CLASSICAL FIELD THEORY (interaction at a distance)

III

1

V FAMILIES OF SYMMETRIES ...

Last time, we associated Noether hamiltonians/charges

$$Q_X \in C^\infty(\mathcal{P}, \mathbb{R})$$

with vector fields $X \in \Gamma(T\mathcal{P})$ on the space of states $\mathcal{P}$ of field theory engendered by ~~continuous~~ smooth symmetries

$$(x, \phi(x)) \longmapsto (\tilde{x}_\lambda(x), \tilde{\phi}_\lambda(x, \phi(x))),$$

$$\lambda \in \mathbb{R}$$

as per

$$X \lrcorner \Omega = -\delta Q_X$$

In so doing, we took into account the linear structure on $\Gamma(T\mathcal{P})$ exclusively.

There is, however, a natural algebraic structure on $\Gamma(T\mathcal{P})$ of higher order...

Indeed, we have the Lie bracket of vector fields, skew-symmetric,

$$[\cdot,\cdot] : \Gamma(TP)\times\Gamma(TP) \longrightarrow \Gamma(TP)$$

satisfying the Jacobi identity.

It seems natural to consider a collection of symmetry vector fields $\{X_A\}_{A\in\overline{1,N}}$ in INVOLUTION with respect to the ~~commutator~~ $[\cdot,\cdot]$, i.e., such that

$$\forall A,B\in\overline{1,N} : [X_A, X_B] = f_{AB}{}^{C} X_C$$

for some constants $f_{AB}{}^{C}\in\mathbb{R}$ (STRUCTURE CONSTANTS). As we want $S := \bigoplus_{A=1}^{N} \langle X_A \rangle$ to be closed substructure of $(\Gamma(TP), [\cdot,\cdot])$, we need to impose the Jacobi identity on S,

(JI) $$f_{AB}{}^{D} f_{DC}{}^{E} + f_{CA}{}^{D} f_{DB}{}^{E} + f_{BC}{}^{D} f_{DA}{}^{E} \stackrel{!}{=} 0$$

As we shall find out soon, this is the case for vector fields induced by actions of multi-parameter (Lie) groups...

In the meantime, let us explore
the field-theoretic realisation
of the symmetries behind the X_i.
We find – for $Q_A \equiv Q_{X_A}$

$$\delta\{Q_A, Q_B\} \equiv \delta(X_A \lrcorner X_B \lrcorner \Omega)$$

$$= \mathcal{L}_{X_A}(X_B \lrcorner \Omega) - X_A \lrcorner \mathcal{L}_{X_B}\Omega + X_A \lrcorner X_B \lrcorner \overset{0}{\overset{\shortparallel}{\delta\Omega}}$$

$$= [X_A, X_B] \lrcorner \Omega + \cancel{\mathcal{L}_a} X_B \lrcorner \underset{\underset{0}{\shortparallel}}{\mathcal{L}_{X_A}\Omega} - X_A \lrcorner \underset{\underset{0}{\shortparallel}}{\mathcal{L}_{X_B}\Omega}$$

(we saw it last time)

$$\equiv [X_A, X_B] \lrcorner \Omega$$, so that

$$\{Q_A, Q_B\} = -Q_{[X_A, X_B]} + \underbrace{C_{AB}}_{\text{constants}}$$

or – just to remove the signs –

$$\tilde{Q}_A \equiv -Q_A \Rightarrow \underbrace{\{\tilde{Q}_A, \tilde{Q}_B\} = \tilde{Q}_{[X_A, X_B]} + C_{AB}}_{= f_{AB}{}^{C} \tilde{Q}_C + C_{AB}}$$

Of course,

$$\mathrm{Jac}(\tilde{Q}_A, \tilde{Q}_B, \tilde{Q}_C) = 0 \Leftarrow (\mathrm{JIf})\ p.\,2,$$

& so we have a Lie algebra as previously. Whenever the constants $C_{AB}^{\ \ ij}$ cannot be consistently ~~set~~ to 0, we speak of ...

Def$\underline{^{n}}$: Let $\tilde{\mathfrak{g}}, \mathfrak{g}, \mathfrak{z}$ be Lie algebras ~~& let mapped to 0~~ & let

$\pi : \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$ - be Lie-algebra epimorphism (surjective homomorphism)

$\jmath : \mathfrak{z} \longrightarrow \tilde{\mathfrak{g}}$ - monomorphism (injective —"—), $\mathfrak{z}$ with $\mathrm{Im}\,\jmath \subset \mathfrak{z}(\tilde{\mathfrak{g}})$ - the centre of $\tilde{\mathfrak{g}}$

~~satisfy~~ with the additional property

$$\pi \circ \jmath = 0,$$

which we write concisely as the SHORT EXACT SEQUENCE

$$0 \longrightarrow \mathfrak{z} \xrightarrow{\ \jmath\ } \tilde{\mathfrak{g}} \xrightarrow{\ \pi\ } \mathfrak{g} \longrightarrow 0$$

(all arrows are Lie-algebra homomorphisms) with – just to reemphasise –

$$\jmath(\mathfrak{z}) \subset \mathfrak{z}(\tilde{\mathfrak{g}}) \equiv \{ X \in \tilde{\mathfrak{g}} \mid \forall_{Y \in \tilde{\mathfrak{g}}} : [X, Y]_{\tilde{\mathfrak{g}}} = 0 \}$$

Then $\tilde{\mathfrak{g}}$ is called a <u>CENTRAL EXTENSION</u> of $\mathfrak{g}$ by $\mathfrak{z}$.

