CLASSICAL FIELD THEORY (interaction at a distance)

Having understood the canonical structure of classical mechanics, we may, next, discuss the main object of our interest, to wit, the theory of physical fields in the canonical paradigm. The general scheme is as follows:

I <u>The lagrangean formulation</u>:

We have a fibre bundle

$$\begin{tikzcd} F \arrow[r] & \mathcal{F} \arrow[d] \\ & X \end{tikzcd}$$

TYPICAL FIBRE — F

TOTAL SPACE — $\mathcal{F}$

BASE — X

over SPACETIME X whose <u>global</u> sections $\phi \in \Gamma(\mathcal{F})$ are ~~[illegible]~~ to be thought of as fields – hence the name: FIELD BUNDLE/SPACE given to it – with local degrees

of freedom quantified by F. (2)
(number, structure, e.g., that of a vector space, or representation of a group etc.)

Since we want to model dynamics of fields over X (which we assume to be a metric space (X, g), differentiable, with $\mathrm{sign}(g) = (1, d)$), we need ~~the~~ field equations of evolution, to be obtained from (an ACTION FUNCTIONAL defining) the DIRAC-FEYNMAN AMPLITUDE

$$\mathcal{A}_{DF} \equiv e^{iS} : \Gamma(\mathcal{F}) \longrightarrow U(1)$$

as its stationary points (minima), where

$$S[\phi] = \int_X \underset{\substack{\text{volume}\\ \text{form on } X}}{\mathrm{Vol}(X)}\, \mathcal{L}(\underset{\substack{\text{(local) coords}\\ \text{on } X}}{x}, \phi, \underset{\substack{\text{derivatives of the field(s) } \phi\\ \text{along the } x}}{\partial\phi})$$

This is the arena of Lagrangean Field Theory.

<u>NB</u>: Differentiating sections of a bundle in a geometrically meaningful manner requires some care — we shall come back to it soon...

To $\mathcal{A}_{DF}$, we apply the standard variational principle (PRINCIPLE of LEAST ACTION) with variations

$$V[\phi] = V^A(\phi) \frac{\delta}{\delta\phi^{A}(\cdot)}$$

enumerates ~~fe~~ field components species

vanishing at ∂X

variation of a field configuration

~~formally~~ geometrically, these are just some local coordinates in F (the typical fibre) of $F \to X$

i.e., $$V[\phi]\restriction_{\partial X} \overset{!}{=} 0$$

We then obtain – in the local coord x^μ on X $\rightsquigarrow \partial_\mu = \frac{\partial}{\partial x^\mu}$

$$V \lrcorner\, \delta S[\phi] = \int_X \mathrm{vol}(X) \Big[V^A(\phi(x)) \frac{\partial \mathcal{L}}{\partial \phi^A}(x,\phi(x),\partial\phi(x)) + \partial_\mu V^A(\phi(x)) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)}(x,\phi(x),\partial\phi(x)) \Big]$$

$$= \int_X \mathrm{vol}(X)\, \partial_\mu \Big(V^A(\phi(\cdot)) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)}(\cdot,\phi(\cdot),\partial\phi(\cdot)) \Big)(x)$$

$$+ \int_X \mathrm{vol}(X)\, V^A(\phi(x)) \Big[\frac{\partial \mathcal{L}}{\partial \phi^A}(x,\phi(x),\partial\phi(x)) - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(x,\phi(x),\partial\phi(x))$$

The first term in the variation is (4)

the integral of the divergence of a vector field over X, so

$$V \lrcorner \delta S[\phi] = \int_{\partial X} \partial_\mu \lrcorner \mathrm{Vol}(X) \cdot \underbrace{V^A(\phi(x)) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)}(x, \phi(x), \partial\phi(x))}_{=0}$$

$$+ \int_X \mathrm{Vol}(X)\, V^A(\phi(x)) \left[\frac{\partial \mathcal{L}}{\partial \phi^A}(x, \phi(x), \partial\phi(x)) - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)}(x, \phi(x), \partial\phi(x)) \right]$$

whence also the Euler-Lagrange Eq^{n}s:

$$\frac{\partial \mathcal{L}}{\partial \phi^A} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)}, \quad A \in \overline{1, \dim F}$$

satisfied by EXTREMAL SECTIONS of $\mathcal{F}$ (CLASSICAL FIELD CONFIGURATIONS)

These are – in analogy with the mechanical model – 2nd-order partial differential equations for the ϕ_*^C as

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)} = \frac{\partial^2 \mathcal{L}}{\partial x^\mu \partial(\partial_\mu \phi^A)} + \frac{\partial^2 \mathcal{L}}{\partial \phi^B \partial(\partial_\mu \phi^A)} \frac{\partial \phi^B}{\partial x^\mu} + \frac{\partial^2 \mathcal{L}}{\partial(\partial_\nu \phi^B) \partial(\partial_\mu \phi^A)} \partial^2_{\mu\nu} \phi^B$$

We shall assume, from now onwards, that solutions to these eq^{n}s are determined – at least locally – in time – by the Cauchy data localised on a hypersurface $x^0 \equiv t = \text{const.}$

$$\left(\phi(t, \vec{x}) \,,\, \partial_t \phi(t, \vec{x}) \right)$$

(NB: spatial $\partial_i \phi(t, \vec{x})$ are fixed by $\phi(t, \vec{x})$)

Here, we have explicitly singled out the 'time' coörd x^0.

Examples :

1° WAVE Eq^{n} : Let ϕ be a scalar field of $\mathcal{F} = X \times \mathbb{R}$. The lagrangean density

$$\mathcal{L}(x, \phi, \partial\phi) := \frac{1}{2v^2}(\partial_t \phi)^2 - \frac{1}{2}\vec{\nabla}\phi \cdot \vec{\nabla}\phi$$

yields, as its E-L eq^{n}s,

$$\frac{1}{v^2}\partial_t^2 \phi - \vec{\nabla}^2 \phi = 0$$

$$\stackrel{\text{def}}{=} \Box_v \phi$$

This is the eq^{n} of a wave travelling at a speed v.

