CLASSICAL FIELD THEORY IN THE TIME OF COVID-19
12. LECTURE BATCH

The Anderson–Brout–Englert–Higgs–Guralnik–Hagen–Kibble effect as an example of the structure-group reduction

Our hitherto exploration of the rich theory of principal and associated bundles has provided us with quasi-algorithmic procedures of construction of lagrangean models of field dynamics with a built-in local symmetry on the basis of global symmetry-invariant dynamics. The procedure leads to a natural emergence of a new species of fundamental fields, to wit, the massless vector (bosonic) gauge field that mediates interactions between currents of matter-field excitations charged with respect to the symmetry (in the sense of Noether) and exhibits its own dynamics typically described by Yang–Mills invariants. The scheme has been confirmed, through innumerable experiments, to neatly and consistently explain the (microscopic) nature of the fundamental interactions, at least up to the currently attainable energy scales – this is, in short, the story of tremendous success of the so-called Standard Model of electroweak and strong interactions[1].

The symmetry of physical phenomena referred to above is a property of the *space* of field configurations (manifesting itself though invariance of the Dirac–Feynman amplitudes defined on these and covariance of the ensuing field equations). It is *not* – at least not *a priori* – inherited by *particular* field configurations. In other words, it may happen – as it, indeed, does in nature – that the isotropy group of a classical field configuration (a critical point of the Dirac–Feynman amplitude, oftentimes termed a (classical) **vacuum** of the theory) is a proper (closed) subgroup of the symmetry group of the Dirac–Feynman amplitude. Whenever this is the case, we speak of a **spontaneous breakdown of symmetry**. It leads to a host of rather peculiar field-theoretic phenomena that we now proceed to describe in the language of fibre bundles with a compatible connection in the experimentally confirmed scenario known under the name of the **Anderson–Brout–Englert–Higgs–Guralnik–Hagen–Kibble effect**, usually referred to as the Higgs effect, in which we are given a field of type F, with the action

$$\lambda_\cdot \ :\ \mathrm{G}\times F\longrightarrow F$$

of the symmetry group G, whose dynamics is determined by the standard Klein–Gordon-type 'kinetic' term associated with a G-invariant metric on F, as in lecture 11 (p. 5), and a G-invariant potential term

$$U\ :\ F\longrightarrow\mathbb{R}\,,\qquad\qquad U\circ\lambda_g=U\,,\qquad g\in\mathrm{G}$$

whose distinguished minimum $f_0\in F$ defines a vacuum of (the F-sector of) the theory with an isotropy group

$$\{\ g\in\mathrm{G}\quad|\quad\lambda_g(f_0)=f_0\ \}\equiv\mathrm{H}\subsetneq\mathrm{G}\,.$$

The latter subgroup is closed by construction, and hence it is a Lie subgroup of G by Cartan's Thm. 4.1. The symmetry orbit of the vacuum,

$$\mathrm{G}\triangleright f_0=\{\ \lambda_g(f_0)\quad|\quad g\in\mathrm{G}\ \}\equiv F_0\subset F\,,$$

[1]Classically, we might freely add the lagrangean description of the gravitational field to the above. However, while the Standard Model admits a largely consistent quantisation, there exists no consistent model of quantum dynamics of the gravitational field to date.

to be called the **vacuum manifold** henceforth, is then equivariantly diffeomorphic with the homogeneous space[2]

$$\mathrm{G/H} \cong \mathrm{G} \triangleright f_0$$

on which the symmetry group acts in an induced manner as

$$[\ell]. \;:\; \mathrm{G} \times \mathrm{G/H} \longrightarrow \mathrm{G/H} \;:\; (k, g\,\mathrm{H}) \longmapsto \ell_k(g)\,\mathrm{H} \equiv (k \cdot g)\,\mathrm{H}\,.$$

It is, consequently, the bundles

$$\begin{array}{ccc} F_0 & \longrightarrow & \mathsf{P}_{\mathrm{G}} \times_{\lambda.} F_0 \\ & & \Big\downarrow{\scriptstyle \pi_{\mathsf{P}_{\mathrm{G}} \times_{\lambda.} F_0} \equiv \pi_0} \\ & & \Sigma \end{array}$$

and

$$\begin{array}{ccc} \mathrm{G/H} & \longrightarrow & \mathsf{P}_{\mathrm{G}} \times_{[\ell].} \mathrm{G/H} \\ & & \Big\downarrow{\scriptstyle \pi_{\mathsf{P}_{\mathrm{G}} \times_{[\ell].} \mathrm{G/H}} \equiv [\pi]} \\ & & \Sigma \end{array}$$

associated with (principal) gauge bundles P_{G} through the induced actions on the single orbits F_0 and $\mathrm{G/H}$, respectively, that shall play a central rôle in the remainder of our discussion. The fundamental geometric mechanism to be encountered in it is heralded by the following

