CLASSICAL FIELD THEORY (interaction at a distance) ①

VI MANIFOLDS with a LIE-GROUP ACTION

We shall be concerned with actions of Lie groups on smooth manifolds, with view to understanding the construction of associated bundles that will play a prime rôle in the gauging of global symmetries in a field theory.

Def.ⁿ 1. Let G be a Lie group & let X be a set with the symmetric group $\mathfrak{S}(X)$ (the group of permutations of X). A group homomorphism

$$\lambda_\cdot : G \longrightarrow \mathfrak{S}(X) : g \longmapsto \lambda_g$$

is termed a LEFT ACTION of GROUP G on X, & the ~~latter~~ pair (X, λ) is referred to as a SET with LEFT GROUP ACTION.

Similarly, any (group) antihomomorphisms

$$\varrho_\cdot : G \to \mathfrak{S}(X) : g \longmapsto \varrho_g$$

is termed a RIGHT ACTION of GROUP G, on X, & the ~~letter~~ pair $(X, \varrho_\cdot)$ is referred to as a SET with RIGHT GROUP ACTION.

We shall often write – by a minor abuse of the ~~notation~~ notation –

$$\lambda_\cdot : G \times X \longrightarrow X : (g, x) \longmapsto \lambda_g(x) \equiv g \triangleright x \equiv \lambda(g, x)$$

&

$$\varrho_\cdot : X \times G \longrightarrow X : (x, g) \longmapsto \varrho_g(x) \equiv x \triangleleft g \equiv \varrho_\cdot(x, g)$$

Let X_A, $A \in \{1, 2\}$ be two sets with the respective left actions λ^A of G, & let $f : X_1 \to X_2$ be a map between them.

We say that f is a LEFT G-EQUIVARIANT MAP if it satisfies the condition expressed by the commutative diagram

$$\begin{tikzcd} G \times X_1 \arrow[r, "\lambda^1"] \arrow[d, "\mathrm{id}_G \times f"'] \arrow[dr, phantom, "\circlearrowright"] & X_1 \arrow[d, "f"] \\ G \times X_2 \arrow[r, "\lambda^2"'] & X_2 \end{tikzcd}$$

Analogously, we define a RIGHT (3)
G-EQUIVARIANT MAP as one for which

$$\begin{tikzcd} X_1 \times G \arrow[r, "\varrho^1"] \arrow[d, "f \times \mathrm{id}_G"'] \arrow[dr, phantom, "\circlearrowright"] & X_1 \arrow[d, "f"] \\ X_2 \times G \arrow[r, "\varrho^2"'] & X_2 \end{tikzcd}$$

commutes.

Whenever $(X, \mathcal{T}(X))$ is a topological space, G is a topological group and λ (resp. ϱ) is continuous, we call (X, λ) (resp. (X, ϱ)) a SPACE with a LEFT (resp. RIGHT) TOPOLOGICAL ACTION of G. A continuous left/right G-equivariant map between spaces with left/right topological actions, is called a LEFT/RIGHT TOPOLOGICALLY G-EQUIVARIANT MAP.

Replacing the topological space $(X, \mathcal{T}(X))$ above with a diff. manifold $(M, \mathcal{A})$, the topological group G with a Lie group G, So the qualifier(s) 'topological' with 'smooth', we obtain, respectively, a ~~DIFF.~~ DIFF. MANIFOLD $\rightarrow ((M, \mathcal{A}), \lambda)$ equipped with a SMOOTH LEFT/RIGHT ACTION, resp. a LEFT/RIGHT SMOOTHLY G-EQUIVARIANT MAP.

The set of (~~topologically~~ smoothly) ~~cont~~ equivariant maps between manifolds X_A, $A \in \{1, 2\}$ shall be denoted as

$$\mathrm{Hom}_G(X_1, X_2).$$

An action of a Lie group on a manifold induces distinguished vector fields on the latter, ~~that~~ we introduce in

Def^n 2. Let $((M, \mathcal{A}), \lambda)$ be a manifold with a smooth left action λ of a Lie group G with a Lie algebra $\mathfrak{g}$. ~~The~~ A LEFT FUNDAMENTAL VECTOR FIELD on M ~~(for the left action~~ λ) is the image of an element $X \in \mathfrak{g}$ under the mapping

$$\mathcal{K}_{\cdot_1}(\cdot_2) \equiv T_{(e,\cdot_2)}\lambda(\cdot_1, 0_{T_{\cdot_2}M}) : \mathfrak{g} \longrightarrow \Gamma(TM)$$

$$: X \longmapsto \underbrace{T_{(e,\cdot_2)}\lambda(X, 0_{T_{\cdot_2}M})}_{\mathcal{K}_X(\cdot_2)},$$

with values

$$\mathcal{K}_X(x) \equiv T_{(e,x)}\lambda(X, 0_{T_xM}) \in T_xM$$

~~The~~ A ~~RIGHT~~ RIGHT FUNDAMENTAL VECTOR FIELD on M is defined analogously.

For both practical & historical reasons, it is worth devoting a moment to a derivation of a formula for the action of a fundamental vector field on the space $C^1(M,\mathbb{R})$ of functions on M. Thus, let $f \in C^1(M,\mathbb{R})$ & compute its derivative along $\mathcal{K}_X$ at $x \in M$,

$$\mathcal{K}_X(f)(x) \equiv T_x f\big(\mathcal{K}_X(x)\big) = T_x f \circ T_{(e,x)}\lambda\big(X, 0_{T_xM}\big)$$

$$= T_{(e,x)}(f\circ\lambda)\big(X, 0_{T_xM}\big) = \frac{d}{dt}\Big|_{t=0}(f\circ\lambda)\big(\mathcal{L}^X_t(e), \Phi_0(t,x)\big)$$

$$= \frac{d}{dt}\Big|_{t=0} f\big(\mathcal{L}^X_t(e)\rhd \overset{x}{\Phi_0}(t,x)\big) = \frac{d}{dt}\Big|_{t=0} f\big((e\cdot\exp(t\rhd X))\rhd x\big)$$

$$= \frac{d}{dt}\Big|_{t=0} f\big(\exp(t\rhd X)\rhd x\big).$$

We may, in particular, take f to be the coordinate function on some neighbourhood $\mathcal{O}_x \ni x$, whereby we obtain an explicit representation of $\mathcal{K}_X$ in the form

$$K_X(x) = \frac{d}{dt}\Big|_{t=0}\left(\mathcal{L}^X_t(e)\triangleright x\right) = \frac{d}{dt}\Big|_{t=0}\left(\exp(t\triangleright X)\triangleright x\right)$$