How do we get that in our picture? Define

$$\tilde{\mathfrak{g}} := \bigoplus_{A=1}^{N} \langle \tilde{Q}_A \rangle \oplus \langle \mathbb{1} \rangle \qquad \underbrace{}_{\text{basis}}$$

$$\mathfrak{g} := \bigoplus_{A=1}^{N} \langle \underline{Q}_A \rangle$$

$$\mathfrak{z} := \langle 1 \rangle \equiv \mathbb{R}$$

&

$$\pi : \tilde{\mathfrak{g}} \to \mathfrak{g} : X^A \tilde{Q}_A + x \mathbb{1} \longmapsto X^A \underline{Q}_A$$

$$\jmath : \mathfrak{z} \to \tilde{\mathfrak{g}} : x \longmapsto x \mathbb{1}$$

One readily checks that these <u>are</u> Lie-algebra homomorphisms.

We then obtain:

Prop$^{\underline{n}}$: In the above notation, the map

$$\tilde{\imath} : \tilde{\mathfrak{g}} \longrightarrow \mathfrak{z} \oplus \mathfrak{g} : \tilde{X} \longmapsto \left(j^{-1}\left(\tilde{X} - \sigma \circ \pi(\tilde{X})\right), \pi(\tilde{X}) \right)$$

written in terms of the linear map

$\sigma : \mathfrak{g} \longrightarrow \tilde{\mathfrak{g}}$ with the property

$\pi \circ \sigma = \mathrm{id}_{\mathfrak{g}}$ (PROVE ITS EXISTENCE!),

is a Lie-algebra isomorphism

if we equip $\mathfrak{z} \oplus \mathfrak{g}$ with the Lie bracket

$$[(Z_1, X_1), (Z_2, X_2)]_{\mathfrak{z}\oplus\mathfrak{g}} := \left(j^{-1}\left([\sigma(X_1), \sigma(X_2)]_{\tilde{\mathfrak{g}}} - \sigma([X_1, X_2]_{\mathfrak{g}}) \right), [X_1, X_2]_{\mathfrak{g}} \right).$$

Proof: Exercise! ((i) show that ↗ is a Lie bracket

(ii) verify $\forall \tilde{X}, \tilde{Y} \in \tilde{\mathfrak{g}}$: $[\tilde{\imath}(\tilde{X}), \tilde{\imath}(\tilde{Y})]_{\mathfrak{z}\oplus\mathfrak{g}} = \tilde{\imath}([\tilde{X}, \tilde{Y}]_{\tilde{\mathfrak{g}}})$

(iii) write out $\tilde{\imath}^{-1}$!

In the proof of the Jacobi identity for $\mathfrak{z} \oplus \mathfrak{g}$, we stumble upon the entity

$$\Theta : \mathfrak{g} \times \mathfrak{g} \xrightarrow[\text{bi-linear !}]{} \mathbb{R} : (X,Y) \longmapsto j^{-1}\left([\sigma(X),\sigma(Y)]_{\tilde{\mathfrak{g}}} - \sigma([X,Y]_{\mathfrak{g}})\right)$$

with the property:

$$\forall X,Y \in \mathfrak{g} : \Theta(Y,X) = -\Theta(X,Y)$$

(a 2-form on $\mathfrak{g}$ (with values in $\mathbb{R}$))

Ex Compute

$$\delta\Theta(X_1,X_2,X_3) \equiv -\Theta([X_1,X_2]_{\mathfrak{g}},X_3) - \Theta([X_3,X_1],X_2) - \Theta([X_2,X_3],X_1)$$

$\hookrightarrow$ the (3-)COBOUNDARY of Θ

that has to $= 0 \implies \Theta$ is a 2-COCYCLE!

An obvious instantiation of such an identity is when

$$\Theta(X,Y) = \mu([X,Y]_{\mathfrak{g}}) \equiv \delta\mu(X,Y)$$

for some $\mu : \mathfrak{g} \xrightarrow[\text{linear}]{} \mathbb{R}$.

Indeed, δ(H) then follows from the Jacobi identity for $\mathfrak{g}$.

(H) $= \delta\mu$ is termed a 2-coboundary.

Sounds familiar? Rightly so!

What does this correspond to physically?

Well, the Noether charges are determined up to constants (in a simple/generic situation). Therefore, we may ask if there exist constants q_A such that

$$\widetilde{Q}_A \longmapsto \widetilde{Q}_A + q_A =: \widehat{Q}_A$$

restores the original Lie algebra:

$$\{\widehat{Q}_A, \widehat{Q}_B\}_\Omega = f_{AB}{}^C \widehat{Q}_C$$

For that, we need

$$\forall A, B \in \mathbb{N} : f_{AB}{}^C q_C \stackrel{!}{=} C_{AB},$$

& this corresponds precisely to (H) $= \delta\mu$.

<u>Exercise</u>: Prove the ~~last~~ veracity of the observation!

Thus, field theoretic realisations of Lie algebras of symmetries (represented by the vector fields X_A) are generically central extensions of the Lie algebras associated with 2-cocycles $g \times g \to \mathbb{R}$, trivial only for 2-COBOUNDARY 2-COCYCLES.