Definition 1. Adopt the hitherto notation, and in particular that of Def. 6.1, and let G_1 and G_2 be Lie groups of which the former is monomorphically embedded in the latter by

$$\jmath_{12} \;:\; \mathrm{G}_1 \hookrightarrow \mathrm{G}_2\,.$$

Given principal bundles $\mathcal{P}_A \coloneqq (\mathsf{P}_{\mathrm{G}_A}, B, \mathrm{G}_A, \pi_{\mathrm{G}_A})$, $A \in \{1,2\}$ over a common base B, we call an arbitrary monomorphism between them

$$\big(I_{12}, \mathrm{id}_B, \jmath_{12}\big) \;:\; \mathcal{P}_1 \longrightarrow \mathcal{P}_2\,,$$

composed of an embedding

$$I_{12} \;:\; \mathsf{P}_{\mathrm{G}_1} \hookrightarrow \mathsf{P}_{\mathrm{G}_2}$$

of the total spaces covering the identity diffeomorphism $\mathrm{id}_B \;:\; B \longrightarrow B$ on the base and of the formerly introduced Lie-group monomorphism $\jmath_{12}$, a **reduction of the structure group of** $\mathcal{P}_1$. Whenever it exists, the image subbundle

$$\big(I_{12}, \mathrm{id}_B, \jmath_{12}\big)(\mathcal{P}_1) \subset \mathcal{P}_2$$

is termed the **reduced bundle**, and we say that the structure group G_2 is **reducible** to G_1.

Thus, we shall demonstrate how a non-punctual nature of the vacuum manifold (*i.e.*, the relation $\mathrm{H} \subsetneq \mathrm{G}$) and existence of a section of either of the two principal bundles written out above leads to a reduction of the structure group of the gauge bundle of the field theory, and systematically reconstruct the field content of a field theory with the reduced gauge group. In so doing, we shall find the following reformulation of the concept of reducibility useful

Proposition 1. Let G_A, $A \in \{1,2\}$ be Lie groups and let $\jmath_{12} \;:\; \mathrm{G}_1 \hookrightarrow \mathrm{G}_2$ be a Lie-group monomorphism. A principal bundle with the structure group G_2 admits a reduction along $\jmath_{12}$ iff there exists a trivialising cover of its base with the corresponding transition maps taking values in $\jmath_{12}(\mathrm{G}_1)$.

[2]This is an elementary result of the theory of spaces with a group action which we leave to the Reader as an easy exercise.

Proof: Adopt the notation of Def. 1. Assume, first, that the principal bundle $\mathcal{P}_2$ admits a reduction and consider a cover $\{\mathcal{O}_i\}_{i\in I}$ of B trivialising for $\mathcal{P}_1$ (and so also for the reduced bundle $(I_{12}, \mathrm{id}_B, \jmath_{12})(\mathcal{P}_1)$), with the associated local trivialisations

$$\tau^1_i \ :\ \pi^{-1}_{\mathsf{P}_1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times \mathrm{G}_1$$

and the corresponding transition maps

$$g^1_{ij} \ :\ \mathcal{O}_{ij} \longrightarrow \mathrm{G}_1 \,.$$

The latter can be used to define manifestly smooth and G_2-equivariant mappings

$$\tau^2_i \ :\ \pi^{-1}_{\mathsf{P}_2}(\mathcal{O}_i) \longrightarrow \mathcal{O}_i \times \mathrm{G}_2 \ :\ p_2 \longmapsto \big(\pi_{\mathsf{P}_2}(p_2), \phi_{\mathsf{P}_2}\big(I_{12} \circ \tau^{1\,-1}_i\big(\pi_{\mathsf{P}_2}(p_2), e\big), p_2\big)\big) \,.$$

These are, in fact, local trivialisations of $\mathcal{P}_2$. Indeed, they are injective,

$$\tau^2_i(p_2') = \tau^2_i(p_2) \quad\Longleftrightarrow\quad \begin{cases} \pi_{\mathsf{P}_2}(p_2') = \pi_{\mathsf{P}_2}(p_2) \\ \phi_{\mathsf{P}_2}\big(I_{12} \circ \tau^{1\,-1}_i\big(\pi_{\mathsf{P}_2}(p_2'), e\big), p_2'\big) = \phi_{\mathsf{P}_2}\big(I_{12} \circ \tau^{1\,-1}_i\big(\pi_{\mathsf{P}_2}(p_2), e\big), p_2\big) \end{cases}$$