~~Ke~~

Fundamental vector fields are intimately related to the left-invariant vector fields on G. Indeed, we have

<u>Prop$^{\underline{n}}$</u> 1. Adopt the hitherto notation. The mapping $K_\cdot : \mathfrak{g} \to \Gamma(TM)$ ($\mathfrak{g} \equiv \mathrm{Lie}\,G$) is a G-equivariant antihomomorphism of Lie algebras, i.e., for any $X, Y \in \mathfrak{g}$ & $g \in G$, we have

left action of G on $\mathfrak{g}$

$$\lambda_{g*} K_X = K_{T_e \mathrm{Ad}_g(X)}$$

left action of G on $\Gamma(TM)$

commutator of vector fields

& $$[K_X, K_Y] = -K_{[X,Y]_{\mathfrak{g}}}$$

Lie bracket on $\mathfrak{g}$

<u>Proof</u>: The first part is proven in a direct calculation using the obvious identity

$$\lambda_g \circ \lambda = \lambda \circ (\mathrm{Ad}_g \times \lambda_g).$$

Thus, we compute, at an arbitrary $x \in M$,

$$\left(\lambda_{g*}\mathcal{K}_X\right)(x) = T_{\lambda_{g^{-1}}(x)}\lambda_g\left(\mathcal{K}_X\left(\lambda_{g^{-1}}(x)\right)\right)$$

$$= T_{g^{-1}\triangleright x}\lambda_g \circ T_{(e,g^{-1}\triangleright x)}\lambda\left(X, 0_{T_{g^{-1}\triangleright x}M}\right)$$

$$= T_{(e,g^{-1}\triangleright x)}\left(\lambda_g \circ \lambda\right)\left(X, 0_{T_{g^{-1}\triangleright x}M}\right)$$

$$= T_{(e,g^{-1}\triangleright x)}\left(\lambda \circ \left(\mathrm{Ad}_g \times \lambda_g\right)\right)\left(X, 0_{T_{g^{-1}\triangleright x}M}\right)$$

$$= T_{(e,x)}\lambda\left(T_e\mathrm{Ad}_g(X), T_{g^{-1}\triangleright x}\lambda_g\left(0_{T_{g^{-1}\triangleright x}M}\right)\right)$$

$$= T_{(e,x)}\lambda\left(T_e\mathrm{Ad}_g(X), 0_{T_xM}\right)$$

$$\equiv \mathcal{K}_{T_e\mathrm{Ad}_g(X)}(x) \quad \checkmark$$

When proving the anti-homomorphic nature of $\mathcal{K}_\cdot$, we derive a relation between the $\mathcal{K}_X$'s & the vector fields $(R_X \oplus 0_{TM}) \in \Gamma(TG) \oplus \Gamma(TM) \equiv \Gamma(T(G \times M))$ (here, 0_{TM} is the zero section of TM).

We use the identity

$$\lambda\circ(p_g\times id_M)=\lambda\circ(id_G\times\lambda_g)$$

& calculate – for any $g\in G$ & $x\in M$ –

$$T_{(g,x)}\lambda\big((R_X, 0_{TM})(g,x)\big)=T_{(g,x)}\lambda\circ(\underbrace{T_e p_g\times id_{T_xM}}_{\overset{\shortparallel}{T_{(e,x)}(p_g\times id_M)}})(X,0_{T_xM})$$

$$=T_{(g,x)}\big(\lambda\circ(p_g\times id_M)\big)(X,0_{T_xM})$$

$$=T_{(e,x)}\big(\lambda\circ(id_G\times\lambda_g)\big)(X,0_{T_xM})$$

$$=T_{(e,g\triangleright x)}\lambda\circ(id_{T_eG}\times T_x\lambda_g)(X,0_{T_xM})$$

$$=T_{(e,g\triangleright x)}\lambda\big(X,T_x\lambda_g(0_{T_xM})\big)$$

$$=T_{(e,g\triangleright x)}\lambda(X,0_{T_{g\triangleright x}M})\qquad \mathcal{K}_X\equiv T_{(e,\cdot)}\lambda\big((R_X,0_{TM})(e,\cdot)\big)$$

$$\Downarrow$$

$$\equiv\mathcal{K}_X\big(\lambda(g,x)\big).\quad\Rightarrow\quad\left[\begin{aligned}\mathcal{K}_X(x)&\equiv\mathcal{K}_X(\lambda(e,x))\\&=T_{(e,x)}\lambda\big((R_X,0_{TM})(e,x)\big)\end{aligned}\right.$$

Consequently, we obtain ...

$$[\mathcal{K}_X, \mathcal{K}_Y](x) \equiv [\mathcal{K}_X, \mathcal{K}_Y] \circ \lambda(e, x)$$

$$= \left[T_{(1|2)} \lambda \left(R_X^{(1)}, 0_{TM}^{(2)} \right), T_{(1|2)} \lambda \left(R_Y(\cdot_1), 0_{TM}(\cdot_2) \right) \right] (e, x)$$

$$= T_{(\cdot_1|\cdot_2)} \lambda \left(\underbrace{[(R_X(\cdot_1), 0_{TM}(\cdot_2)), (R_Y(\cdot_1), 0_{TM}(\cdot_2))]}_{\text{on } T(G \times M)} \right) (e, x)$$

$$\equiv T_{(\cdot_1|\cdot_2)} \lambda \left(\left([R_X, R_Y](\cdot_1), [0_{TM}, 0_{TM}](\cdot_2) \right) \right) (e, x)$$

$$= T_{(\cdot_1|\cdot_2)} \lambda \left(\left(-R_{[X,Y]}(\cdot_1), 0_{TM}(\cdot_2) \right) \right) (e, x)$$

$$= -\left(T_{(\cdot_1|\cdot_2)} \lambda \left(R_{[X,Y]}, 0_{TM} \right)(\cdot_1, \cdot_2) \right) (e, x)$$

$$= -\mathcal{K}_{[X,Y]}(x)$$

Passing to the global level, we find

Th^m 1. [The Equivariant Rank Theorem]
Let $((M_A, \mathcal{A}_A), \lambda^A)$, $A \in \{1, 2\}$ be manifolds with the respective smooth actions λ^A (left) of the Lie groups G. Assume λ^1 transitive, i.e., $\forall\, x, y \in M_1\ \exists\, g \in G :\ y = \lambda^1(g, x)$.

Let $F \in \mathrm{Hom}_G(M_1, M_2)$ be smooth. Then F has constant rank, & so preimages of points in M_2 along F are closed submanifolds embedded in M_1.