2° The Schrödinger Eq^{n} : Let ψ be a complex field of $\mathcal{F} = X \times \mathbb{C}$.

The lagrangean density

$$\mathcal{L}(\underset{(t,\vec{x})}{x}, \psi, \partial\psi) := \frac{i}{2}\overset{\text{complex conjugate of } \psi}{\bar{\psi}}\,\partial_t \psi - \frac{i}{2}(\partial_t \bar{\psi})\psi - \frac{1}{2m}\vec{\nabla}\bar{\psi} \cdot \vec{\nabla}\psi - \bar{\psi}\, V(t, \vec{x})\, \psi$$

yields, as its E-L eq^{n}s,

$$\frac{i}{2}\partial_t \psi - V(t,\vec{x})\psi = -\frac{i}{2}\partial_t \psi - \frac{1}{2m}\vec{\nabla}^2\psi ,$$

or

$$\boxed{i\partial_t \psi = \left(-\frac{1}{2m}\vec{\nabla}^2 + V(t,\vec{x})\right)\psi}$$

which is the familiar Schrödinger eq^{n} for the ~~pa~~ wave-function ψ.

II The hamiltonian approach: (8)

Let us define the lagrangian of our field theory as

time parameter $\Rightarrow$ $\Phi(t) \equiv \Phi_t$, $\dot{\Phi}(t) \equiv \partial_t \Phi_t$

$$L[t, \underset{\underset{\phi(t,\vec{x})}{\parallel\!\!\!\!\!\!\!\!}}{\Phi(t)}, \underset{\underset{\partial_t \phi(t,\vec{x})}{\parallel}}{\dot{\Phi}(t)}] := \int d\mathrm{vol}(\Sigma_t) \mathcal{L}(t, \vec{x}; \phi(t,\vec{x}), \partial\phi(t,\vec{x}))$$

$\vec{x} \in \Sigma_t$

where Σ_t is an equitemporal slice, i.e., a Cauchy hypersurface (of $t = \text{const}$) on which we have Cauchy data as above $\phi|_{\Sigma_t}$, $\partial_t\phi|_{\Sigma_t}$

The lagrangian is a functional of these data ($\phi|_{\Sigma_t}$ & $\partial_t\phi|_{\Sigma_t}$ are functions on Σ_t, & L is a 'function' of these functions). By analogy with mechanics, we define the momentum conjugate to Φ as

$$\Pi_{t_A}(\vec{x}) := \frac{\delta L}{\delta \dot{\Phi}_t(\cdot)}(\vec{x}),$$

where the functional derivative of a functional F of functions $f^k(\cdot)$ over Σ_t is defined by the formula:

$$V^k(f)\frac{\delta}{\delta f^k(\cdot)} \lrcorner\, \delta F(f^k)$$

$$:=: \int_{\Sigma_t} Vol(\Sigma_t)\, V^k(f)\, \frac{\delta F}{\delta f^k(\cdot)}(f).$$

If, as we shall assume, we can express the 'velocities' $\dot{\Phi}_t$ in terms of the 'momenta' Π_t for a given Φ_t, we may define the hamiltonian of the field theory:

$$H(t, \Phi_t, \Pi_t) := \int_{\Sigma_t} Vol(\Sigma_t)\, \Pi_t^A(\vec{x})\, \dot{\Phi}_t(\vec{x}) - L(t, \Phi_t, \dot{\Phi}_t)$$

The E-L eq$^{\underline{n}}$s now take on the form of Hamilton's eq$^{\underline{n}}$s :

$$\begin{cases} \dfrac{d}{dt}\Phi_t^A(\vec{x}) = \dfrac{\delta H}{\delta \Pi_{tA}}(\vec{x}) \\[2ex] \dfrac{d}{dt}\Pi_{tA}(\vec{x}) = -\dfrac{\delta H}{\delta \Phi_t^A}(\vec{x}) \end{cases}$$

∞-dimensional Fréchet manifold (typically)

Once again, we define the PHASE SPACE (or SPACE of STATES) $\mathcal{P}$ of the theory as the space of solutions to the E-L eq$^{\underline{n}}$s, naturally parametrised — that's what we assume! — by pairs (Φ_t, Π_t). We endow it with the Poisson structure :

$$\forall F, G \in C^\infty(\mathcal{P}, \mathbb{R}) : \{F, G\} := \int_{\Sigma_t} d\vec{x} \left(\frac{\delta F}{\delta \Phi_t^A} \frac{\delta G}{\delta \Pi_{tA}} - \frac{\delta F}{\delta \Pi_{tA}} \frac{\delta G}{\delta \Phi_t^A} \right)(\vec{x})$$

functionals!

We shall verify later on that the Right Hand Side (RHS) of the above is independent of t.

We have the familiar-looking formulae:

$$\begin{cases} \dfrac{d}{dt}\Phi_t^A = \{\Phi_t^A, H\} \\ \dfrac{d}{dt}\Pi_{tA} = \{\Pi_{tA}, H\} \end{cases},$$

& more generally,

explicit dependence on time!

$$\forall F \in C^\infty(\mathbb{R}\times\mathcal{P}, \mathbb{R}) : \frac{d}{dt}F = \partial_t F + \{F, H\}.$$

III The geometric approach

Let ϕ_* be a classical field configuration. We then obtain:

$$0 = V \lrcorner \, \delta S_V[\phi_*] = \int_{\partial V} \partial_\mu \lrcorner \mathrm{Vol}(X)\, V^A(\phi_*(x))\, \pi^\mu_A(x),$$

S_V: restriction to some volume $\subset X$

where $\pi^\mu_A(x) := \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)}(x, \phi_*(x), \partial\phi_*(x))$.