$$\Longleftrightarrow\quad \begin{cases} \pi_{\mathsf{P}_2}(p_2') = \pi_{\mathsf{P}_2}(p_2) \\ \phi_{\mathsf{P}_2}\big(I_{12} \circ \tau^{1\,-1}_i\big(\pi_{\mathsf{P}_2}(p_2), e\big), p_2'\big) = \phi_{\mathsf{P}_2}\big(I_{12} \circ \tau^{1\,-1}_i\big(\pi_{\mathsf{P}_2}(p_2), e\big), p_2\big) \end{cases}$$

$$\Longleftrightarrow\quad \begin{cases} \pi_{\mathsf{P}_2}(p_2') = \pi_{\mathsf{P}_2}(p_2) \\ \phi_{\mathsf{P}_2}\big(p_2', p_2\big) = e \end{cases} \quad\Longleftrightarrow\quad p_2' = p_2 \,,$$

and hence also surjective (recall the local model of P_2). Their explicit inverses read

$$\tau^{2\,-1}_i \ :\ \mathcal{O}_i \times \mathrm{G}_2 \longrightarrow \pi^{-1}_{\mathsf{P}_2}(\mathcal{O}_i) \ :\ (x,g) \longmapsto r_g \circ I_{12} \circ \tau^{1\,-1}_i(x,e) \,.$$

The corresponding transition maps,

$$g^2_{ij} \ :\ \mathcal{O}_{ij} \longrightarrow \mathrm{G}_2 \,,$$

can be extracted from a direct calculation (carried out for arbitrary $(x,g) \in \mathcal{O}_{ij} \times \mathrm{G}$):

$$\begin{aligned} \tau^2_i \circ \tau^{2\,-1}_j(x,g) &= \tau^2_i \circ r^{\mathsf{P}_2}_g \circ I_{12} \circ \tau^{1\,-1}_j(x,e) = (\mathrm{id}_B \times \wp_g) \circ \tau^2_i \circ I_{12} \circ \tau^{1\,-1}_j(x,e) \\ &= \big(\pi_{\mathsf{P}_2} \circ I_{12} \circ \tau^{1\,-1}_j(x,e), \phi_{\mathsf{P}_2}\big(I_{12} \circ \tau^{1\,-1}_i\big(\pi_{\mathsf{P}_2} \circ I_{12} \circ \tau^{1\,-1}_j(x,e), e\big), I_{12} \circ \tau^{1\,-1}_j(x,e)\big) \cdot g\big) \\ &= \big(\pi_{\mathsf{P}_1} \circ \tau^{1\,-1}_j(x,e), \phi_{\mathsf{P}_2}\big(I_{12} \circ \tau^{1\,-1}_i\big(\pi_{\mathsf{P}_1} \circ \tau^{1\,-1}_j(x,e), e\big), I_{12} \circ \tau^{1\,-1}_j(x,e)\big) \cdot g\big) \\ &= \big(x, \phi_{\mathsf{P}_2}\big(I_{12} \circ \tau^{1\,-1}_i\big(x,e\big), I_{12} \circ \tau^{1\,-1}_j(x,e)\big) \cdot g\big) \\ &= \big(x, \phi_{\mathsf{P}_2}\big(I_{12} \circ \tau^{1\,-1}_i\big(x,e\big), I_{12} \circ \tau^{1\,-1}_i\big(x, g^1_{ij}(x)\big)\big) \cdot g\big) \\ &= \big(x, \phi_{\mathsf{P}_2}\big(I_{12} \circ \tau^{1\,-1}_i\big(x,e\big), I_{12} \circ r^{\mathsf{P}_1}_{g^1_{ij}(x)} \circ \tau^{1\,-1}_i\big(x,e\big)\big) \cdot g\big) \\ &= \big(x, \phi_{\mathsf{P}_2}\big(I_{12} \circ \tau^{1\,-1}_i\big(x,e\big), r^{\mathsf{P}_2}_{\jmath_{12} \circ g^1_{ij}(x)} \circ I_{12} \circ \tau^{1\,-1}_i\big(x,e\big)\big) \cdot g\big) = \big(x, \jmath_{12} \circ g^1_{ij}(x) \cdot g\big) \,, \end{aligned}$$

that is

$$g^2_{ij} \equiv \jmath_{12} \circ g^1_{ij} \ :\ \mathcal{O}_{ij} \longrightarrow \jmath_{12}(\mathrm{G}_1) \subset \mathrm{G}_2 \,,$$

as claimed.