Proof: Transitivity of λ^1 in conjunction with G-equivariance of F imply, for any $x_1, x_2 \in M_1$,

$$T_{x_2}F \circ T_{x_1}\lambda^1_{g_{21}} = T_{x_1}\left(F \circ \lambda^1_{g_{21}}\right) = T_{x_1}\left(\lambda^2_{g_{21}} \circ F\right)$$

$$= T_{F(x_1)}\lambda^2_{g_{21}} \circ T_{x_1}F ,$$

where $g_{21} \in G$ is such that $x_2 = \lambda^1_{g_{21}}(x_1)$. But $\lambda^1_{g_{21}}$ & $\lambda^2_{g_{21}}$ are diffeomorphisms, & so both $T_{x_1}\lambda^1_{g_{21}}$ & $T_{F(x_1)}\lambda^2_{g_{21}}$ are invertible, whence

$$\mathrm{rk}\, T_{x_2}F = \mathrm{rk}\, T_{x_1}F \quad \forall x_1, x_2 \in M_1$$

$$\Rightarrow \mathrm{rk}\, T_{\cdot}F = \mathrm{const} \quad \text{over } M_1 .$$

SEE: NIEZBĘDNIK.

The last statement follows from the Constant Rank Theorem. $\square$

In physics, we are oftentimes interested in projecting out (Lie-)group action, i.e., passing from a given manifold M to its orbit space $M/G = \{G \rhd x \mid x \in M\}$. This is what happens in the symplectic reduction of (classical) spaces of states with respect to local-symmetry groups &c in the gauging of global symmetries. Therefore, it is of utmost significance to ~~understand~~ identify the circumstances in which the passage $M \to M/G$ is 'smooth', i.e., leaves us with a smooth manifold M/G. Instrumental in it is the concept introduced in

Def$^{\underline{n}}$ 3. We call an action (left)

$$\lambda : G \times X \to X$$

of a topological group G on a topological space $(X, \mathcal{T}(X))$ PROPER iff the mapping

$$\Lambda := (\lambda, pr_2) : G \times X \longrightarrow X \times X$$
$$: (g,x) \longmapsto (\lambda(g,x), x)$$

is proper (i.e, preimages along Λ of compact sets are compact).

×

The property defined above can be reinterpreted as

<u>Prop^n</u> 2. Let G be a topological group & let $(X, \mathcal{T}(X))$ be a Hausdorff topological space. An action $\lambda : G \times X \to X$ is proper iff for ever $K \subset X$ compact the set

$$G(K) := \{ g \in G \mid \lambda_g(K) \cap K \neq \emptyset \}$$

is compact.

<u>Proof</u>: First, assume λ proper & pick up $K \subset X$ compact. The set

$$G(K) = \{ g \in G \mid \exists x \in K : gx \in K \}$$
$$= \{ g \in G \mid \exists x \in K : \Lambda(g,x) \in K \times K \}$$
$$= pr_1 (\Lambda^{-1}(K \times K))$$

is compact as a continuous (pr_1 is continuous) image of the preimage of a cartesian square $K \times K$ of a compact K ($K \times K$ is compact in the product topology on $X \times X$) along the proper map Λ.

Conversely, let $G(K)$ be compact for any $K \subset X$ compact. Consider an arbitrary compact subset $\widetilde{K} \subset X \times X$. Its continuous images $pr_A(\widetilde{K}) \subset X$, $A \in \{1,2\}$ are compact, & so this property is inherited by their union

$$K_{12} := pr_1(\widetilde{K}) \cup pr_2(\widetilde{K}) \subset X.$$

But, clearly, whenever $(g,x) \in G \times X$ satisfies $\Lambda(g,x) \in \widetilde{K}$, we have $\Lambda(g,x) \in K_{12} \times K_{12}$, & so

$$\Lambda^{-1}(\widetilde{K}) \subset \Lambda^{-1}(K_{12} \times K_{12}) \equiv \{(g,x) \in G \times K_{12} \mid g \triangleright x \in K_{12}\} \subset G(K_{12}) \times K_{12}$$

Now, a classic theorem stating compactness of ~~a~~ closed subsets in a topological space implies ~~that~~ $\Lambda^{-1}(\tilde{K})$ is compact.

Indeed, this set is closed as a continuous preimage of a ~~clos~~ compact set $\tilde{K}$ which – in virtue of the theorem ~~of~~ stating closedness of ~~compact~~ compact subsets in a Hausdorff space ($X \times X$ inherits hausdorffness from X) – is <u>closed</u>.

But $G(K_{12}) \times K_{12}$ is compact as a cartesian product of compacts K_{12} (see: above) & $G(K_{12})$ (by assumption on Λ), & so $\Lambda^{-1}(\tilde{K})$ is a closed subset in a compact set which – by the same theorem(s) – is compact.

$\square$

In practice, it often proves convenient (16)
to rephrase the above property as

Prop.^{n} 3. An action $\lambda : G \times X \to X$ as above on a locally-precompact topological space $(X, \mathcal{T}(X))$ (i.e., such that every point $x \in X$ is contained in some ~~A~~ precompact set ($\Leftrightarrow$ ~~[illegible]~~ whose closure is compact) that contains some open neighbourhood of x and 'normal' ~~standard~~ spaces have this property !!!))

is proper iff the convergence of an arbitrary sequence of points

$\lambda(g_\cdot, x_\cdot) : \mathbb{N} \to \cancel{X} : n \mapsto g_n \triangleright x_n$,

defined for an arbitrary convergent sequence $x_\cdot : \mathbb{N} \to \cancel{X} : n \mapsto x_n$ & an arbitrary sequence $g_\cdot : \mathbb{N} \to \cancel{X} G : n \mapsto g_n$ (not necessarily convergent!) implies existence of a convergent sub-sequence in $g_\cdot$.

Proof: Assume, firstly that Λ is proper & take sequences $x_\cdot : \mathbb{N} \to X$ & $g_\cdot : \mathbb{N} \to G$ such that $x := \lim_{n\to\infty} x_n \in X$, $y := \lim_{n\to\infty} (g_n \blacktriangleright x_n) \in X$ exist. Taking into account local precompactness of X, we choose arbitrary precompact subsets $\mathcal{U} \ni x$ & $V \ni y$ containing x resp. y together with some open neighbourhoods of ~~the~~ respective points.

