm H}}\mathsf{P}_{\rm G}\longrightarrow{\rm H}\ :\ (p_1,p_2)\longmapsto\phi_{\mathsf{P}_{\rm G}}(p_1,p_2)\,,$$

that is manifestly smooth, we conclude, on the basis of Prop. 2, the veracity of the second part of the thesis. □

We shall now state a convenient criterion of triviality of a principal bundle, to be compared with the analogous criterion for line bundles, established before.

Proposition 5. There exists a one-to-one correspondence between local sections of class C^∞ of a principal bundle and its local trivialisations In particular, a principal bundle is globally trivialisable iff it has a global section.

Proof: To a local section $\sigma\ :\ \mathcal{O}\longrightarrow\pi_{\mathsf{P}_{\rm G}}^{-1}(\mathcal{O})\subset\mathsf{P}_{\rm G},\ \mathcal{O}\in\mathscr{T}(B)$, we associate a (local) trivialisation

$$\tau_\sigma\ :\ \pi_{\mathsf{P}_{\rm G}}^{-1}(\mathcal{O})\longrightarrow\mathcal{O}\times{\rm G}\ :\ p\longmapsto\big(\pi_{\mathsf{P}_{\rm G}}(p),\phi_{\mathsf{P}_{\rm G}}\big(\sigma\circ\pi_{\mathsf{P}_{\rm G}}(p),p\big)\big)$$

with the desired properties, that is invertible,

$$\tau_\sigma^{-1}\ :\ \mathcal{O}\times{\rm G}\longrightarrow\pi_{\mathsf{P}_{\rm G}}^{-1}(\mathcal{O})\ :\ (x,g)\longmapsto\sigma(x)\triangleleft g\,,$$

and G-equivariant,

$$\begin{aligned}\tau_\sigma(p\triangleleft g)\ &\equiv\ \big(\pi_{\mathsf{P}_{\rm G}}(p\triangleleft g),\phi_{\mathsf{P}_{\rm G}}\big(\sigma\circ\pi_{\mathsf{P}_{\rm G}}(p\triangleleft g),p\triangleleft g\big)\big)=\big(\pi_{\mathsf{P}_{\rm G}}(p),\phi_{\mathsf{P}_{\rm G}}\big(\sigma\circ\pi_{\mathsf{P}_{\rm G}}(p),p\triangleleft g\big)\big)\\ &=\ \big(\pi_{\mathsf{P}_{\rm G}}(p),\phi_{\mathsf{P}_{\rm G}}\big(\sigma\circ\pi_{\mathsf{P}_{\rm G}}(p),p\big)\cdot g\big)=\big(\pi_{\mathsf{P}_{\rm G}}(p),\phi_{\mathsf{P}_{\rm G}}\big(\sigma\circ\pi_{\mathsf{P}_{\rm G}}(p),p\big)\big)\triangleleft g\equiv\tau_\sigma(p)\triangleleft g\,.\end{aligned}$$

And conversely, to an arbitrary local trivialisation $\tau\ :\ \pi_{\mathsf{P}_{\rm G}}^{-1}(\mathcal{O})\xrightarrow{\cong}\mathcal{O}\times{\rm G}$, we associate a (local) section

$$\sigma_\tau\ :\ \mathcal{O}\longrightarrow\pi_{\mathsf{P}_{\rm G}}^{-1}(\mathcal{O})\ :\ x\longmapsto\tau^{-1}(x,e)\,.$$

The above assignments are mutually inverse. Indeed, on the one hand,

$$\forall_{x\in\mathcal{O}}\ :\ \sigma_{\tau_\sigma}(x)=\tau_\sigma^{-1}(x,e)=\sigma(x)\triangleleft e=\sigma(x)\,,$$

and on the other,

$$\forall_{p\in\pi_{\mathsf{P}_{\rm G}}^{-1}(\mathcal{O})}\ :\ \tau_{\sigma_\tau}(p)=\big(\pi_{\mathsf{P}_{\rm G}}(p),\phi_{\mathsf{P}_{\rm G}}\big(\sigma_\tau\circ\pi_{\mathsf{P}_{\rm G}}(p),p\big)\big)=\big(\pi_{\mathsf{P}_{\rm G}}(p),\phi_{\mathsf{P}_{\rm G}}\big(\tau^{-1}\big(\pi_{\mathsf{P}_{\rm G}}(p),e\big),p\big)\big)\,,$$

but since

$$p\equiv\tau^{-1}\big(\pi_{\mathsf{P}_{\rm G}}(p),e\big)\triangleleft\phi_{\mathsf{P}_{\rm G}}\big(\tau^{-1}\big(\pi_{\mathsf{P}_{\rm G}}(p),e\big),p\big)\,,$$