Conversely, suppose that there is a covering $\{\mathcal{O}_i\}_{i\in I}$ of B trivialising for $\mathcal{P}_2$ such that the corresponding local trivialisations

$$\tau^2_i \ :\ \pi^{-1}_{\mathsf{P}_2}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times \mathrm{G}_2$$

yield transition maps

$$g^2_{ij} \;:\; \mathcal{O}_{ij} \longrightarrow \jmath_{12}(\mathrm{G}_1) \subset \mathrm{G}_2 \,.$$

The latter give rise to the unique smooth mappings

$$g^1_{ij} \;:\; \mathcal{O}_{ij} \longrightarrow \mathrm{G}_1$$

with the property

$$g^2_{ij} = \jmath_{12} \circ g^1_{ij}$$

which we may use to reconstruct a principal bundle $\mathcal{P}_1$ over B with the g^1_{ij} as the transition maps, along the lines of (the proof of) the Clutching Theorem of Lecture 1 (pp. 29–31). Thus, we have

$$\mathsf{P}_1 \equiv \big(\bigsqcup_{i\in I} (\mathcal{O}_i \times \mathrm{G}_1)\big)/_{g^1_{..}}$$

with the projection to the base

$$\pi_{\mathsf{P}_1} \;:\; \big(\bigsqcup_{i\in I} (\mathcal{O}_i \times \mathrm{G}_1)\big)/_{g^1_{..}} \longrightarrow B \;:\; [(x,g,i)] \longmapsto x$$

and local trivialisations

$$[\tau^1_i] \;:\; \pi^{-1}_{\mathsf{P}_1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times \mathrm{G}_1 \;:\; [(x,g,i)] \longmapsto (x,g)\,.$$

The manifestly injective diffeomorphisms

(1) $$\Phi_i := \tau^{2\,-1}_i \circ (\mathrm{id}_{\mathcal{O}_i} \times \jmath_{12}) \circ [\tau^1_i] \;:\; \pi^{-1}_{\mathsf{P}_1}(\mathcal{O}_i) \longrightarrow \pi^{-1}_{\mathsf{P}_2}(\mathcal{O}_i)\,, \qquad i \in I$$

glue over the intersections $\mathcal{O}_{ij} \ni x$,

$$\begin{aligned}\Phi_j\big([(x,g,i)]\big) &\equiv \Phi_j\big([(x,g^1_{ji}(x)\cdot g,j)]\big) \equiv \tau^{2\,-1}_j\big(x,\jmath_{12}\big(g^1_{ji}(x)\cdot g\big)\big) = \tau^{2\,-1}_j\big(x,g^2_{ji}(x)\cdot \jmath_{12}(g)\big) \\ &= \tau^{2\,-1}_i\big(x,g^2_{ij}(x)\cdot g^2_{ji}(x)\cdot \jmath_{12}(g)\big) = \tau^{2\,-1}_i\big(x,\jmath_{12}(g)\big) \equiv \Phi_i\big([(x,g,i)]\big)\,,\end{aligned}$$

and so give rise to a globally smooth embedding

$$I_{12} \;:\; \mathsf{P}_1 \hookrightarrow \mathsf{P}_2\,, \qquad I_{12}\restriction_{\pi^{-1}_{\mathsf{P}_1}(\mathcal{O}_i)} \equiv \Phi_i$$

with the desired properties. □

With the above maths in hand, we are finally ready to discuss the physics.

The configurational aspect. We begin by introducing the main protagonist of our story in

Definition 2. Adopt the hitherto notation. A **Higgs field of type** F_0 **with the gauge symmetry of type** P_{G} is an arbitrary global section

$$H \in \Gamma\big(\mathsf{P}_{\mathrm{G}} \times_{\lambda_\cdot} F_0\big)$$

of the **Higgs vacuum bundle** $(\mathsf{P}_{\mathrm{G}} \times_{\lambda_\cdot} F_0, \Sigma, \pi_0)$ over the spacetime Σ.

We shall, now, spend some time developing basic intuitions as to geometric implications of existence of a Higgs field. To this end, we *fix* a G-equivariant diffeomorphism

$$\mu_* \;:\; F_0 \xrightarrow{\cong} \mathrm{G}/\mathrm{H}_*\,,$$

which amounts to[3] distinguishing a point $f_* \equiv \mu^{-1}_*(\mathrm{H}_*)$ with the isotropy group $\mathrm{G}_{f_*} \equiv \mathrm{H}_*$, and pass to the local description of the Higgs field H. Thus, consider a local section

$$\sigma \;:\; \mathcal{O} \longrightarrow \mathsf{P}_{\mathrm{G}}$$

[3]Every G-equivariant map $\mu_* \;:\; F_0 \xrightarrow{\cong} \mathrm{G}/\mathrm{H}_*$ distinguishes the point $f_* := \mu^{-1}_*(\mathrm{H}_*)$ whose isotropy group $\mathrm{G}_{f_*} \ni g$ is determined by the set of equivalences: $\lambda_g(f_*) = f_* \iff g_*\,\mathrm{H}_* \equiv [\ell]_g(\mathrm{H}_*) = [\ell]_g \circ \mu_*(f_*) = \mu_*\big(\lambda_g(f_*)\big) = \mu_*(f_*) = \mathrm{H}_* \iff g \in \mathrm{H}_*$, *i.e.*, $\mathrm{G}_{f_*} \equiv \mathrm{H}_*$. Conversely, with an arbitrary point $f_* \in F_0$ with the corresponding isotropy group $\mathrm{G}_{f_*} = \mathrm{H}_*$, we associate the map $\mu_* \;:\; F_0 \longrightarrow \mathrm{G}/\mathrm{H}_* \;:\; \lambda_g(f_*) \longmapsto g\,\mathrm{H}_*$ which is manifestly well-defined and G-equivariant.