Convergence of $x_\cdot$ to x (resp. of $\lambda(g_\cdot, x_\cdot)$ to y) means that almost all terms of the sequence are in $\mathcal{U}$ (resp. V), & so all the more in the compact set $\overline{\mathcal{U}}$ (resp. $\overline{V}$). But this implies
closure of $\mathcal{U}$
that almost all terms of the sequence $\Lambda(g_\cdot, x_\cdot) : \mathbb{N} \to X \times X : n \mapsto (g_n \blacktriangleright x_n, x_n)$ are in the compact set $\overline{\mathcal{U}} \times \overline{V}$, & so almost all terms

of $(g_\cdot, x_\cdot) : \mathbb{N} \to G \times X : n \mapsto (g_n, x_n)$

the sequence are in the subset $\Lambda^{-1}(\overline{\mathcal{U}} \times \overline{V})$ which is ~~do~~ compact due to the assumed proper-ness of Λ. Consequently, there exists (~~a suggested~~ a convergent sub-sequence of $(g_\cdot, x_\cdot)$ which gives us the convergent sub-sequence of $g_\cdot$ sought after.

Conversely, assume that the implication from the end of the statement of the proposition is satisfied. Fix a compact $K \subset X \times X$ & pick up an arbitrary sequence $(g_\cdot, x_\cdot)$ as above in $\Lambda^{-1}(K)$. Its image in $X \times X$ along Λ is contained in the compact subset K, & so we may extract from it a convergent sub-sequence

The preimage of that subsequence (19)
along Λ intersects with $(g_\cdot, x_\cdot)$ &
so defines a subsequence of $\hat{\jmath}$ from
the statement of the proposition.
In this way, we obtain a sub-sequence
of $(g_\cdot, x_\cdot)$ in $\Lambda^{-1}(K) \subset G \times X$ which is
convergent (in $G \times X$!!!) in the product
topology, but $\Lambda^{-1}(K)$ is closed as
a continuous preimage of a compacts
& hence closed (see: above) subset K,
therefore the limit of this convergent
sub-sequence is in $\Lambda^{-1}(K)$. This
implies that $\Lambda^{-1}(K)$ is compact. $\square$

<u>Corollary</u> 1. Any topological action of
a <u>compact</u> topological group is proper.

<u>Corollary</u> 2. Any topological action of a <u>closed</u>
subgroup of a Lie group, obtained through
restriction of the right / left regular action of
G on itself is proper.

We now come to the main point of the lecture ...

Def$^{\underline{n}}$ 4. Let (X, λ) be a set with an action $\lambda : G \times X \to X$ of a group G. The set

$$X/G := \{ [x]_G = G \rhd x \mid x \in X \}$$

is called the ~~orbi~~ ORBISPACE of λ.

We have the simple

Lemma 1. : Let $((X, \mathcal{T}(X)), \lambda)$ be a topological space with a topological action λ of a topological group G. The canonical projection

$$\pi_{X/G} : X \to X/G : x \longmapsto [x]_G$$

is an OPEN (images of opens are open) map ~~wrt~~ with respect to the quotient topology ~~of~~ on X/G.

Proof: Consider $\mathcal{O} \subset X$ open in X. Its image $\pi_{X/G}(\mathcal{O})$ is – by definition of the topology on X/G – open in X/G iff its preimage

$\pi_{X/G}^{-1}(\pi_{X/G}(\mathcal{O})) = \{\lambda(g,x) \mid (g,x) \in G \times \mathcal{O}\} \equiv G \triangleright \mathcal{O}$

is open in X. But this is the case

as $\quad G \triangleright \mathcal{O} \equiv \bigcup_{g \in G} \lambda_g(\mathcal{O})$

is a union of homeomorphic images

$\lambda_g(\mathcal{O})$ of the open $\mathcal{O}$. $\square$

We thus arrive at the all-important...

Thm 2. [The Quotient Manifold Theorem]
Whenever the action $\lambda : G \times M \to M$ of a Lie group G on a manifold $(M, \hat{A})$ is smooth, free & proper!, the orbispace M/G is a topological space of dimension $\dim M - \dim G$ equipped with an essentially (up to iso) unique smooth structure for which the canonical projection $\pi_{M/G} : M \to M/G$ is a smooth surjective submersion. The orbispace with this structure is called the QUOTIENT MANIFOLD.

Proof: We begin by identifying the structure of a topological manifold on the orbispace M/G. The topology on M/G is a quotient topology induced from M along $\pi_{M/G}$: A subset $\mathcal{O} \subset M/G$ is open iff its preimage $\pi_{M/G}^{-1}(\mathcal{O})$ is open in M. This is a Hausdorff topology. Indeed, consider the equivalence relation R_λ on M defined ~~by~~ as

$$R_\lambda := \Lambda(G \times M) \subset M \times M,$$

i.e., with the ~~[illegible]~~ ~~distinct~~ disjoint equivalence classes given by orbits of λ. Let $x, y \in M$ be such ~~that~~ $\pi_{M/G}(x_1) \neq \pi_{M/G}(\cancel{x}_2)$, so that $(x_1, x_2) \notin R_\lambda$. In virtue of the theorem stating closedness of a ~~[illegible]~~ proper map with a precompact codomain, the subset $R_\lambda \subset M \times M$ is closed ($\Leftarrow$ ~~[illegible]~~ $G \times M$ is closed!), so there exists an open ~~[illegible]~~ neighbourhood

$\mathcal{O}_{(x_1,x_2)} \ni (x_1,x_2)$ with the property $\mathcal{O}_{(x_1,x_2)} \cap \mathcal{R}_\lambda = \emptyset$. By the definition of the product topology (on $M\times M$), such an open set is a union of a family of cartesian products of opens in M, $\mathcal{O}_{(x_1,x_2)} = \bigcup_{i\in I} \mathcal{O}_i^1 \times \mathcal{O}_i^2$, & so picking up any one of these, say $\mathcal{O}_i^1 \times \mathcal{O}_i^2 \equiv \mathcal{O}_1 \times \mathcal{O}_2$, we obtain the relations $\mathcal{O}_\alpha \ni x_\alpha$, $\alpha\in\{1,2\}$ & $(\mathcal{O}_1\times\mathcal{O}_2)\cap\mathcal{R}_\lambda = \emptyset$, whence also $\pi_{M/G}(\mathcal{O}_\alpha) \ni \pi_{M/G}(x_\alpha)$, with $\pi_{M/G}(\mathcal{O}_1) \cap \pi_{M/G}(\mathcal{O}_2) = \emptyset$. In the light of Lemma 1, the $\pi_{M/G}(\mathcal{O}_\alpha)$ are open neighbourhoods of the respective x_α in M/G.

In the next step, we decompose M into orbits of λ & associate with this decomposition adapted local (coordinate) charts in which coordinates charting directions transverse to (nearby) orbits are ultimately used in the construction of an atlas on M/G.