so that

$$\begin{aligned}\tau(p)\ &=\ \tau\big(\tau^{-1}\big(\pi_{\mathsf{P}_{\rm G}}(p),e\big)\triangleleft\phi_{\mathsf{P}_{\rm G}}\big(\tau^{-1}\big(\pi_{\mathsf{P}_{\rm G}}(p),e\big),p\big)\big)=\tau\circ\tau^{-1}\big(\pi_{\mathsf{P}_{\rm G}}(p),e\big)\triangleleft\phi_{\mathsf{P}_{\rm G}}\big(\tau^{-1}\big(\pi_{\mathsf{P}_{\rm G}}(p),e\big),p\big)\\ &=\ \big(\pi_{\mathsf{P}_{\rm G}}(p),e\big)\triangleleft\phi_{\mathsf{P}_{\rm G}}\big(\tau^{-1}\big(\pi_{\mathsf{P}_{\rm G}}(p),e\big),p\big)=\big(\pi_{\mathsf{P}_{\rm G}}(p),\phi_{\mathsf{P}_{\rm G}}\big(\tau^{-1}\big(\pi_{\mathsf{P}_{\rm G}}(p),e\big),p\big)\big)\,,\end{aligned}$$

we obtain

$$\tau_{\sigma_\tau}(p)=\tau(p)\,.$$

□

Remark 2. It deserves to be emphasised that the last part of the statement of the above proposition does *not* apply to fibre bundles in general. In order to convince onself of that, it suffices to note that every vector bundle $\mathbb{V}$ has a global section, to wit, the zero section $\mathbf{0}_{\mathbb{V}}$, but not all of them are trivial.

Next, we establish a convenient local description of those morphisms between principal bundles which cover the identity diffeomorphism on the base (it is among these that we subsequently recognise the so-called "gauge transformations").

Theorem 1. Adopt the notation of Def. 1. Any morphism $(\Phi,\ \mathrm{id}_B, \mathrm{id}_{\mathrm{G}})$ between principal bundles $(\mathsf{P}^\alpha_{\mathrm{G}}, B, \mathrm{G}, \pi_{\mathsf{P}^\alpha_{\mathrm{G}}})$, $\alpha \in \{1,2\}$ with the respective local trivialisations $\tau^\alpha_i\ :\ \pi^{-1}_{\mathsf{P}^\alpha_{\mathrm{G}}}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times \mathrm{G}$ (associated with the common trivialising cover $\mathscr{O} = \{\mathcal{O}_i\}_{i\in I}$) and transition maps $g^\alpha_{ij}\ :\ \mathcal{O}_{ij} \longrightarrow \mathrm{G}$), described by the commutative diagram

$$\begin{array}{ccc} \mathsf{P}^1_{\mathrm{G}} & \xrightarrow{\ \Phi\ } & \mathsf{P}^2_{\mathrm{G}} \\ {\scriptstyle \pi_{\mathsf{P}^1_{\mathrm{G}}}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathsf{P}^2_{\mathrm{G}}}} \\ B & \underset{\mathrm{id}_B}{=\!=\!=} & B \end{array} ,$$

determines a family $\{h_i\}_{i\in I}$ of locally smooth mappings

$$h_i\ :\ \mathcal{O}_i \longrightarrow \mathrm{G}\,, \quad i \in I$$

such that

$$\forall_{x\in\mathcal{O}_{ij}}\ :\ g^2_{ij}(x) = h_i(x)\cdot g^1_{ij}(x)\cdot h_j(x)^{-1}\,. \tag{3}$$

Conversely, every such family defines a unique morphism of the type described.

Proof: We leave it as an exercise for the Reader. □

Proposition 6. Adopt the notation of Def. 1. The subcategory

$$\mathbf{GrpBun}_{\mathrm{G}}(B\,|\,\mathrm{id}_B)$$

of the category $\mathbf{GrpBun}_{\mathrm{G}}(B)$ of principal bundles with the base B and the structure group G with the same object class as $\mathbf{GrpBun}_{\mathrm{G}}(B)$ but with morphisms with the identity component on the base, $f = \mathrm{id}_B$ (and on the structure group, $\varphi = \mathrm{id}_{\mathrm{G}}$) is a **groupoid**, *i.e.*, a category with all morphisms invertible (that is, isomorphisms).

Proof: We leave it as an exercise for the Reader. □

Principal bundles play a fundamentaln rôle in the description of (local) symmetries of physical theories. We shall have more to say about this soon. Besides, they give a natural point of departure for novel physically relevant mathematical constructions, which we review next.