We formalise the rel^n between 2-COCYCLES up to 2-COBOUNDARIES & CENTRAL EXTENSIONS of LIE ALGEBRAS in a separate note.

x

Can we always remove the CAs? No! An example is provided by the ~~..~~ wave theory of p. 6 of Lecture Notes II (ex. 1°)

Example:

Given a solution φ to the wave eq^n ~~of~~: $\Box_v \varphi = 0$,

we may consider a transformation:

$$(x, \phi) \longmapsto (x, \phi + \lambda\varphi)$$

(for $\lambda \in \mathbb{R}$ constant)

This is a symmetry (global) of the (lagrangean) theory ~~from~~ p. 6 LN II, with space-of-states ~~transformation~~ incarnation:

$$P \ni \phi \longmapsto \hat{\phi}_\lambda = \phi + \lambda\varphi \in P.$$

Indeed, we find

$$\mathcal{L}(x, \hat{\phi}_\lambda, \partial\hat{\phi}_\lambda) = \frac{1}{2v^2}(\partial_t\phi + \lambda\partial_t\varphi)^2 - \frac{1}{2}(\nabla\phi + \lambda\nabla\varphi)^2$$

$$= \mathcal{L}(x, \phi, \partial\phi) + \frac{\lambda}{v^2}\partial_t\phi\,\partial_t\varphi - \lambda\nabla\phi\cdot\nabla\varphi + O(\lambda^2)$$

$$= \mathcal{L}(x, \phi, \partial\phi) + \partial_t\left(\frac{\lambda}{v^2}\phi\,\partial_t\varphi\right) + \nabla(-\lambda\phi\nabla\varphi)$$

$$- \frac{\lambda}{v^2}\phi\,\partial_t^2\varphi + \lambda\phi\nabla^2\varphi + O(\lambda^2)$$

$$= \mathcal{L}(x, \phi, \partial\phi) + \partial_t\left(\frac{\lambda}{v^2}\phi\,\partial_t\varphi\right) + \nabla(-\lambda\phi\nabla\varphi) + O(\lambda^2)$$

so that we have

$$\mathcal{L}(x, \hat{\Phi}_\lambda, \partial\hat{\Phi}_\lambda) - \mathcal{L}(x, \phi, \partial\phi)$$
$$= \partial_\mu K^\mu_\lambda(x, \phi) + O(\lambda^2)$$

for
$$\begin{cases} K^0_\lambda(x,\phi) = \frac{\lambda}{v^2}\phi\,\partial_t\varphi\,, \\ K^i_\lambda(x,\phi) = -\lambda\,\phi\,\partial^i\varphi\,. \end{cases}$$

Consequently, we compute (as on p. 26 of LN II)

$$Q_{X_{\varphi_1}}[\phi,\pi] = \int_{\Sigma_t} d\vec{x}\left(\frac{1}{v^2}\Phi_t\,\partial_t\varphi(t,\cdot) - \Pi_t\,\varphi(t,\cdot)\right)(\vec{x})$$
$$\equiv Q_{\varphi_1}[\phi,\pi]\,,$$

& so also

$$\{Q_{\varphi_1}, Q_{\varphi_2}\}_\Omega[\phi,\pi] = \int_{\Sigma_t} d\vec{x}\,(\varphi_1\partial_t\varphi_2 - \varphi_2\partial_t\varphi_1)(t,\vec{x})$$
$$\not\equiv 0 \quad \text{(generically)}$$

even though we start with an abelian symmetry!

Recall (from p. 37 of LN 0) that we have 12

$$[\hat{Q}_A, \hat{Q}_B] = i\,\widehat{\{Q_A, Q_B\}_\Omega},$$

quantum Noether hamiltonians

& so the central term comes over to the quantum theory.

It is a <u>common mistake</u> among physicists to claim that central extensions of symmetry (Lie-)algebras are a quantum phenomenon.

The appearance of the $\{Q_A\}_{A\in 1,N}$ raises a host of natural questions:

(1) Do the Lie algebras integrate to anything (just as vector fields in involution may sometimes integrate to a foliation by integral leaves)?

(2) Where do they come from?

(3) How can we render the global symmetries represented by the Φs <u>local</u> (in the spirit of local field theory)?

Answers to all ~~of~~ these questions call for the introduction of elementary concepts from the theory of Lie groups & Lie algebras, to which we turn next...

VI ELEMENTS OF LIE-GROUP THEORY

~~The~~ The basic object of our consideration is defined in

Def.ⁿ 1 A <u>TOPOLOGICAL GROUP</u> is a group

$$\left(G, \, m \equiv \cdot \, , \, \mathrm{Inv} \equiv (\cdot)^{-1}, \, \bullet \mapsto e\right)$$

whose support G (a set) is a topological space, & whose structural maps are continuous. A <u>TOPOLOGICAL SUBGROUP</u> of $(G, m, \mathrm{Inv}, \bullet \mapsto e)$ is a topological group ~~whose~~ $(H, m\restriction_{H\times H}, \mathrm{Inv}\restriction_{H}, \bullet \mapsto e)$ whose support $H \subset G$ is a topological subspace of G.

A <u>TOPOLOGICAL-GROUP HOMOMORPHISM</u> ~~between~~ (between topological groups) is a homomorphism of groups that is continuous with respect to the topologies on its domain & codomain.

Analogously, a <u>LIE GROUP</u> is a group whose support is a C^∞-manifold, & whose structural maps are C^∞.

A <u>LIE SUBGROUP</u> is a subgroups whose support is a C^∞-submanifold.

A <u>LIE-GROUP HOMOMORPHISM</u> is a group homomorphism that is smooth with respect to the differential structures on its domain & codomain.

<u>Examples of Lie groups</u>:

(1) $\mathbb{R}^{\times n}$ with component-wise addition.