Our point of departure is a demonstration of the ~~fact~~ that ~~we do have~~ the orbits are, indeed, submanifolds ~~~~ smoothly embedded in M. For ~~that~~ purpose, we consider the smooth map:

$$\Omega_x := \lambda(\cdot, x) : G \to M : g \mapsto \lambda(g,x)$$

given for a fixed $x \in M$. The map satisfies the obvious identity

$$\Omega_x(G) \equiv [x]_G .$$

It is manifestly G-equivariant,

$$\forall g \in G : \Omega_x \circ l_g = \lambda_g \circ \Omega_x ,$$

i.e. it intertwines the left regular action of G on itself with λ. However, the former is transitive, so ~~that~~ we may invoke Th^{m} 1. to conclude ~~that~~ Ω_x has a constant rank. Moreover, it is injective as $(g_2 \triangleright x \equiv) \Omega_x(g_2) = \Omega_x(g_1) (\equiv g_1 \triangleright x) \iff g_2^{-1} \cdot g_1 \triangleright x = x$ implies – due to the free character of λ – that $g_2 = g_1$,

& so – in virtue of the theorem about immersivity of a smooth injection of constant rank (which follows directly from the Constant – Rank Theorem) — we infer that Ω_x is an immersion. Furthermore, whenever $K \subset M$ is compact, & hence closed (due to hausdorffness of M), its continuous preimage $\Omega_x^{-1}(K)$ is closed in G, but any of its elements $g \in \Omega_x^{-1}(K)$ satisfies $g \triangleright x \in K$, & so

$$(g \triangleright (K \cup \{x\})) \cap (K \cup \{x\}) = ((g \triangleright K) \cup \{g \triangleright x\}) \cap (K \cup \{x\}) \supset \{g \triangleright x\} \neq \emptyset,$$

which implies $\Omega_x^{-1}(K) \subset G(K \cup \{x\})$, the latter ~~set~~ being compact by Req $\underline{^{\circ}}$ 2., & this leads us to conclude that is compact. This, in turn, implies that Ω_x is proper & as such, a smooth embedding (as every

injective immersion which is proper
— cp Mezbgdnik Rozmaitości').

Fix (arbitrarily) a point $x \in M$,
& so also its preimage $e \in G$ along Ω_x,
& local charts $\kappa_e : \mathcal{O}_e \to \mathbb{R}^D$, $D = \dim G$
on an open neighbourhood $\mathcal{O}_e$ of $e \in G$
as well as $\kappa_x : \mathcal{O}_x \to \mathbb{R}^N$, $N = \dim M$
on ~~some~~ an open neighbourhood $\mathcal{O}_x$ of $x \in M$
in which Ω_x has the canonical presentation (for an immersion), i.e.,

$$G \triangleright \{x\} \cap \mathcal{O}_x = \kappa_x^{-1}(\mathcal{U}_x \times \{0_n\}),$$
$$n = N - D$$

where $\mathcal{U}_x \in \mathcal{T}(\mathbb{R}^D)$ is a homeomorphic image of a fragment of the orbit contained in $\mathcal{O}_x$. Let

$$\Delta_x := \kappa_x^{-1}(\{0_D\} \times \mathbb{R}^n)$$

be a submanifold in $\mathcal{O}_x$ transversal to (the fragment of) the said orbit $G \triangleright \{x\}$, defining a decomposition of the tangent

$$T_xM = T_x(G\triangleright\{x\}) \oplus T_x\Delta_x$$

in which – in the light of the immersivity of Ω_x – we identify

$$(1)\quad T_e\Omega_x(T_eG) \equiv T_x(G\triangleright\{x\}) \subset T_xM.$$

Denote
$$\delta_x := \lambda\restriction_{G\times\Delta_x} : G\times\Delta_x \to M.$$

We shall demonstrate that δ_x is a diffeomorphism on a neighbourhood of $(e,x)\in G\times\Delta_x$. To this end, we employ the smooth immersion

$$\imath_x : G \to G\times\Delta_x : g \mapsto (g,x)$$

& ~~decom~~ factorise Ω_x as

$$\Omega_x = \delta_x\circ\imath_x\,,$$

giving
$$T_e\Omega_x = T_{(e,x)}\delta_x\circ T_e\imath_x\,,$$

& so also – by Eq. (1) –

$$T_{(e,x)}\delta_x\big(T_{(e,x)}(G\times\Delta_x)\big) \equiv T_{(e,x)}\delta_x(T_eG\oplus T_x\Delta_x) \supset T_{(e,x)}\delta_x(T_eG\oplus\{0_{T_x\Delta_x}\})$$
$$\equiv T_{(e,x)}\delta_x\circ T_e\imath_x(T_eG) = T_e\Omega_x(T_eG) = T_x(G\triangleright\{x\}).$$

Next, introduce the smooth embedding

$$r_e : \Delta_x \to G\times\Delta_x : y \mapsto (e,y)$$

of the submanifold transversal to (the local fragment of) the orbit $G\triangleright\{x\}$ to be able to decompose the smooth (canonical) injection

$$j_{\Delta_x} : \Delta_x \to M$$

as

$$j_{\Delta_x} = \delta_x \circ r_e ,$$

& so also $T_y j_{\Delta_x} = T_{(e,y)}\delta_x \circ T_y r_e$.

In view of the obvious identity

$$T_x r_e (T_x\Delta_x) = \{0_{T_eG}\}\oplus T_x\Delta_x$$

we obtain, this time, the relation

$$T_{(e,x)}\delta_x\left(T_{(e,x)}(G\times\Delta_x)\right) \supset T_{(e,x)}\delta_x\left(\{0_{T_eG}\}\oplus T_x\Delta_x\right) = T_{(e,x)}\delta_x\circ T_x r_e(T_x\Delta_x)$$

$$= T_x j_{\Delta_x}(T_x\Delta_x) \equiv T_x\Delta_x (\subset T_xM),$$

& so – finally – conclude that

$$T_{(e,x)}\delta_x\left(T_{(e,x)}(G\times\Delta_x)\right) \supset T_x(G\triangleright\{x\})\oplus T_x\Delta_x \,(\equiv T_xM)$$