In other words, we have the following equality of 1-forms on $\mathcal{P}$:

$$\delta S_V \overset{(MF)}{=} \alpha_{\partial V}, \text{ where}$$

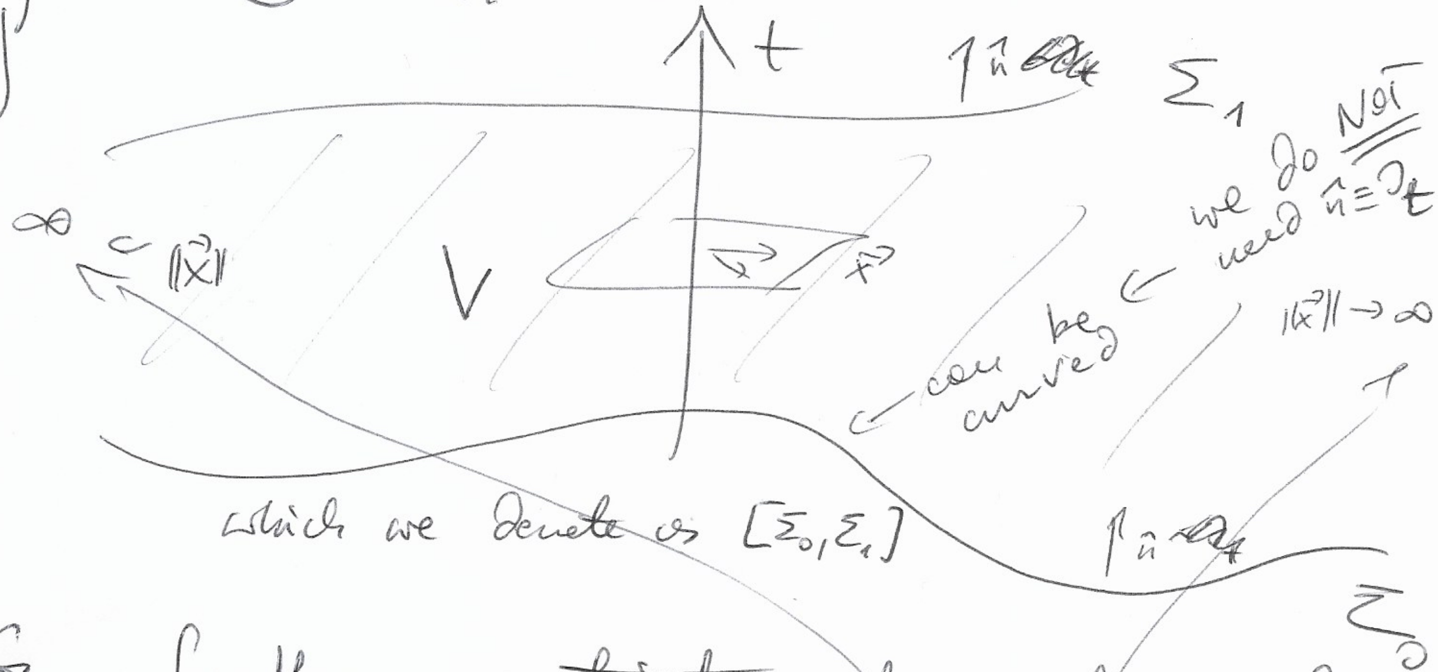
$$\alpha_\Sigma[\phi,\pi] := \int_\Sigma \partial_\mu \lrcorner \mathrm{Vol}(X)\, \delta\phi^A_*(x)\, \pi^\mu_A(x)$$

(ACTION 1-FORM).

Thus, α_Σ is exact if $\Sigma = \partial V$ for some volume $V \subset X$.

Let, next, V be sandwiched (13)

between two (e.g., equitemporal slices ⇒) Cauchy hypersurfaces Σ_0 & Σ_1 ×)



which we denote as $[\Sigma_0, \Sigma_1]$

& further restrict to classical field configurations that drop off to 0 sufficiently fast at $\|\vec{x}\| \to \infty$ (so that the integrals to follow vanish). Our master formula (MF) then rewrites as

orientation!

$$\delta S\Big|_{[\Sigma_0,\Sigma_1]} = \alpha_{\Sigma_1} - \alpha_{\Sigma_0}, \quad \text{so that}$$

restriction to quickly vanishing fields!

$$0 = \delta^2 S_{[\Sigma_0,\Sigma_1]} = \delta\alpha_{\Sigma_1} - \delta\alpha_{\Sigma_0}$$

We may, therefore, define a closed 2-form on the (sub)space of sm. & quickly vanishing (at $\|\vec{x}\| \to \infty$) classical field configurations:

we shall drop π henceforth!

$$\Omega[\phi(x), \pi(x)] = \delta\alpha_{\Sigma} = \int_{\Sigma} \partial_\mu \lrcorner \mathrm{Vol}(\Sigma)\, \delta\pi^\mu_A(x) \wedge \delta\phi^A(x)$$

any one!

Here, the RHS is independent of the choice of ~~Σ~~ the Cauchy hypersurface Σ!

If we take simply $\Sigma \equiv \Sigma_t$ (equitemporal slice), we reobtain

$$\Omega = \int_{\Sigma_t} \cancel{dx}\, \mathrm{Vol}(\Sigma_t)\, \delta\Pi_{tA}(\vec{x}) \wedge \delta\Phi^A_t(\vec{x})$$

Thus, if we can parametrise class. sol^{n}s faithfully to the E-L eq^{n}s by pairs (Φ_t, Π_t), we have a manifestly non-degenerate de Rham 2-cocycle on P $\Longrightarrow$ SYMPLECTIC STRUCTURE!

IV SYMMETRIES

We extend the analogy with mechanics to the description of symmetries.

To this end, we consider transformations

$$(x, \phi(x)) \xmapsto{(\tilde{x}\circ pr_1, \tilde{\phi})} (\tilde{x}^{\#}(x), \tilde{\phi}^{\#}(x,\phi))$$? is this ok? Let's see...