(2) $\mathbb{R}/2\pi\mathbb{Z} \simeq \mathbb{S}^1 \equiv U(1) \equiv \{u \in \mathbb{C} \mid |u| = 1\}$ (unital circle)

(3) $SU(2) \equiv \{u \in \mathbb{C}(2) \mid u^\dagger u = \mathbb{1} = u u^\dagger, \det u = 1\}$
$\simeq \mathbb{S}^3$ (unital 3-sphere)

(4) $GL(n;\mathbb{R}) \equiv \{m \in \mathbb{R}(n) \mid \det m \neq 0\}$

induced structure (top & diff.) $\leftarrow$ open subset in $\mathbb{R}^{n^2}$!

Our study of Lie groups begins with

Lemma 1

Let G be a topological group. For every $g \in G$ and every open neighbourhood $\mathcal{O}_e \in \mathcal{T}(G)$ ← topology of $e \in G$, there exists an open neighbourhood $\mathcal{O}_g \in \mathcal{T}(G)$ of g s.th.

$$f(\mathcal{O}_g \times \mathcal{O}_g) \equiv m \circ (\mathrm{Inv} \times \mathrm{id}_G)(\mathcal{O}_g \times \mathcal{O}_g) \subset \mathcal{O}_e .$$

Proof: The map f is continuous as a superposition of —— maps. Therefore, there exist opens $\mathcal{O}_1, \mathcal{O}_2 \ni g$ such that $f(\mathcal{O}_1 \times \mathcal{O}_2) \subset \mathcal{O}_e$. Define $\mathcal{O}_g := \mathcal{O}_1 \cap \mathcal{O}_2$ to get the thesis. $\square$

— × —

We shall, next, pass to the tangent ...

Prop^n 1 Let $(G, m, \mathrm{Inv}, \mathrm{pt} \mapsto e)$ be a Lie group. The quadruple $(TG, Tm, T\mathrm{Inv}, (\cdot, 0) \mapsto 0_{T_eG} \in T_eG)$ is a Lie group, termed the TANGENT LIE GROUP. The ~~canonical~~ base projection $\pi_{TG} : TG \to G$ & the ZERO SECTION $0_{TG} : G \to TG$ (gives ~~the~~ 0_{T_gG} over every g) are Lie-group homomorphisms s.th. $\boxed{\pi_{TG} \circ 0_{TG} = \mathrm{id}_G.}$ (1)

Proof: The assignment $M \to TM$ is functorial, i.e., for $M_1 \xrightarrow{f} M_2$ we get $TM_1 \xrightarrow{Tf} TM_2$, & $T(f_1 \circ f_2) = Tf_1 \circ Tf_2$ as well as $T\mathrm{id}_M = \mathrm{id}_{TM}$. Therefore, T transports to the tangent Lie group not only the components of the structure of G, but also the axiomatic relations (expressed by commutative diagrams) that define the structure. We only need to prove the homomorphicity of π_{TG} & 0_{TG} & the identity ...

We have – for any $g, h \in G$ –

$$T_{(g,h)} m : T_{(g,h)}(G \times G) \cong T_g G \oplus T_h G \longrightarrow T_{m(g,h)} G,$$

& so $\pi_{TG} \circ Tm = m \circ (\pi_{TG} \times \pi_{TG})$,

but that is precisely homomorphicity!

& since m is smooth, so is Tm,

hence we obtain a Lie-group homomorphism.

Next, define one-sided 'actions'

$l_g : G \circlearrowleft : h \longmapsto m(g,h)$ (LEFT REGULAR ACTION)

$p_g : G \circlearrowleft : h \longmapsto m(h,g)$ (RIGHT ACTION $-''-$).

Pick up local charts $\kappa_g \equiv (x^1, x^2, \ldots, x^n) : \mathcal{O}_g \xrightarrow{\cong} U_g \subset \mathcal{J}(\mathbb{R}^{\times n})$ (for $n \equiv \dim G$) on a neighbourhood $\mathcal{O}_g$ of g,

$\kappa_h \equiv (y^1, y^2, \ldots, y^n) : \mathcal{O}_h \xrightarrow{\cong} U_h \in \mathcal{J}(\mathbb{R}^n)$ on $\mathcal{O}_h$ of h,

& $\kappa_{g \cdot h} \equiv (z^1, z^2, \ldots, z^n) : \mathcal{O}_{g \cdot h} \xrightarrow{\cong} U_{g h} \in \mathcal{J}(\mathbb{R}^n)$ on $\mathcal{O}_{gh}$ of $g \cdot h$,

to obtain – for any $V \equiv V^i \frac{\partial}{\partial x^i}(g) \in T_g G$ & $W = W^a \frac{\partial}{\partial y^a}(h) \in T_h G$ – the identity

$$T_{(g,h)} m (V, W) = V^i \frac{\partial (z^\mu \circ m \circ (\kappa_g^{-1} \times \kappa_h^{-1}))}{\partial x^i}(\kappa_g(g), \kappa_h(h)) \frac{\partial}{\partial x^\mu}(g \cdot h)$$
$$+ W^a \frac{\partial (z^\mu \circ m \circ (\kappa_g^{-1} \times \kappa_h^{-1}))}{\partial y^a}(\kappa_g(g), \kappa_h(h)) \frac{\partial}{\partial x^\mu}(g \cdot h)$$