but also $\subset T_xM$, whence necessarily

$$T_{(e,x)}\delta_x\left(T_{(e,x)}(G\times\Delta_x)\right) = T_xM .$$

This implies surjectivity of $T_{(e,x)}\delta_x$, but we (29)
also have
$$\dim_{\mathbb{R}} T_{(e,x)}(G\times\Delta_x) = \dim_{\mathbb{R}} T_e G + \dim_{\mathbb{R}} T_x\Delta_x$$
$$= \dim_{\mathbb{R}} T_x(G\triangleright\{x\}) + \dim_{\mathbb{R}} T_x\Delta_x$$
$$\equiv \dim_{\mathbb{R}} T_x M\,,$$
so $T_{(e,x)}\delta_x$ is a bijection. At this stage, we may invoke the Inverse-Function Theorem to state existence of an open neighbourhood V_x of $(e,x)\in G\times\Delta_x$ mapped diffeomorphically by (a restriction of) δ_x onto some neighbourhood $\widetilde{O}_x$ of $x\in M$. Given the character of the topology on $G\times\Delta_x$ (~~the keeping~~ cp. ~~with the argumentation from~~ p. 23), we see ~~that~~ the former neighbourhood can be chosen in the product form $V_x = W_e\times W_x$ & ~~with~~ from each of the factors, W_e & W_x, we may take out precompact preimages of open balls $B^D(\widetilde{\kappa}_e(e)=0_D;\varepsilon_e)\equiv\widetilde{\kappa}_e(\widetilde{W}_e)$, $\varepsilon_e>0$ & – respectively – $B^n(\widetilde{\kappa}_x(x)=0_n;\varepsilon_x)\equiv\widetilde{\kappa}_x(\widetilde{W}_x)$, $\varepsilon_x>0$ along local ~~coor~~ coordinate charts $\widetilde{\kappa}_e: W_e\to\mathbb{R}^D$ &

– respectively – $\widetilde{\kappa}_x : \widetilde{W}_x \to \mathbb{R}^n$. In the remainder of our discussion, we focus our attention on the product neighbourhood $\widetilde{V}_x = \widetilde{W}_e \times \widetilde{W}_x$. ...

In the next step, we show that the neighbourhood $\widetilde{W}_x \subset \Delta_x$ can be chosen sufficiently small for every orbit of λ to intersect it in at most one point. Let us assume, on the contrary ~~conversely~~, that this cannot be done: Consider a countable basis of neighbourhoods of x in $\widetilde{W}_x$ consisting of preimages, along $\widetilde{\kappa}_x$, of a family of open balls $B^n(0_n; r_k)$, $r_k := \frac{1}{E(\frac{1}{\epsilon_x})+k}$, $k \in \mathbb{N}$. In each of them, $B_k \equiv \widetilde{\kappa}_x^{-1}(B^n(0_n; r_k))$, there are two points: x_k & $y_k \neq x_k$ from the same λ-orbit, i.e., $y_k = g_k \triangleright x_k$ for some $g_k \in G$. Given the form of the basis of neighbourhood, chosen, the corresponding/ensuing sequence of points in (the precompact) $\widetilde{W}_x$

converge to

$$\lim_{k\to\infty} x_k = x = \lim_{k\to\infty}(g_k \triangleright x_k) ,$$

& so – in virtue of Prop$^{\text{n}}$ 3. (& due to the assumed proper-ness of λ) – the sequence $g_\cdot$ determined by the pair $(x_\cdot, y_\cdot)$ contains a sub-sequence convergent to some $g \in G$. Continuity of λ then enables us to write

$$g \triangleright x = \cancel{\lim}\ \lambda\Big(\lim_{k\to\infty}(g_k, x_k)\Big) = \lim_{k\to\infty} \lambda(g_k, x_k)$$

$$\equiv \lim_{k\to\infty} y_k = x ,$$

but λ is assumed free, whence necessarily $g = e$. On the other hand, for sufficiently large $k_\ell \in \mathbb{N}$, we have $g_{k_\ell} \in \mathcal{W}_e$ with $g_{k_\ell} \neq e$ (as $y_{k_\ell} \neq x_{k_\ell}$), & so

$$\lambda(g_{k_\ell}, x_{k_\ell}) \equiv y_{k_\ell} \equiv \lambda(e, y_{k_\ell}) ,$$

which – by injectivity of $\delta_x \restriction_{\widetilde{\mathcal{W}}_e \times \widetilde{\mathcal{W}}_e} \equiv \lambda \restriction_{\widetilde{\mathcal{W}}_e \times \widetilde{\mathcal{W}}_e}$

implies $(g_{k_\ell}, x_{k_\ell}) = (e, y_{k_\ell})$ ↯

Thus, a neighbourhood with the desired Φ property always exists.

Consider the composite diffeomorphism

$$\phi_x := (\tilde{\kappa}_e \times \tilde{\kappa}_x) \circ (\delta_x \restriction_{\tilde{V}_x})^{-1} : \tilde{O}_x \xrightarrow{\cong} \tilde{V}_x \equiv \tilde{W}_e \times \tilde{W}_x$$
$$\downarrow \cong$$
$$B^N(0_D; \varepsilon_e) \times B^n(0_n; \varepsilon_x)$$

We shall demonstrate that it defines a local chart compatible with λ in a natural manner: orbits of λ ~~are~~ correspond to hypersurfaces parametrised by the first N coordinates. We can certainly interpret ϕ_x in this way as both: its domain & its codomain are open. It, therefore, suffices to verify ~~that~~ the description of orbits is as stated above.

We have

$$\phi_x^{-1} : B^D(0_D;\varepsilon_e) \times B^n(0_n;\varepsilon_x) \to \tilde{O}_x$$

$$: (\xi, \zeta) \longmapsto \delta_x(\tilde{\kappa}_e^{-1}(\xi), \tilde{\kappa}_x^{-1}(\zeta)) = \lambda(\tilde{\kappa}_e^{-1}(\xi), \tilde{\kappa}_x^{-1}(\zeta)),$$

so we readily establish that an arbitrary hypersurface $\zeta = \zeta_* = \text{const}$ is contained in a single orbit since its diffeomorphic preimage in M along ϕ_x satisfies

$$\phi_x^{-1}(B^D(0_D;\varepsilon_e) \times \{\zeta_*\}) = \lambda(\tilde{W}_e \times \{\tilde{\kappa}_x^{-1}(\zeta_*)\})$$

$$\subset \lambda(G \times \{\tilde{\kappa}_x^{-1}(\zeta_*)\}) \equiv G \rhd \{\tilde{\kappa}_x^{-1}(\zeta_*)\}.$$

This shows that an arbitrary orbit $G \rhd \{y\}$, $y \in M$ intersects $\tilde{O}_x$ along a union of (fragments of) hypersurfaces described by $\zeta = \zeta_i = \text{const}$, $i \in I_y$ (in coordinates) for some index set I_y. But $\tilde{W}_x$ was chosen such that an arbitrary orbit intersects it in <u>at most</u> one point, whence $|I_y| \leq 1$.