of the gauge bundle P_{G} over some open set $\mathcal{O} \subset \Sigma$, and use the corresponding local trivialisation

$$\tau_\sigma \;:\; \pi_{\mathsf{P}_{\mathrm{G}}}^{-1}(\mathcal{O}) \xrightarrow{\cong} \mathcal{O} \times \mathrm{G}$$

(*cp* Prop. 6.5) to induce a local trivialisation

$$\widetilde{\tau}_\sigma \;:\; \pi_{\mathsf{P}_{\mathrm{G}} \times_{\lambda_\cdot} F_0}^{-1}(\mathcal{O}) \xrightarrow{\cong} \mathcal{O} \times F_0$$

(as in Def. 7.1) and to define a smooth mapping

$$\chi \coloneqq \mathrm{pr}_2 \circ \widetilde{\tau}_\sigma \circ H \;:\; \mathcal{O} \longrightarrow F_0 \,.$$

In other words, we have the local form

$$H \;:\; \mathcal{O} \longrightarrow \mathsf{P}_{\mathrm{G}} \times_{\lambda_\cdot} F_0 \;:\; x \longmapsto [(\tau_\sigma^{-1}(x,e), \chi(x))] \,,$$

or, equivalently,

$$\chi \equiv H_\sigma \,,$$

that is χ is a local presentation of the Higgs field in the gauge σ in the sense of Def. 8.5. Using these, we define a subset

$$\mathsf{P}^H_{\mathrm{H}_*} \restriction_{\mathcal{O}} \coloneqq \left\{ \tau_\sigma^{-1}(x,g) \quad | \quad g \in \pi_{\mathrm{G}/\mathrm{H}_*}^{-1}\big(\mu_* \circ \chi(x)\big) \right\} \equiv \left\{ \tau_\sigma^{-1}(x,g) \quad | \quad \mu_* \circ \chi(x) = g\,\mathrm{H}_* \right\} \subset \mathsf{P}_{\mathrm{G}} \restriction_{\mathcal{O}} .$$

As shall be argued below, $\mathsf{P}^H_{\mathrm{H}_*} \restriction_{\mathcal{O}}$ is a restriction, to $\mathcal{O}$, of a reduction $\mathsf{P}^H_{\mathrm{H}_*} \hookrightarrow \mathsf{P}_{\mathrm{G}}$ induced by the Higgs field H. Indeed, let $\sigma_A \;:\; \mathcal{O}_A \longrightarrow \mathsf{P}_{\mathrm{G}},\ A \in \{1,2\}$ be two local sections of the gauge bundle over open sets $\mathcal{O}_A \subset B$ with a non-empty intersection $\mathcal{O}_{12} \equiv \mathcal{O}_1 \cap \mathcal{O}_2 \ni x$. In virtue of global smoothness of H, we readily establish, for the respective local presentations $\chi_A \;:\; \mathcal{O}_A \longrightarrow F_0$,

$$[(\tau_{\sigma_1}^{-1}(x,e), \chi_1(x))] \equiv H(x) \equiv [(\tau_{\sigma_2}^{-1}(x,e), \chi_2(x))] = [(\tau_{\sigma_1}^{-1}(x,e), \lambda_{g_{12}(x)} \circ \chi_2(x))] \,,$$

or

$$\chi_1 \restriction_{\mathcal{O}_{12}} = \lambda_{g_{12}(\cdot)} \circ \chi_2 \restriction_{\mathcal{O}_{12}}$$

where

$$g_{12} \;:\; \mathcal{O}_{12} \longrightarrow \mathrm{G}$$

is the transition mapping of P_{G} relating the two local trivialisations τ_{σ_A}. Now, in consequence of the assumed transitivity of the action of G on F_0, we find two smooth maps

$$\gamma_A \;:\; \mathcal{O}_A \longrightarrow \mathrm{G} \,, \qquad A \in \{1,2\}$$