in which we recognise

$$T_{(g,h)}m\,(V,W) = V^i\,(T_g p_h)^\mu{}_i\,\frac{\partial}{\partial z^\mu(g\cdot h)} + W^a\,(T_h l_g)^\mu{}_a\,\frac{\partial}{\partial z^\mu(g\cdot h)}$$

so that

$$\boxed{T_{(g,h)}m \equiv T_g p_h \circ pr_1 + T_h l_g \circ pr_2} \quad (2)$$

From that we derive – for any $g,h\in G$ –

$$Tm\circ(0_{TG}\times 0_{TG})(g,h) \equiv T_{(g,h)}m\,(0_{T_gG}, 0_{T_hG})$$

$$= T_g p_h\,(0_{T_gG}) + T_h l_g\,(0_{T_hG}) = 0_{T_{g\cdot h}G} + 0_{T_{g\cdot h}G}$$

$$= 0_{T_{g\cdot h}G} \equiv 0_{TG}\circ m\,(g,h) \Longleftrightarrow$$ homomorphicity of 0_{TG} (clearly, 0_{TG} is smooth)

Identity (1) is obvious. $\square$

We shall also need

Prop.^n 2

$$\boxed{\begin{aligned} T_g\,\mathrm{Inv} &= -T_e l_{g^{-1}}\circ T_g p_{g^{-1}} \\ &\equiv -T_e p_{g^{-1}}\circ T_g l_{g^{-1}} \end{aligned}} \quad (3)$$

} trivial (commutativity of l. & p.)

Proof: Take the tangent of

$$m\circ(\mathrm{Id}_G\times \mathrm{Inv})\circ\Delta = \eta = m\circ(\mathrm{Inv}\times\mathrm{Id}_G)\circ\Delta$$

for $\Delta : G\to G\times G : g\mapsto (g,g)$ & $\eta : G\circlearrowleft : g\mapsto e$

to obtain

$$T m \circ (\mathrm{id}_{TG} \times T\mathrm{Inv}) \circ T\Delta = 0 = Tm \circ (T\mathrm{Inv} \times \mathrm{id}_{TG}) \circ T\Delta,$$ 20

& so – for any $g \in G$ & $V \in T_g G$ –

(2) $\to$
$$T_g p_{g^{-1}}(V) + T_{g^{-1}} l_g \circ T_g \mathrm{Inv}(V) = 0_{T_eG} = T_{g^{-1}} p_g \circ T_g \mathrm{Inv}(V) + T_g l_{g^{-1}}(V)$$

$\Downarrow$

~~Tg~~ thesis. $\square$

Next, we consider

<u>Prop.$^{\underline{n}}$ 3.</u> In the hitherto notation define the <u>ADJOINT ACTION</u>

$$\mathrm{Ad}_\cdot : \underset{\text{acts}}{G} \times \underset{\text{is being acted upon}}{G} \longrightarrow G : (g, h) \longmapsto g \cdot h \cdot g^{-1}.$$

The map
$$T_e l_\cdot : G \ltimes_{T_e\mathrm{Ad}_\cdot} T_e G \longrightarrow TG$$
$$: (g, X) \longmapsto T_e l_g(X)$$

is a Lie-group isomorphism.

Here, $G \ltimes_{T_e\mathrm{Ad}_\cdot} T_e G$ is the semi-direct group structure ~~of~~ on $G \times T_e G$ defined by

$$(g_1, X_1) \cdot (g_2, X_2) := \left(g_1 \cdot g_2,\ T_e \mathrm{Ad}_{g_2^{-1}}(X_1) + X_2\right).$$

$\mathrm{Ad}_g : G \circlearrowleft : h \mapsto g h g^{-1} \Longrightarrow T_e \mathrm{Ad}_g : T_e G \circlearrowleft$

Proof: Using (2), we derive

$$T_{(g,e)}m\circ(0_{TG}\times id_{T_eG})(g,X) = T_{(g,e)}(0_{T_gG}, X)$$

$$= \underbrace{T_g p_e(0_{T_gG})}_{=0} + T_e l_g(X) = T_e l_g(X) = T_e l_\cdot(g,X),$$

i.e., $\underline{T_e l_\cdot \equiv T_{(g,e)}m\circ(0_{TG}\times id_{T_eG})}$ (4)

which implies smoothness of $T_e l_\cdot$.

But (clearly!) $(T_e l_\cdot)^{-1} = (\pi_{TG}, T_{\pi_{TG}(\cdot)} l_{Inv\circ\pi_{TG}(\cdot)}(\cdot))$,

as seen from

$$(\pi_{TG}, T_{\pi_{TG}(\cdot)} l_{Inv\circ\pi_{TG}(\cdot)}(\cdot))\circ T_e l_\cdot(g,X)$$

$$\equiv (\pi_{TG}\circ T_e l_g(X), T_{(\pi_{TG}\circ T_e l_g(X))} l_{Inv\circ\pi_{TG}\circ T_e l_g(X)}\circ T_e l_g(X))$$

$$\equiv (g, T_g l_{g^{-1}}\circ T_e l_g(X)) \equiv (g,X)$$

&

$$T_e l_\cdot\circ(\pi_{TG}, T_{\pi_{TG}(\cdot)} l_{Inv\circ\pi_{TG}(\cdot)}(\cdot))(v^{\in TG})$$

$$= T_e l_{\pi_{TG}(v)}\circ T_{\pi_{TG}(v)} l_{Inv\circ\pi_{TG}(v)}(v)$$

$$\underset{\text{chain rule!}}{\equiv} T_{\pi_{TG}(v)}(l_{\pi_{TG}(v)}\circ l_{\pi_{TG}(v)^{-1}})(v) = T_{\pi_{TG}(v)} l_e(v) = v.$$

Hence, $T_e l.$ is a diffeomorphism ($(T_e l.)^{-1}$ is manifestly smooth!). It remains to verify its homomorphicity...