& define the transformed field:

$$\hat{\phi}^{\#}(\tilde{x}(x)) = \tilde{\phi}^{\#}(x, \phi(x))$$

Let us, first, understand the above inscription... We have transcribed it directly from mechanics, where

$$\begin{tikzcd} \mathcal{F} = X^{[t_0,t_1]} \times M \arrow[d, "\pi_{\mathcal{F}} \equiv pr_1"'] \\ X \end{tikzcd}$$

M ← manifold in which the particle(s) propagates(-)

I.e., we have a trivial field bundle, with global sections

$$\begin{tikzcd} X \times M \arrow[d, "\pi_{\mathcal{F}}"] \\ X \arrow[u, bend left, "\varphi \equiv (\mathrm{id}_X \times q)"] \end{tikzcd}$$

as previously

For such global sections of the trivial bundle, we have

$$(t,g) \longmapsto (\tilde{t}(x), \tilde{g}(t,g))$$

$$\begin{tikzcd}
X \times M \arrow[rr, "(\tilde{t}\circ pr_1,\, \tilde{g})"] \arrow[d, "\pi_F"] \arrow[drr, phantom, "\circlearrowleft"] & & X \times M \arrow[d, "\pi_F"] \\
X \arrow[u, bend left, "\varphi = (id_X,\, g)"] \arrow[rr, "\tilde{t}"'] & & X \arrow[u, bend right, "(id_X,\, \hat{g}) = \hat{\varphi}"']
\end{tikzcd}$$

as previously

$$t \longmapsto \tilde{t}(t)$$

$\Downarrow$ commutativity of the diagram yields

$$(id_X, \hat{g}) \circ \tilde{t} = (\tilde{t}\circ pr_1, \tilde{g}) \circ (id_X, g)$$

$$\Downarrow pr_2$$

$$\hat{g} \circ \tilde{t} = \tilde{g} \circ (id_X, g) \quad // \text{ evaluate on } t$$

$$\hat{g}(\tilde{t}(t)) = \tilde{g}(t, g(t)),$$

which is what we showed before!

Thus, our formula is OK iff (17)
we think of ϕ as a map
from X into the typical fibre F
(not $\mathcal{F}$!), & take $\mathcal{F} = X \times F$. Otherwise,
we should consider general
field-bundle morphisms

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\Phi"] \arrow[d, "\pi_{\mathcal{F}}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{F} \arrow[d, "\pi_{\mathcal{F}}"] \\ X \arrow[r, "\hat{f}"'] & X \end{tikzcd}$$

On the other hand, every bundle
is locally trivial, & so
we shall – for the sake of
simplicity – keep the simplified
description ...

Thus, with the above understanding, (18)

we consider

$$\left(x, \underset{\in F}{\phi(x)}\right) \longmapsto \left(\tilde{x}(x), \underset{\in F}{\tilde{\phi}}(x,\phi)\right)$$

&

$$\hat{\phi}\circ\tilde{x}(x) \overset{\text{def}}{:=} \tilde{\phi}(x,\phi(x))$$

Assume, furthermore, that there exist functions $K^\mu : X \times \underset{\overset{\shortparallel}{\mathcal{F}}}{F} \longrightarrow \mathbb{R}$

such that

$$\mathcal{L}\left(\tilde{x}, \hat{\phi}\circ\tilde{x}(\cdot), \partial\hat{\phi}\circ\tilde{x}(\cdot)\right)\frac{\partial(\tilde{x})}{\partial(x)}(\cdot) \overset{!}{=} \mathcal{L}(x,\phi(\cdot),\partial\phi(\cdot)) + \partial_\mu K^\mu(x,\phi(\cdot))$$

where is the jacobian of the diffeomorphism $x \mapsto \tilde{x}(x)$.

Upon restricting ~~the action~~ $\int \mathrm{vol}(X)$ to a volume $V \subset X$, we obtain

$$\int_V \mathrm{vol}(x)\frac{\partial(\tilde{x})}{\partial(x)}\mathcal{L}(\tilde{x},\hat{\phi}\circ\tilde{x},\partial\hat{\phi}\circ\tilde{x}) = S_V[\phi] + \int \mathrm{vol}(X)\,\partial_\mu K^\mu(\cdot,\phi)$$

$$\underline{\underline{\text{L}}}\ \int_{\tilde{x}(V)} \mathrm{vol}(x)\,\mathcal{L}(x,\hat{\phi},\partial\hat{\phi}) = S^V_{\tilde{x}(V)}[\hat{\phi}]$$

Hence, we find

$$S_{\tilde{X}(V)}[\hat{\phi}] = S_V[\phi] + \int_V \mathrm{Vol}(x)\, \partial_\mu K^\mu(\cdot, \phi(\cdot))$$

But the last term is a total divergence integrated over volume V, which – by Stokes' Th$^{\underline{m}}$ – is equal to the flux of K through ∂V, i.e.,

$$\boxed{S_{\tilde{X}(V)}[\hat{\phi}] = S_V[\phi] + \int_{\partial V} \partial_\mu \lrcorner \mathrm{Vol}(x)\, K^\mu(\cdot, \phi(\cdot))}$$

(with the induced orientation of the boundary of V!)

$\Downarrow$ as before (in mechanics)

transformations $(\tilde{x}_{\mathrm{opr.}}, \hat{\phi})$ induce mappings

$$\mathcal{P} \ni \phi \longmapsto \hat{\phi} \in \mathcal{P}$$

on the phase space of the theory.
(i.e., if ϕ is critical for $S_{(V)}$, $\hat{\phi}$ is critical for $S_{(\tilde{X}(V))}$)

A natural question arises: Are these also (pre-)symplectomorphisms?