satisfying, at every point $y \in \mathcal{O}_A$,

$$\chi_A(y) = \lambda_{\gamma_A(y)}(f_0) \,,$$

so that

$$\lambda_{\gamma_1(x)}(f_0) \equiv \chi_1(x) = \lambda_{g_{12}(x)} \circ \chi_2(x) = \lambda_{g_{12}(x) \cdot \gamma_2(x)}(f_0) \,,$$

or

$$g_{12} \cdot \gamma_2 \restriction_{\mathcal{O}_{12}} = \gamma_1 \cdot h_{12} \restriction_{\mathcal{O}_{12}} \tag{2}$$

for some smooth map

$$h_{12} \;:\; \mathcal{O}_{12} \longrightarrow \mathrm{H}_* \,.$$

Since, moreover,

$$\mu_* \circ \chi_A(y) \equiv \mu_*\big(\lambda_{\gamma_A(y)}(f_0)\big) = [\ell]_{\gamma_A(y)}\big(\mu_*(f_0)\big) \equiv \gamma_A(y)\,\mathrm{H}_* \,,$$

we have, for any $h \in \mathrm{H}_*$,

$$\tau_{\sigma_A}^{-1}\big(y, \gamma_A(y) \cdot h\big) \in \mathsf{P}^H_{\mathrm{H}_*} \restriction_{\mathcal{O}_A} \,,$$

and so we may define local sections of the would-be reduction as

$$\underline{\sigma}_A \;:\; \mathcal{O}_A \longrightarrow \mathsf{P}^H_{\mathrm{H}_*} \;:\; y \longmapsto \tau_{\sigma_A}^{-1}\big(\cdot, \gamma_A(\cdot)\big) \equiv r_{\gamma_A(\cdot)}\big(\sigma_A(\cdot)\big) \,,$$

with the corresponding local trivialisations

$$\underline{\tau}_A \ :\ \left(\pi_{\mathsf{P}_{\mathrm{G}}}\restriction_{\mathsf{P}^H_{\mathrm{H}_*}}\right)^{-1}(\mathcal{O}_A) \longrightarrow \mathcal{O}_A \times \mathrm{H}_* \ :\ \tau^{-1}_{\sigma_A}\big(y,g\big) \longmapsto \big(y, \gamma_A(y)^{-1}\cdot g\big)\,. \tag{3}$$

From this, we readily derive the transition mappings,

$$\begin{aligned}\underline{\tau}_2^{-1}(x,h) &\equiv \tau^{-1}_{\sigma_2}\big(x,\gamma_2(x)\cdot h\big) = \tau^{-1}_{\sigma_1}\big(x, g_{12}(x)\cdot\gamma_2(x)\cdot h\big) = \tau^{-1}_{\sigma_1}\big(x,\gamma_1(x)\cdot\big(h_{12}(x)\cdot h\big)\big)\\ &\equiv \underline{\tau}_1^{-1}\big(x, h_{12}(x)\cdot h\big)\,,\end{aligned}$$

and so $\mathsf{P}^H_{\mathrm{H}_*}$ does, indeed, have $\mathrm{H}_* \ni h_{12}(x)$ as the structure group.

Remaining on the current level of intuitiveness, we may enquire as to the nature of the dependence of the (would-be) reduction upon the arbitrary choices made along the way, to wit, that of the G-equivariant modelling μ_* of the symmetry orbit F_0 of the vacuum and that of the local gauge σ of the gauge bundle P_{G} under reduction. For the former, note that given two points $f_*^{(A)} \in F_0,\ A \in \{1,2\}$, there exists an element $g_{21} \in \mathrm{G}$ such that

$$f_*^{(2)} = \lambda_{g_{21}}\big(f_*^{(1)}\big)\,,$$

and then

$$\mathrm{H}_*^{(2)} \equiv \mathrm{G}_{f_*^{(2)}} = \mathrm{Ad}_{g_{21}}\big(\mathrm{G}_{f_*^{(1)}}\big) \equiv \mathrm{Ad}_{g_{21}}\big(\mathrm{H}_*^{(1)}\big)\,.$$

The two restrictions

$$\mathsf{P}^H_{\mathrm{H}_*^{(A)}}\restriction_{\mathcal{O}} = \big\{\ \tau_\sigma^{-1}(x,g) \quad|\quad \mu_*^{(A)}\circ\chi(x) = g\,\mathrm{H}_*^{(A)}\ \big\}\,, \qquad A \in \{1,2\}\,,$$

defined in terms of the respective G-equivariant diffeomorphisms

$$\mu_*^{(A)} \ :\ F_0 \xrightarrow{\cong} \mathrm{G}/\mathrm{H}_*^{(A)} \ :\ \lambda_g\big(f_*^{(A)}\big) \longmapsto g\,\mathrm{H}_*^{(A)}\,,$$