Thus, the local chart ϕ_x is -indeed- compatible with $\mathcal{A}$ in the aforementioned sense & can be used to chart the orbit space on a neighbourhood of $\pi_{M/G}(x) \equiv [x]_G$.

We shall now construct a local chart on the latter neighbourhood (of $[x]_G$), remembering that openness of $\pi_{M/G}(\widetilde{\mathcal{O}}_x) \ni [x]_G$ in the quotient topology is ensured by openness of $\pi_{M/G}$ established in Lemma 1. Denote a local section through x transverse to orbits contained in $\widetilde{\mathcal{O}}_x$ as

$$\widetilde{\Delta}_x := \phi_x^{-1}\left(\{0_D\} \times B^n(0_n; \varepsilon_x)\right)$$

& note that the restriction of the canonical projection

$$\pi_{M/G}\restriction_{\widetilde{\Delta}_x} : \widetilde{\Delta}_x \to \pi_{M/G}(\widetilde{\mathcal{O}}_x)$$

is a bijection by the assumption made

with regard to the nature of the intersection of orbits of λ with $\widetilde{\mathcal{O}}_x$ & the confirmed compatibility of ϕ_x with this action.
Moreover, whenever $W \subset \widetilde{\Delta}_x$ is open, its image in the orbispace,

$$\pi_{M/G}(W) \equiv \pi_{M/G} \circ \phi_x^{-1}\left(\{O_D\} \times pr_2 \circ \phi_x(W)\right)$$

$$= \pi_{M/G} \circ \phi_x^{-1}\left(B^D(O_D; \varepsilon_e) \times pr_2 \circ \phi_x(W)\right)$$

is manifestly open as an image, along a composition of open maps, of a ~~set~~ subset $B^D(O_D; \varepsilon_c) \times pr_2 \circ \phi_x(W)$ which is open in the product topology — indeed, ϕ_x is a homeomorphism, & so is open, & pr_2 of $\phi_x(W)$ (the latter being open in the product topology, & hence a union of a finite family of cartesian products of opens) is a similar union of projections (to the second component) of opens, & so also open in $\mathbb{R}^n$.

Altogether, then, the restriction $\pi_{M/G}|_{\widetilde{\Delta}_x}$ is a homeomorphism — its inverse shall be denoted as

$$\sigma_{[x]} := \left(\pi_{M/G}|_{\widetilde{\Delta}_x}\right)^{-1} : \pi_{M/G}(\widetilde{O}_x) \xrightarrow{\cong} \widetilde{\Delta}_x .$$

Define

$$\psi_{[x]} := pr_2 \circ \phi_x \circ \sigma_{[x]} : \pi_{M/G}(\widetilde{O}_x) \xrightarrow{\cong} \widetilde{\Delta}_x \xrightarrow{\cong} \{0_D\} \times B^n(0_n; \varepsilon_x) \xrightarrow{\cong} B^n(0_n; \varepsilon_x),$$

explicitly ~~[illegible]~~ homeomorphically mapping the open neighbourhood $\pi_{M/G}(\widetilde{O}_x)$ onto the open ball $B^n(0_n; \varepsilon_x) \subset \mathbb{R}^n$, i.e., a local(ly) C^0-smooth (continuous) chart.

We emphasise that the local (coordinate) presentation of the canonical projection determined by the local chart ϕ_x over $\widetilde{O}_x$ & the attendant local chart $\psi_{[x]}$ over $\pi_{M/G}(\widetilde{O}_x)$ takes the simple form

$$\psi_{[x]} \circ \pi_{M/G} \circ \phi_x^{-1} : B^D(0_D; \varepsilon_0) \times B^n(0_n; \varepsilon_x) \twoheadrightarrow B^n(0_n; \varepsilon_x),$$

& so is manifestly submersive.

Consequently, $\sigma_{[x]}$ acquires a natural interpretation of a local section of the submersion $\pi_{M/G}$. Thus, if we <u>induce</u> the manifold structure on M/G from that on M (as above) using the sub-atlas on M given by all possible local charts on M compatible with λ & all local sections of $\pi_{M/G}$ over images – with respect to these sections – of their domains in M, then the only things that remains to complete the proof is a demonstration of the ~~corresponding~~ ensuing smoothness of transition maps defined on non-empty intersections of neighbourhoods in M/G, & ultimately – uniqueness of the ~~thus~~ obtained smooth structure.

Let, then, $\psi_{[x_A]} : \pi_{M/G}(\widetilde{\mathcal{O}}_{x_A}) \xrightarrow{\cong} B^n(0_n; \varepsilon_{x_A})$, $A \in \{1,2\}$ be two local charts satisfying $\pi_{M/G}(\widetilde{\mathcal{O}}_{x_1}) \cap \pi_{M/G}(\widetilde{\mathcal{O}}_{x_2}) \neq \emptyset$

& associated with local maps

$$\phi_{x_A} : \widetilde{\mathcal{O}}_{x_A} \xrightarrow{\cong} B^D(0_D; \varepsilon_{e;A}) \times B^n(0_n; \varepsilon_{x_A}).$$

The condition $\pi_{M/G}(\widetilde{\mathcal{O}}_{x_1}) \cap \pi_{M/G}(\widetilde{\mathcal{O}}_{x_2}) \neq \emptyset$ for the projections of the neighbourhoods $\widetilde{\mathcal{O}}_{x_A}$ of the x_A in M means that there exist points $y_A \in \widetilde{\mathcal{O}}_{x_A}$ belonging to the same orbit,

$$y_2 = g_{21} \triangleright y_1 \text{ for some } g_{21} \in G,$$

hence we may – without any loss of generality – assume that

$y_A = x_A$ & we have $x_2 = g_{21} \triangleright x_1$

(if need be, we shift (the 'centres' of) the maps ϕ_{x_A}). Assume, first, that $g_{21} = e$. We may, then, write ~~all the the~~ the transition maps between local charts $\phi_{x_1} = (\xi_1, \zeta_1)$ & $\phi_{x_2} = (\xi_2, \zeta_2)$ directly over $\mathcal{O}_{12} := \widetilde{\mathcal{O}}_{x_1} \cap \widetilde{\mathcal{O}}_{x_2} \ni y$. As intersections of an arbitrary orbit with each neighbourhood correspond to a constant value of ζ_1 (& similarly for ζ_2), the smooth (by construction) transition map between these charts the form