Let us presuppose, as before, (20)

that V is bounded by $\partial V = \Sigma_1 \cup (-\Sigma_0)$
orientation,

so that

$$S_{\tilde{x}(V)}[\hat{\phi}] - S_V[\phi] = \left(\int_{\Sigma_1} - \int_{\Sigma_0}\right) \partial_\mu \lrcorner \mathrm{Vol}(x) K^\mu(\,;\phi(\cdot))$$

If we assume, moreover, that $\tilde{x}$ transforms Cauchy hyperspaces spatially infinite! Σ_k, $k \in \{0,1\}$ into (respective) new Cauchy hypersurfaces $\tilde{\Sigma}_k \equiv \tilde{x}(\Sigma_k)$, $k \in \{0,1\}$, we obtain δ from the above

$$\alpha_{\tilde{\Sigma}_1}[\hat{\phi}, \hat{\pi}] - \alpha_{\Sigma_1}[\phi, \pi] - \delta \int_{\Sigma_1} \partial_\mu \lrcorner \mathrm{Vol}(x) K^\mu(\cdot, \phi(\cdot))$$

$$= \alpha_{\tilde{\Sigma}_0}[\hat{\phi}, \hat{\pi}] - \alpha_{\Sigma_0}[\phi, \pi] - \delta \int_{\Sigma_0} \partial_\mu \lrcorner \mathrm{vol}(x) K^\mu(\,;\phi(\cdot))$$

So so — reasoning as before — we must have

$$\alpha_{\tilde{\Sigma}}[\hat{\phi}, \hat{\pi}] - \alpha_\Sigma[\phi, \pi] = \delta \int_{\Sigma} \partial_\mu \lrcorner \mathrm{vol}(x) K^\mu(\cdot, \phi(\cdot)) \quad (**)$$

for <u>any</u> hypersurface (spatially infinite) Σ, whence —as anticipated—

$$\delta\Omega[\hat{\phi},\hat{\pi}] = \delta\alpha_{\tilde{\Sigma}}[\hat{\phi},\hat{\pi}] = \delta\alpha_{\Sigma}[\phi,\pi] = \Omega[\phi,\pi]$$

independent of $\tilde{\Sigma}$!

Such canonical transformations ((pre-) symplectomorphisms) on $\mathcal{P}$ shall be termed <u>SYMMETRIES</u> of the field theory $\mathcal{A}_{DF}$.

These can be global (cover id_X) or local ($\tilde{X} \neq id_X$), discrete or continuous. We shall, next, study the continuous ones at some length, with view to deriving —after E. Noether— the associated conservation laws.

V Noether currents & charges (22)

Consider a 1-parameter (smooth*) family of symmetries

* ≡ the dependence on λ is smooth!

$$(x, \phi(x)) \longmapsto (\tilde{x}_\lambda(x), \tilde{\phi}_\lambda(x, \phi(x)))$$

$\lambda \in \mathbb{R}$

which – for $|\lambda| \approx 0$ – is described by vector fields $\Xi \in \Gamma(TX)$ & $\Psi \in \Gamma(TF)$ as per

in local coords (which is where we have a vector structure!) [from $\mathbb{R}^{\dim X}$ resp. $\mathbb{R}^{\dim F}$]

$$\begin{cases} \tilde{x}_\lambda(x) = x + \lambda \Xi(x) \lrcorner dx + O(\lambda^2) \\ \tilde{\phi}_\lambda(x, \phi) = \phi + \lambda \Psi(x, \phi) \lrcorner \delta\phi + O(\lambda^2) \end{cases},$$

$\leftarrow$ vertical!!! (x unchanged here!)

& with $\mathcal{K}^\mu_\lambda(x, \phi(\cdot)) = \lambda \kappa^\mu(x, \phi) + O(\lambda^2)$.

We then have

$$\hat{\phi}_\lambda(\tilde{x}_\lambda(x)) = \hat{\phi}_\lambda(x + \lambda \Xi(x) + O(\lambda^2)) = \hat{\phi}_\lambda(x) + \lambda \Xi(x) \lrcorner d\hat{\phi}_\lambda(x) + O(\lambda^2)$$

$\hat{\phi}_\lambda(x) = \phi(x) + O(\lambda)$! ; $d\hat{\phi}_\lambda(x) = d\phi(x) + O(\lambda)$

$$= \tilde{\phi}_\lambda(x, \phi(x)) = \phi(x) + \lambda \Psi(x, \phi) \lrcorner \delta\phi + O(\lambda^2),$$

or

$$\vec{\phi}_\lambda(x) = \phi(x) + \lambda\left(\Psi(x,\phi) \lrcorner \delta\phi - \Xi^\mu(x)\partial_\mu\phi(x)\right) + O(\lambda^2)$$

We thus obtain a vector field

$$X[\phi,\pi] = \left(\Psi^A(x,\phi) - \Xi^\mu(x)\partial_\mu\phi^A(x)\right)\frac{\delta}{\delta\phi^A}$$

generating an infinitesimal canonical transformation on $\mathcal{P}$, i.e., such that

$$\mathcal{L}_X \Omega = 0$$
$$= X \lrcorner \underbrace{\delta\Omega}_{=0} + \delta(X \lrcorner \Omega).$$

But if Ω is non-degenerate, we have the sol$^{\underline{n}}$

$$X \lrcorner \Omega = -\delta Q_X$$

The NOETHER HAMILTONIAN/CHARGE on $\mathcal{P}$

Let us derive an explicit expression for Q_X ...