are related by the diffeomorphism[4] with restrictions

$$\Phi_{21}\restriction_{\mathcal{O}} \ :\ \mathsf{P}^H_{\mathrm{H}_*^{(1)}}\restriction_{\mathcal{O}} \xrightarrow{\cong} \mathsf{P}^H_{\mathrm{H}_*^{(2)}}\restriction_{\mathcal{O}} \ :\ \tau_\sigma^{-1}(x,g) \longmapsto r_{g_{21}^{-1}}\big(\tau_\sigma^{-1}(x,g)\big)\,,$$

or

$$\Phi_{21} \equiv r_{g_{21}^{-1}}\restriction_{\mathsf{P}^H_{\mathrm{H}_*^{(1)}}}\,,$$

covering the identity diffeomorphism on the base. This we augment with the isomorphisms

$$\mathrm{Ad}_{g_{21}} \ :\ \mathrm{H}_*^{(1)} \xrightarrow{\cong} \mathrm{H}_*^{(2)}$$

between the respective structure groups, to obtain an isomorphism

$$\Big(r_{g_{21}^{-1}}\restriction_{\mathsf{P}^H_{\mathrm{H}_*^{(1)}}}, \mathrm{id}_\Sigma, \mathrm{Ad}_{g_{21}}\Big) \ :\ \Big(\mathsf{P}^H_{\mathrm{H}_*^{(1)}}, \Sigma, \mathrm{H}_*^{(1)}, \pi_{\mathsf{P}_{\mathrm{G}}}\restriction_{\mathsf{P}^H_{\mathrm{H}_*^{(1)}}}\Big) \xrightarrow{\cong} \Big(\mathsf{P}^H_{\mathrm{H}_*^{(2)}}, \Sigma, \mathrm{H}_*^{(2)}, \pi_{\mathsf{P}_{\mathrm{G}}}\restriction_{\mathsf{P}^H_{\mathrm{H}_*^{(2)}}}\Big)$$

between the reductions. Indeed, upon denoting the defining actions of the respective structure groups (induced through restriction) as

$$r^{(A)}_{\cdot} \ :\ \mathsf{P}^H_{\mathrm{H}_*^{(A)}} \times \mathrm{H}_*^{(A)} \longrightarrow \mathsf{P}^H_{\mathrm{H}_*^{(A)}} \ :\ \big(\tau_\sigma^{-1}(x,g), h_A\big) \longmapsto r_{h_A}\big(\tau_\sigma^{-1}(x,g)\big)\,,$$

we readily verify the commutativity of the diagram

$$\begin{array}{ccc} \mathsf{P}^H_{\mathrm{H}_*^{(1)}} \times \mathrm{H}_*^{(1)} & \xrightarrow{\ r^{(1)}_{\cdot}\ } & \mathsf{P}^H_{\mathrm{H}_*^{(1)}} \\ \Big\downarrow{\scriptstyle r_{g_{21}^{-1}}\restriction_{\mathsf{P}^H_{\mathrm{H}_*^{(1)}}}\times\mathrm{Ad}_{g_{21}}} & & \Big\downarrow{\scriptstyle r_{g_{21}^{-1}}\restriction_{\mathsf{P}^H_{\mathrm{H}_*^{(1)}}}} \\ \mathsf{P}^H_{\mathrm{H}_*^{(2)}} \times \mathrm{H}_*^{(2)} & \xrightarrow[\ r^{(2)}_{\cdot}\ ]{} & \mathsf{P}^H_{\mathrm{H}_*^{(2)}} \end{array}\,.$$

[4]Here, smoothness is understood with respect to the natural submanifold structure implicit in our intuitive discussion that shall become transparent later on.

Passing to the question of dependence of the reduction upon the choice of the gauge, we consider two local gauges

$$\sigma_{(A)}\ :\ \mathcal{O}\longrightarrow \mathsf{P}_{\rm G}\,,\qquad A\in\{1,2\}$$

over the *same* neighbourhood $\mathcal{O}\subset\Sigma$, related by a gauge transformation

$$\Gamma_{(12)}\ :\ \mathcal{O}\longrightarrow {\rm G}$$

as

$$\sigma_{(2)}(\cdot)=r_{\Gamma_{(12)}(\cdot)}\big(\sigma_{(1)}(\cdot)\big)\,,$$

whence also, for arbitrary $(x,g)\in\mathcal{O}\times{\rm G}$,

$$\tau_{\sigma_{(1)}}\circ\tau_{\sigma_{(2)}}^{-1}(x,g)=\big(x,\Gamma_{(12)}(x)\cdot g\big)\,.$$