$$\phi_{x_2} \circ \left(\phi_{x_1} \restriction \mathcal{O}_{12}\right)^{-1} : \phi_{x_1}(\mathcal{O}_{12}) \xrightarrow{\cong} \phi_{x_2}(\mathcal{O}_{12})$$

$$: (\xi_1(y), \zeta_1(y)) \longmapsto \left(F_1 \circ (\xi_1, \zeta_1)(y), F_2 \circ \zeta_1(y)\right)$$

for some F_1, F_2 smooth. In particular,

$$\zeta_2(y) = F_2 \circ \zeta_1(y) \quad , \quad y \in \mathcal{O}_{12} ,$$

& so we obtain the <u>manifestly smooth</u>

$$\psi_{[x_2]} \circ \left(\psi_{[x_1]} \restriction \pi_{M/G}(\mathcal{O}_{12})\right)^{-1} : \psi_{[x_1]} \circ \pi_{M/G}(\mathcal{O}_{12}) \xrightarrow{\cong} \psi_{[x_2]} \circ \pi_{M/G}(\mathcal{O}_{12})$$

$$: \mathrm{pr}_2 \circ \phi_{x_1} \circ \left(\pi_{M/G} \restriction \tilde{\Delta}_{x_1}\right)^{-1}(\pi_{M/G}(y)) \equiv \zeta_1(y) \mapsto \zeta_2(y) = F_2(\zeta_1(y)) .$$

Passing to the case $g_{21} \neq e$, we conclude that while it is <u>not</u> possible (a priori) to relate ϕ_{x_1} & ϕ_{x_2} over $\mathcal{O}_{12}$ (the set could be empty), we may, nevertheless, replace the local chart ϕ_{x_2} & the local section $\sigma_{[x_2]}$ entering the definition of $\psi_{[x_2]}$ over $\pi_{M/G}(\mathcal{O}_{x_2})$ with another one from the same ~~atl~~ sub-atlas on M resp. set of <u>local</u> sections, jointly inducing on M/G the same local chart $\psi_{[x_2]}$ over $\pi_{M/G}(\mathcal{O}_{x_2})$ but, at the same time, take the points from its domain through an open subset in M that does intersect $\mathcal{O}_{x_1}$ in a neighbourhood of x_1, which brings us back to the previously considered case. Indeed, use the automorphism $\lambda_{g_{21}}$ to define

$$\phi^{21}_x := \phi_{x_2}\circ\lambda_{g_{21}} : \lambda_{g_{21}^{-1}}(\widetilde{\mathcal{O}}_{x_2}) \xrightarrow{\cong} B^D(0_D;\varepsilon_{x_2}/2)\times B^n(0_n;\varepsilon_{x_2}/2)$$

on the open (by construction) neighbourhood $\lambda_{g_{21}^{-1}}(\widetilde{\mathcal{O}}_{x_2})$ of x_1. Since $\lambda_{g_{21}}$ takes orbits to orbits, ϕ^{21}_x is, again, a local chart compatible with $\mathcal{A}$. We augment this definition with a suitable redefinition of the local section of the canonical projection onto $\pi_{M/G}(\widetilde{\mathcal{O}}_{x_2})$, to wit,

$$\sigma^{21}_{[x_2]} := \lambda_{g_{21}^{-1}} \circ \sigma_{[x_2]} .$$

The last definition makes sense as

$$\pi_{M/G} \circ \sigma^{21}_{[x_2]} \equiv \left(\pi_{M/G} \circ \lambda_{g_{21}^{-1}}\right) \circ \sigma_{[x_2]}$$

$$= \pi_{M/G} \circ \sigma_{[x_2]} = \mathrm{id}_{\pi_{M/G}(\mathcal{O}_{x_2})} ,$$

so – as anticipated – reproduce, in conjunction with the new local chart, the very same local chart on $\pi_{M/G}(\mathcal{O}_{x_2})$, as demonstrated in

$$\psi^{21}_{[x_2]} := \mathrm{pr}_2 \circ \phi_{x_2} \circ \sigma^{21}_{[x_2]} = \mathrm{pr}_2 \circ \phi_{x_2} \circ \lambda_{g_{21}} \circ \lambda_{g_{21}^{-1}} \circ \sigma^{21}_{[x_2]}$$

$$= \mathrm{pr}_2 \circ \phi_{x_2} \circ \sigma^{21}_{[x_2]} \equiv \psi_{[x_2]} .$$

This ~~ends~~ concludes the proof of existence of the smooth structure stated in the theorem. We still have to prove its uniqueness.

To this end, consider two such (arbitrary) structures over the same orbispace M/G, denoting them as $(M/G)_A$, $A \in \{1,2\}$ for a convenient distinction. By assumption the canonical projection $\pi_{M/G} : M \to (M/G)_A$, $A \in \{1,2\}$

is a smooth surjective submersion for both cases, & so we may apply to it the statement of the Theorem on the Quasi-Universal Property of a Submersion (Niezbędnik Rozmaitości), & that in two ways: We may write down the (trivially) commutative diagram

$$\begin{tikzcd} & (M/G)_2 \\ M \arrow[ur, "\pi_{M/G}"] \arrow[r, "\pi_{M/G}"'] & (M/G)_1 \arrow[u, "\mathrm{id}_{M/G}"'] \end{tikzcd}$$

in which we treat $\pi_{M/G} : M \twoheadrightarrow (M/G)_1$ as a submersion & $\pi_{M/G} : M \twoheadrightarrow (M/G)_2$ (the <u>same</u> map) as a 'reference' map whose smoothness ensures ~~the~~ smoothness of $\mathrm{id}_{M/G}$. Upon swapping $(M/G)_1$ with $(M/G)_2$, we conclude that $\mathrm{id}_{M/G}$ is also smooth when viewed as a map $(M/G)_2 \to (M/G)_1$, which altogether implies equivalence between the two smooth structures.

Phew... $\square$

By way of illustration of the Theorem, of ~~an~~ a physically relevant example, we state (without proof)

Thm 3. Let G be a Lie group, & $H \subseteq G$ - its closed subgroup. There exists a unique structure of a smooth manifold on the quotient space G/H, with the quotient topology induced from G, for which the canonical projection $\pi_{G/H} : G \longrightarrow G/H$ is a surjective submersion. The natural left action

$$[\ell]. \; G \times (G/H) \longrightarrow G/H : (\tilde{g}, gH) \longmapsto (\tilde{g} \cdot g)H$$

is smooth with respect to this structure. The smooth G-manifold $(G/H, [\ell].)$ is called the SMOOTH HOMOGENEOUS SPACE of G.