We have – as established before –
(for ϕ critical $\Rightarrow \hat{\phi}_\lambda$ also critical)

$$S_{\tilde{x}_\lambda(V)}[\hat{\phi}_\lambda] - S_V[\phi] = \int_{\partial V} \partial_\mu \lrcorner \mathrm{vol}(X)\, K^\mu_\lambda(x,\phi_\lambda)$$

split it!

$$= \left(S_{\tilde{x}_\lambda(V)}[\hat{\phi}_\lambda] - S_{\tilde{x}_\lambda(V)}[\phi]\right)$$ fibre/vertical variation (up to $O(\lambda^2)$)

$$+ \left(S_{\tilde{x}_\lambda(V)}[\phi] - S_V[\phi]\right)$$ base/horizontal variation

But

$$S_{\tilde{x}_\lambda(V)}[\hat{\phi}_\lambda] - S_{\tilde{x}_\lambda(V)}[\phi]$$

because this is already $O(\lambda)$

$$= S_V[\hat{\phi}_\lambda] - S_V[\phi] + O(\lambda^2)$$

$$= \lambda\, X \lrcorner\, \delta S_V[\phi] + O(\lambda^2)$$

$$\equiv \lambda\, X \lrcorner\, \alpha_{\partial V}[\phi] + O(\lambda^2)$$

So

$$S_{\tilde{X}_\lambda(V)}[\phi] - S_V[\phi]$$

$$= \Big(\int_{\tilde{X}_\lambda(V)} - \int_V\Big)\, \underbrace{\mathrm{vol}(X)\, \mathcal{L}(x, \phi, \partial\phi)}_{\Lambda}$$

$$\equiv \int_V \left(\tilde{X}_\lambda^{*} \Lambda - \Lambda\right) \equiv \lambda \int_V \mathcal{L}_{\Xi} \Lambda + \mathcal{O}(\lambda^2) \qquad (*)$$

However, Λ is of top degree ($\equiv \dim X$), so also $d\Lambda \equiv 0$, whence

$$(*) = \lambda \int_V d(\Xi \lrcorner \Lambda) + \mathcal{O}(\lambda^2)$$

$$= \lambda \int_{\partial V} \Xi \lrcorner \Lambda + \mathcal{O}(\lambda^2),$$

so that – all in all –

$$\mathcal{X} \lrcorner\, \alpha_{\partial V}[\phi] + \int_{\partial V} \Xi \lrcorner \Lambda = \int_{\partial V} \partial_\mu \lrcorner \mathrm{vol}(X)\, K^\mu(\cdot, \phi(\cdot))$$

$$\Updownarrow$$

$$\int_{\partial V} \partial_\mu \lrcorner \mathrm{vol}(X)\, J^\mu_X(\cdot) = 0$$

for (CCC)

$$J^\mu_X(x) = K^\mu(x, \phi(x)) - \mathcal{L}(x, \phi(x), \partial\phi(x))\, \Xi^\mu(x) + \pi^\mu_A(x) \left(-\Psi^A(x, \phi(x)) + \Xi^\nu \partial_\nu \phi^A(x) \right)$$

The NOETHER CURRENT

$$= \int_V \mathrm{vol}(X)\, \partial_\mu J^\mu_X(\cdot)$$

This being so $\forall V \subset X$, we infer the CURRENT-CONSERVATION LAW:

$$\boxed{\partial_\mu J^\mu_X = 0}$$

We have just proven

Th$^{\underline{m}}$ [E. Noether]

Every 1-parameter family of smooth symmetries of a field theory corresponds to a conserved current, given by (CC).

Let $V = [\Sigma_0, \Sigma_1]$ as before, & consider fields vanishing (quickly) at $\|\vec{x}\| \to \infty$. We then obtain

$$\int_{\Sigma_1} \partial_\mu \lrcorner \, \mathrm{vol}(x)\, J^\mu_{\mathcal{X}}(\cdot) = \int_{\Sigma_0} \partial_\mu \lrcorner \, \mathrm{vol}(x)\, J^\mu_{\mathcal{X}}(\cdot)$$

$$\Downarrow$$

We have the CONSERVED CHARGE

$$\boxed{Q_{\mathcal{X}}[\Phi] := \int_{\Sigma} \partial_\mu \lrcorner \, \mathrm{vol}(x)\, J^\mu_{\mathcal{X}}(\cdot)}$$

In particular, upon choosing (28)

$\Sigma = \Sigma_t$ (equitemporal slice),

we obtain

$$Q_X[\phi,\pi] = \int_{\Sigma_t} d\vec{x}\; J^0_X(t,\vec{x}),$$

independent of t ($\Leftrightarrow$ conserved).

This is the most fundamental consequence of the Noether Th$^{\underline{m}}$.

We shall now demonstrate that Q_X is the charge we have been after all along.

To this end, return to (**) from p. (20) & compute – in the case in hand – (& under the assumption that $\tilde{X}_\lambda(\Sigma)=\Sigma_\lambda$ Cauchy hypersurface)

$$\alpha_{\tilde{X}_\lambda(\Sigma)}[\hat{\Phi}_\lambda,\hat{\pi}_\lambda] - \alpha_\Sigma[\phi,\pi] = \delta\int_\Sigma \partial_{\mu\nu}\mathrm{Vol}(X)\,\kappa^\mu_\lambda(\phi/\pi)$$

$$\stackrel{\llcorner}{=} \left(\alpha_{\tilde{X}_\lambda(\Sigma)}[\hat{\phi}_\lambda,\hat{\pi}_\lambda] - \alpha_{\tilde{X}_\lambda(\Sigma)}[\phi,\pi]\right) + \left(\alpha_{\tilde{X}_\lambda(\Sigma)}[\phi,\pi] - \alpha_\Sigma[\phi,\pi]\right)$$

$\Leftarrow$ grouped as previously the $\hat{S}_\nu$

with partial results:

$$\alpha_{\tilde{x}_\lambda(\Sigma)}[\hat{\phi}_\lambda, \hat{\pi}_\lambda] - \alpha_{\tilde{x}_\lambda(\Sigma)}[\phi, \pi] = \alpha_\Sigma[\hat{\phi}_\lambda, \hat{\pi}_\lambda] - \alpha_\Sigma[\phi, \pi] + \mathcal{O}(\lambda^2)$$

$$= \lambda \, \mathcal{L}_X \, \alpha_\Sigma[\phi, \pi] + \mathcal{O}(\lambda^2)$$

$$= \lambda \left(X \lrcorner \, \delta\alpha_\Sigma[\phi, \pi] + \delta(X \lrcorner \, \alpha_\Sigma)[\phi, \pi] \right) + \mathcal{O}(\lambda^2)$$