We compute – for any $g,h\in G$ & $X,Y\in T_eG$ –

$$T_e l.\big((g,X)\cdot(h,Y)\big) \equiv T_e l.\big(g\cdot h,\ T_e\underbrace{\mathrm{Ad}_{h^{-1}}}_{\mathrm{Ad}_G}(X)+Y\big)$$

$$= T_e l_{g\cdot h}\big(T_e\mathrm{Ad}_{h^{-1}}(X)+Y\big) = T_e(l_g\circ l_h\circ l_{h^{-1}}\circ p_h)(X) + T_e(l_g\circ l_h)(Y)$$

$$= T_e(p_h\circ l_g)(X) + T_e(l_g\circ l_h)(Y)$$

$$= T_g p_h\big(T_e l_g(X)\big) + T_h l_g\big(T_e l_h(Y)\big)$$

$$\equiv \big(T_g p_h\circ pr_1 + T_h l_g\circ pr_2\big)\big(T_e l.(g,X),\ T_e l.(h,Y)\big)$$

$$\underset{(2)}{\equiv} T_{(g,h)} m\circ(T_e l.\times T_e l.)\big((g,X),(h,Y)\big),$$

which is the desired identity! $\square$

We may now introduce ...

Def$^{\underline{n}}$ 2. A LEFT-INVARIANT VECTOR FIELD (LI) on a Lie group G is a vector field $\mathcal{L} \in \Gamma(TG)$ with the property

$$\forall g \in G : \ l_{g*}\mathcal{L} = \mathcal{L},$$

$$\left(\text{i.e.}, \quad T_h l_g (\mathcal{L}(h)) = \mathcal{L}(g \cdot h) \ . \right)$$

Analogously, a RIGHT-INVARIANT VECTOR FIELD (RI) on a Lie group G is a vector field $\mathcal{R} \in \Gamma(TG)$ with the property

$$\forall g \in G : \ p_{g*}\mathcal{R} = \mathcal{R}.$$

Prop$^{\underline{n}}$ 4: Sets

$$\mathfrak{X}_L(G) := \{ \mathcal{L} \in \Gamma(TG) \mid \mathcal{L} \text{ left-invariant} \}$$

$$\mathfrak{X}_R(G) := \{ \mathcal{R} \in \Gamma(TG) \mid \mathcal{R} \text{ right-invariant} \}$$

are $\mathbb{R}$-linear subspaces in $\Gamma(TG)$, & the commutator of vector fields closes on each of them.

The pair

$$\left(\mathfrak{X}_L(G), [\cdot,\cdot]\restriction_{\mathfrak{X}_L(G)\times\mathfrak{X}_L(G)} \right)$$

is termed the (Lie) <u>ALGEBRA</u> of <u>LI VECTOR FIELDS on G</u>, and the pair

$$\left(\mathfrak{X}_R(G), [\cdot,\cdot]\restriction_{\mathfrak{X}_R(G)\times\mathfrak{X}_R(G)} \right)$$

is termed the (Lie) <u>ALGEBRA</u> of <u>RI VECTOR FIELDS on G</u>.

<u>Proof</u>: ~~The~~ $\mathbb{R}$-linearity is obvious (the defining cond$^{\underline{n}}$ is $\mathbb{R}$-linear!).
We have - for $\mathcal{L}_1, \mathcal{L}_2 \in \mathfrak{X}_L(G)$ -

$$l_{g*}[\mathcal{L}_1, \mathcal{L}_2] \overset{\text{course on Diff. Geom.!}}{=} [l_{g*}\mathcal{L}_1, l_{g*}\mathcal{L}_2] = [\mathcal{L}_1, \mathcal{L}_2]$$

& similarly for $\mathcal{R}_1, \mathcal{R}_2 \in \mathfrak{X}_R(G)$ & ρ_{g*}. □

How are the two ~~...~~ $\mathbb{R}$-linear spaces related? The answer is given in

<u>Prop$^{\underline{n}}$</u> 5. The map $T_\cdot \mathrm{Ad}_\cdot : \mathfrak{X}_R(G) \longrightarrow \mathfrak{X}_L(G)$

$$: \mathcal{R}(\cdot) \longmapsto T_\cdot \mathrm{Ad}_\cdot(\mathcal{R}(\cdot))$$

is an isomorphism. There exist, moreover, canonical isomorphisms $H_\cdot : \mathfrak{g} \equiv \mathrm{Der}_e^1 C^1(G,\mathbb{R}) \xrightarrow{\cong} \mathfrak{X}_H(G)$, $H \in \{L, R\}$
with the property $L_\cdot \circ (R_\cdot)^{-1} \equiv T_\cdot \mathrm{Ad}_\cdot$.

These induce on $\mathfrak{g} \cong T_e G$ a Lie bracket

$$[\cdot,\cdot]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : (X_1, X_2) \longmapsto [L_{X_1}, L_{X_2}](e).$$

The Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}) \equiv \mathrm{Lie}\, G$ is called the <u>TANGENT LIE ALGEBRA of G</u>.