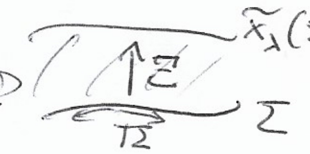
$$\equiv \lambda \left(X \lrcorner \, \Omega[\phi, \pi] + \delta(X \lrcorner \, \alpha_\Sigma)[\phi, \pi] \right) + \mathcal{O}(\lambda^2)$$

&

$$\alpha_{\tilde{x}_\lambda(\Sigma)}[\phi, \pi] - \alpha_\Sigma[\phi, \pi] \equiv \delta S_{[\Sigma, \tilde{x}_\lambda(\Sigma)]}[\phi]$$

orientation on $\Phi_\Xi([0,\lambda],\Sigma)$ $\Leftarrow$ $\left(\Xi, D_\Sigma\right)^*$ comes first!

$\tilde{x}_\lambda(\Sigma) \equiv \Phi_\Xi(\lambda, \Sigma)$



we can think of Ω as the flow of Σ $\Phi_\Xi([0,\lambda],\Sigma)$ along Ξ by time λ, i.e.,

$$\equiv \delta \int_{\Phi_\Xi([0,\lambda],\Sigma)} \mathrm{vol}(x) \, \mathcal{L}(\cdot, \phi(\cdot), \partial\phi(\cdot))$$

$$\equiv \delta \int_\Sigma \lambda \, \Xi \lrcorner \, \mathrm{vol}(\cancel{x}) \, \mathcal{L}(\cdot, \phi(\cdot), \partial\phi(\cdot)) + \mathcal{O}(\lambda^2)$$

$$\equiv \lambda \, \delta \int_\Sigma \partial_\mu \lrcorner \, \mathrm{vol}(X) \, \Xi^\mu(\cdot) \, \mathcal{L}(\cdot, \phi(\cdot), \partial\phi(\cdot))$$

Altogether, we obtain

$$\mathcal{X} \lrcorner \Omega[\phi, \pi] + \delta(\mathcal{X} \lrcorner \alpha_\Sigma)[\phi, \pi]$$

$$+ \delta \int_\Sigma \partial_\mu \lrcorner \mathrm{vol}(x)\, \Xi^\mu(\cdot)\, \mathcal{L}(\cdot, \phi(\cdot), \partial\phi(\cdot))$$

$$= \delta \int_\Sigma \partial_\mu \lrcorner \mathrm{vol}(x)\, K^\mu(x, \phi(\cdot)),$$ i.e.,

$$\mathcal{X} \lrcorner \Omega = -\delta\Big(\mathcal{X} \lrcorner \alpha_\Sigma - \int_\Sigma \partial_\mu \lrcorner \mathrm{vol}(x)\big(\Xi^\mu(\cdot)\mathcal{L}(\cdot,\phi,\partial\phi) + K^\mu(\cdot, \phi(\cdot))\big)\Big)$$

which is – indeed – the previous result!

We shall conclude the lecture with the discussion of more or less universal examples ...

Examples

1° Universal:

Consider $(x,\phi) \mapsto (\overset{=:\hat{x}_\lambda(x)}{x+\lambda\Xi}, \phi)$ $(+O(\lambda^2))$

where Ξ is a constant vector in X,

so that $\dfrac{\partial(x+\lambda\Xi)}{\partial x} = \dfrac{\partial x}{\partial x} = 1$.

We then have

$$\hat{\phi}_\lambda(x+\lambda\Xi) = \phi(x) \underset{+O(\lambda^2)}{,} \text{ \& so also}$$

$$(\partial_\mu\hat{\phi}_\lambda)(x+\lambda\Xi) = (\partial_\mu\phi)(x), \text{ whence}$$

$$\mathcal{L}(\hat{x}_\lambda(x), \hat{\phi}_\lambda\circ\hat{x}_\lambda, \partial\hat{\phi}_\lambda\circ\hat{x}_\lambda) = \mathcal{L}(x+\lambda\Xi, \phi, \partial\phi)$$

which yields $\mathcal{K}^\mu_\lambda = 0$.

The corresponding ~~Noether~~ current reads

$$J^\mu_\Xi(x) = -\mathcal{L}(\phi, \phi(x), \partial\phi(x))\Xi^\mu + \Xi^\nu\partial_\nu\phi^A(x)\pi^\mu_A(x)$$

$$\equiv T^\mu{}_\nu(x)\Xi^\nu, \text{ where}$$

$$\boxed{T^\mu{}_\nu(x) := \pi^\mu{}_A(x)\,\partial_\nu\phi^A(x) - \mathcal{L}(x,\phi(x),\partial\phi(x))\,\delta^\mu_\nu}$$

is the ENERGY-MOMENTUM TENSOR

For $\Sigma = \Sigma_t$, we recover the component charges in the standard form:

$$E := \int_{\Sigma_t} d\vec{x}\, T^0{}_0(t,\vec{x})$$

$$= \int_{\Sigma_t} d\vec{x} \left(\Pi_{tA}(\vec{x})\, \dot{\Phi}_t(\vec{x}) - \mathcal{L}(x, \Phi_t(\vec{x}), \dot{\Phi}_t(\vec{x}), \vec{\nabla}\Phi_t(\vec{x})) \right)$$

$\Downarrow$ HAMILTONIAN !!!

This is the ENERGY

$$P_i := \int_{\Sigma_t} d\vec{x}\, T^0{}_i(t,\vec{x}) = \int_{\Sigma_t} d\vec{x}\, \Pi_{tA}(\vec{x})\, \partial_i \Phi^A_t(\vec{x})$$

MOMENTUM of the field

$\Downarrow$

$T^0{}_0$ – energy density

$T^0{}_i$ – momentum density

In concrete examples:

For the wave field:

$$\begin{cases} E = \dfrac{1}{2} \displaystyle\int_{\Sigma_t} d\vec{x} \left[\dfrac{1}{v^2} \left(\dot{\Phi}_t \right)^2 (t,\vec{x}) + \vec{\nabla}\Phi_t \cdot \vec{\nabla}\Phi_t (t,\vec{x}) \right] \\[2ex] \vec{P} = \dfrac{1}{v^2} \displaystyle\int_{\Sigma_t} d\vec{x} \; \dot{\Phi}_t \, \vec{\nabla}\Phi_t (t,\vec{x}) \end{cases}$$

For the probability field (wave function):
without potential!

$$\begin{cases} E = \displaystyle\int_{\Sigma_t} d\vec{x} \; \overline{\Psi}(t,\vec{x}) \left(-\dfrac{1}{2m} \vec{\nabla}^2 \right) \Psi(t,\vec{x}) \\[2ex] \vec{P} = i \displaystyle\int_{\Sigma_t} d\vec{x} \; \overline{\Psi}(t,\vec{x}) \, \vec{\nabla} \Psi(t,\vec{x}) \end{cases}$$

2° Non-universal yet important

(A) Consider the scalar field ϕ with a Lagrangian density depending on ϕ only through its derivatives, i.e.,

$$\mathcal{L}(x,\phi,\partial\phi) \equiv \underline{\mathcal{L}}(x,\partial\phi).$$

We may then consider the transformation

$$(x,\phi) \longmapsto (x,\phi+\lambda),$$

with $\lambda \in \mathbb{R}$ (small)

We have

$$\hat{\phi}(\tilde{x}_\lambda(x)) = \phi(x) + \lambda, \quad \partial\hat{\phi}(\tilde{x}) = \partial\hat{\phi}(x) = \partial\phi(x),$$

$$\stackrel{L}{=} \hat{\phi}(x)$$

& so

$$\mathcal{L}(\tilde{x}_\lambda(x), \hat{\phi}_\lambda \circ \tilde{x}_\lambda, \partial\hat{\phi}_\lambda \circ \tilde{x}_\lambda) = \underline{\mathcal{L}}(x,\partial\phi) \equiv \mathcal{L}(x,\phi,\partial\phi),$$

whence $K^\mu_\lambda = 0$

The attendant conserved current is

$$J^\mu_X(x) = -\pi^\mu(x)$$

& the Noether charge is

$$Q_X[\phi,\pi] = -\int_{\Sigma_t} d\vec{x}\, \Pi_t(t,\vec{x})$$

In particular, for the wave field, we get

$$Q_X[\phi,\pi] = -\frac{1}{v^2}\int_{\Sigma_t} d\vec{x}\, \dot{\mathcal{L}}_t(\vec{x})$$

(B) Consider a <u>complex</u> scalar field $\phi : X \to \mathbb{C}$ with the conjugate $\bar{\phi}$. We may contemplate the transformation (global symmetry)

$$(x,\phi,\bar{\phi}) \longmapsto (x, e^{i\lambda}\phi, e^{-i\lambda}\bar{\phi}), \quad \lambda\in\mathbb{R}$$

$$\sqsubseteq (x, \phi + i\lambda\phi, \bar{\phi} - i\lambda\bar{\phi}) + O(\lambda^2)$$

The transformed fields are

$$\begin{array}{ccc}
\hat\phi_\lambda\circ\tilde x_\lambda(x) = e^{i\lambda}\phi(x) & & \\
\shortparallel & & \\
\hat\phi_\lambda(x) & \& & \hat{\bar\phi}_\lambda\circ\tilde x_\lambda(x) = e^{-i\lambda}\bar\phi(x) \\
 & & \shortparallel \\
 & & \hat{\bar\phi}_\lambda(x)
\end{array} ,$$

& so

$$\begin{array}{l}
\mathcal{L}\left(\tilde x_\lambda(x),\ \hat\phi_\lambda\circ\tilde x_\lambda,\ \partial\hat\phi_\lambda\circ\tilde x_\lambda,\ \hat{\bar\phi}_\lambda\circ\tilde x_\lambda,\ \partial\hat{\bar\phi}_\lambda\circ\tilde x_\lambda\right) \\
\shortparallel \\
\mathcal{L}\left(x,\ e^{i\lambda}\phi,\ e^{-i\lambda}\bar\phi,\ e^{i\lambda}\partial\phi,\ e^{-i\lambda}\partial\bar\phi\right)
\end{array}$$

Thus, if only

$$\mathcal{L}\left(x,\ e^{i\lambda}\phi,\ e^{-i\lambda}\bar\phi,\ e^{i\lambda}\partial\phi,\ e^{-i\lambda}\partial\bar\phi\right)$$
$$\overset{!}{=} \mathcal{L}\left(x,\ \phi,\ \bar\phi,\ \partial\phi,\ \partial\bar\phi\right),$$

we obtain a symmetry (global) with $K^\mu_\lambda \equiv 0$, with the corresponding conserved current $J^\mu_\chi(x) = i\left(\bar\pi^\mu\bar\phi - \pi^\mu\phi\right)$

so the Noether charge

$$Q_X[\psi,\pi] = i \int_{\Sigma_t} d\vec{x} \left(\bar{\Pi}_t(\vec{x}) \bar{\Phi}_t(\vec{x}) - \Pi_t(\vec{x}) \Phi_t(\vec{x}) \right)$$

In particular, for the probability field, we find

$$Q_X[\psi,\pi] = \int_{\Sigma_t} d\vec{x} \; |\psi(t,x)|^2$$

We have recovered the conservation of probability as a consequence of ~~the~~ global U(1)-symmetry!

$$\psi(x) \mapsto e^{i\alpha} \psi(x)$$

This is simply independence of probability-density dynamics of U(1)-rotations, or - in other words - its dependence on <u>rays</u> exclusively

We shall encounter more examples in future.