<u>Proof</u>: Consider $\mathcal{R} \in \mathfrak{X}_R(G)$ to obtain – for any $g, h \in G$ –

$$\left(l_{g*}\left(T_\cdot \mathrm{Ad}_\cdot(\mathcal{V}(\cdot))\right)\right)(h) \equiv T_{g^{-1}\cdot h}\, l_g\left(T_\cdot \mathrm{Ad}_\cdot(\mathcal{V}(\cdot))(g^{-1}\cdot h)\right)$$

$$= T_{g^{-1}\cdot h}\, l_g \circ T_{g^{-1}\cdot h}\, \mathrm{Ad}_{g^{-1}\cdot h}\left(\mathcal{V}(g^{-1}\cdot h)\right)$$

$$= T_{g^{-1}\cdot h}\left(\cancel{l_g \circ l_{g^{-1}}} \circ l_h \circ p_{h^{-1}\cdot g}\right)\left(\mathcal{V}(g^{-1}\cdot h)\right)$$

$$= T_e l_h \circ \left(T_{g^{-1}\cdot h}\, p_{h^{-1}\cdot g}\left(\mathcal{V}(g^{-1}\cdot h)\right)\right)$$

$$\underset{RI}{=} T_e l_h\left(\mathcal{V}(e)\right) \equiv T_e l_h \circ T_h p_{h^{-1}}\left(\mathcal{V}(h)\right)$$

$$\equiv \left(T_\cdot \mathrm{Ad}_\cdot\left(\mathcal{V}(\cdot)\right)\right)(h),$$ which is what we want!

Our search for the isomorphism L. (26)
starts with the following observation:
$\forall \mathcal{L} \in \mathfrak{X}_L(G)$:

$$\mathcal{L}(g) \equiv (l_{g*}\mathcal{L})(g) = T_{g^{-1}\cdot g}\, l_g\left(V(g^{-1}\cdot g)\right)$$

$$\equiv T_e\, l_g\left(V(e)\right) \equiv \underset{\mathfrak{g}}{\underset{\cap}{V(e)}} \circ l_g^* \leftarrow \text{pull-back on } C^\infty(G,\mathbb{R})$$

~~Therefore~~ Accordingly, we postulate

$$L_\cdot : \mathfrak{g} \Rightarrow \mathfrak{X}_L(G) : X \longmapsto X \circ l_\cdot^* \equiv L_X$$

So check – for any $X \in \mathfrak{g}$ –
(the other way round)

$$l_{g*} L_X \equiv T_{l_{g^{-1}}(\cdot)}\, l_g\left(L_X(l_{g^{-1}}(\cdot))\right)$$

$$\equiv L_X(l_{g^{-1}}(\cdot)) \circ l_g^* \equiv X \circ l^*_{l_{g^{-1}}(\cdot)} \circ l_g^*$$

$$= X \circ (l_g \circ l_{l_{g^{-1}}(\cdot)})^* = X \circ l^*_{g\cdot g^{-1}\cdot(\cdot)} = X \circ l_\cdot^* \equiv L_X,$$

which is the desired left-invariance!

~~From~~ the basis of our earlier computation,
we postulate, furthermore, ...

$(L_\cdot)^{-1} \equiv ev_e : \mathfrak{X}_L(G) \longrightarrow \mathfrak{g} : V \longmapsto V(e)$

& check

$$id_{\mathfrak{X}_L(G)}(V) \equiv V = T_\cdot \ell_{(\cdot)}(V) \circ \ell^* \equiv V(e) \circ \ell^*_\cdot$$

$$\equiv ev_e(V) \circ \ell^*_\cdot \equiv L_\cdot \circ ev_e(V)$$

as well as

$$ev_e \circ L_\cdot(X) \equiv L_X(e) \equiv X \circ \ell^*_e \equiv X\,.$$

Upon replacing $\ell \longmapsto p$, we similarly obtain

$$R_\cdot : \mathfrak{g} \to \mathfrak{X}_R(G) : X \longmapsto X \circ p^*_\cdot \equiv R_X$$

& $R_\cdot^{-1} = ev_e$. This immediately yields the expected result:

$$(L_\cdot \circ R_\cdot^{-1}(\mathcal{R}))(g) \equiv L_{R_\cdot^{-1}(\mathcal{R})}(g)$$

$$\equiv L_{\mathcal{R}(e)}(g) \equiv \mathcal{R}(e) \circ \ell^*_g = T_e \ell_g(\mathcal{R}(e))$$

$$\equiv T_e \ell_g \circ (T_g p_{g^{-1}} \circ T_e p_g(\mathcal{R}(e)))$$

$$= (T_e \ell_g \circ T_g p_{g^{-1}})(T_e p_g(\mathcal{R}(e)))$$

$$\equiv T_g Ad_g(\mathcal{R}(g)) \equiv (T_\cdot Ad_\cdot(\mathcal{R}(\cdot)))(g)\,.$$

The last thing to be checked is the vanishing of the Jacobiator of $[\cdot,\cdot]_{\mathfrak{g}}$ as defined above ...

But we have

$$L_{[X_1,X_2]_{\mathfrak{g}}} \equiv L_{[L_{X_1},L_{X_2}](e)} \equiv L_{ev_e([L_{X_1},L_{X_2}])}$$

$$= L_\cdot \circ \varepsilon_\cdot^{-1}([L_{X_1},L_{X_2}]) = [L_{X_1},L_{X_2}],$$

& so

$$\mathrm{Jac}_{\mathfrak{g}}(X_1,X_2,X_3) \equiv [[X_1,X_2]_{\mathfrak{g}},X_3]_{\mathfrak{g}} + [[X_3,X_1]_{\mathfrak{g}},X_2]_{\mathfrak{g}} + [[X_2,X_3]_{\mathfrak{g}},X_1]_{\mathfrak{g}}$$

$$\equiv [L_{[X_1,X_2]_{\mathfrak{g}}}, L_{X_3}](e) + [L_{[X_3,X_1]_{\mathfrak{g}}}, L_{X_2}](e) + [L_{[X_2,X_3]_{\mathfrak{g}}}, L_{X_1}](e)$$

$$= [[L_{X_1},L_{X_2}],L_{X_3}](e) + [[L_{X_3},L_{X_1}],L_{X_2}](e) + [[L_{X_2},L_{X_3}],L_{X_1}](e)$$

$$\equiv \mathrm{Jac}_{\mathfrak{X}(TG)}(L_{X_1},L_{X_2},L_{X_3})(e) = 0 \quad \square$$

STANDARD NOTATION (& nomenclature):

Let $\dim G \equiv N < \infty$ & choose a basis $\mathcal{T} \equiv \{t_A\}_{A \in \overline{1,N}}$ in $\mathfrak{g}$. **STRUCTURE CONSTANTS** of $\underline{\mathrm{Lie}\,G}$ in $\mathcal{T}$ are numbers f

$$f_{AB}{}^{C} \equiv -f_{BA}{}^{C} \in \mathbb{R}, \quad A,B,C \in \overline{1,N}$$

defined by **STRUCTURE EQUATIONS** of $\mathrm{Lie}(G)$

$$[t_A, t_B] = f_{AB}{}^{C}\, t_C, \quad A,B \in \overline{1,N}$$

$\mathfrak{g} \Rightarrow$ decomposes in $\mathcal{T}$!

We now (re)-obtain **JACOBI IDENTITIES**

$$f_{AB}{}^{D} f_{DC}{}^{E} + f_{CA}{}^{D} f_{DB}{}^{E} + f_{BC}{}^{D} f_{DA}{}^{E} = 0$$

as an equivalent statement of ~~the Jacobi~~

$$\mathrm{Jac}_{\mathfrak{g}} \equiv 0.$$

We denote (in what follows):

$$\left| \begin{aligned} L_A &:= T_e l_\cdot (t_A) \\ R_A &:= T_e p_\cdot (t_A) \end{aligned} \right. , \quad A \in \overline{1,N}$$

The last ~~fact~~ that we discuss is

just like T in Man !

Th$^{\underline{m}}$ 1 [Functoriality of Lie$(\cdot)$]

The assignment $\underset{\text{Lie group}}{G} \longmapsto \underset{\text{its tangent Lie algebra}}{\mathrm{Lie}(G)}$

canonically extends to a (convenient) functor with the morphism component

$$\left(G_1 \xrightarrow{X} G_2\right) \overset{\mathrm{Lie}}{\longmapsto} \left(\underset{\mathfrak{g}_1}{\overset{\shortparallel}{\mathrm{Lie}\,G_1}} \xrightarrow{T_{e_1}X \equiv \mathrm{Lie}(X)} \underset{\mathfrak{g}_2}{\overset{\shortparallel}{\mathrm{Lie}\,G_2}}\right).$$

Proof: First of all, note that) is well defined, & so all we need to verify is its (Lie-algebra) homomorphicity, i.e., $[\cdot,\cdot]_{\mathfrak{g}_2} \circ (\mathrm{Lie}X \times \mathrm{Lie}X)$

$= \mathrm{Lie}X \circ [\cdot,\cdot]_{\mathfrak{g}_1}$.

Consider – for any $g_1 \in G_1$ – the identity

$$\underline{X \circ l_{g_1} = l_{X(g_1)} \circ X}$$

& compute – for any $f \in C^\infty(G_2, \mathbb{R})$ & $X \in \mathfrak{g}_1$ –

$$\left(T_{g_1}X\left(L_X(g_1)\right)\right)(f) \equiv L_X(g_1) \circ X^*(f) \equiv X \circ l_{g_1}^* \circ X^*(f)$$

$$= X \circ (X \circ l_{g_1})^*(f) = X \circ \left(l_{X(g_1)} \circ X\right)^*(f)$$

$$= (X \circ X^*) \circ l_{X(g_1)}^*(f) \equiv T_{e_1}X(X) \circ l_{X(g_1)}^*(f) \equiv L_{T_{e_1}X(X)}(X(g_1))(f)$$

whence $$T_\cdot \chi\left(L_X(\cdot)\right) \equiv L_{T_{e_1}\chi(X)}\left(\chi(\cdot)\right),$$

& so also – for any $X_1, X_2 \in \mathfrak{g}$ –

$$\left[T_{e_1}\chi(X_1), T_{e_1}\chi(X_2)\right]_{\mathfrak{g}_2} \equiv \left[L_{T_e\chi(X_1)}, L_{T_e\chi(X_2)}\right]_{\Gamma(TG_2)}(e_2)$$

$$= \left[L_{T_e\chi(X_1)}, L_{T_e\chi(X_2)}\right]_{\Gamma(TG_2)}\left(\chi(e_1)\right)$$

$$= T_{e_1}\chi\left(\left[L_{X_1}, L_{X_2}\right]_{\Gamma(TG_1)}(e_1)\right) \equiv T_{e_1}\chi\left([X_1, X_2]_{\mathfrak{g}_1}\right) \quad \square$$

Next, we shall get back from the infinitesimal/tangential level to the global one on G by integrating the distinguished H-invariant vector fields ...