Write

$$\chi_{(A)}\coloneqq H_{\sigma_{(A)}}\,,$$

so that

$$\begin{aligned}[(\tau_{\sigma_{(1)}}^{-1}(x,e),\chi_{(1)}(x))] &= H(x)=[(\tau_{\sigma_{(2)}}^{-1}(x,e),\chi_{(2)}(x))]=[(r_{\Gamma_{(12)}(x)}(\tau_{\sigma_{(1)}}^{-1}(x,e)),\chi_{(2)}(x))]\\ &= [(\tau_{\sigma_{(1)}}^{-1}(x,e),\lambda_{\Gamma_{(12)}(x)}(\chi_{(2)}(x)))]\,,\end{aligned}$$

and

$$\mathsf{P}_{{\rm H}_*}^{H(A)}\restriction_{\mathcal{O}}=\big\{\ \tau_{\sigma_{(A)}}^{-1}(x,g)\quad\big|\quad \mu_*\circ\chi_{(A)}(x)=g\,{\rm H}_*\ \big\}\,.$$

Given that

$$\chi_{(A)}(\cdot)=\lambda_{\gamma_{(A)}(\cdot)}(f_*)$$

for some smooth profiles

$$\gamma_{(A)}\ :\ \mathcal{O}\longrightarrow{\rm G}\,,$$

we conclude that

$$\Gamma_{(12)}\cdot\gamma_{(2)}=\gamma_{(1)}\cdot h_{(12)}$$

for some smooth profile

$$h_{(12)}\ :\ \mathcal{O}\longrightarrow{\rm H}_*\,,$$

and, therefore, (the inverses of) the ensuing local trivialisations of the associated reductions,

$$\underline{\tau}_{(A)}\ :\ \big(\pi_{\mathsf{P}_{\rm G}}\restriction_{\mathsf{P}_{{\rm H}_*}^{H(A)}}\big)^{-1}(\mathcal{O})\longrightarrow\mathcal{O}\times{\rm H}_*\ :\ \tau_{\sigma_{(A)}}^{-1}\big(y,g\big)\longmapsto\big(y,\gamma_{(A)}(y)^{-1}\cdot g\big)$$

become related as

$$\begin{aligned}\underline{\tau}_{(2)}^{-1}(x,h) &\equiv \tau_{\sigma_{(2)}}^{-1}\big(x,\gamma_{(2)}(x)\cdot h\big)=\tau_{\sigma_{(1)}}^{-1}\big(x,\Gamma_{(12)}(x)\cdot\gamma_{(2)}(x)\cdot h\big)\\ &= \tau_{\sigma_{(1)}}^{-1}\big(x,\gamma_{(1)}(x)\cdot h_{(12)}(x)\cdot h\big)\equiv\underline{\tau}_{(1)}^{-1}(x,h_{(12)}(x)\cdot h)\,.\end{aligned}$$

We infer that the above gauge transformation induces an automorphism[5] of the reduction, with local data $h_{(12)}$.

Thus, all in all, the *isomorphism class* of $\mathsf{P}_{{\rm H}_*}^{H}$ is manifestly independent of the arbitrary choices involved in its constructions. This constatation permits us, in particular, to abstract from the geometric peculiarities of the vacuum manifold F_0 and consider its fixed model

$$F_0\equiv{\rm G}/{\rm H}$$

with

$$\lambda_.\equiv[\ell].$$

[5]We urge the Reader to verify that the local data do, indeed, satisfy the relevant identity (6.3) upon invoking Eq. (2).

in what follows.

We formalise the intuition developed above in

Theorem 1 (On the Reduction of a Principal Bundle through the Higgs Mechanism)**.** Adopt the hitherto notation and let H be a closed subgroup of the Lie group G. An arbitrary principal bundle $(\mathsf{P}_{\mathrm{G}}, \Sigma, \mathrm{G}, \pi_{\mathsf{P}_{\mathrm{G}}})$ with the structure group G admits a reduction of the latter along the canonical injection

$$\jmath_{\mathrm{H}} \ :\ \mathrm{H} \hookrightarrow \mathrm{G}$$

iff there exists a Higgs field of type $\mathrm{G/H}$ with the gauge symmetry of type P_{G}.

Proof: Assume, first, that a principal bundle $(\mathsf{P}_{\mathrm{G}}, \Sigma, \mathrm{G}, \pi_{\mathsf{P}_{\mathrm{G}}})$ admits the reduction, so that – in virtue of Prop. 1 – there exists an open cover $\{\mathcal{O}_i\}_{i\in I}$ of Σ together with local sections

$$\sigma_i \ :\ \mathcal{O}_i \longrightarrow \mathsf{P}_{\mathrm{G}},\ i \in I$$

such that the corresponding local trivialisations

$$\tau_i \equiv \tau_{\sigma_i} \ :\ \pi_{\mathsf{P}_{\mathrm{G}}}^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times \mathrm{G}$$

yield smooth transition maps

$$h_{ij} \ :\ \mathcal{O}_{ij} \longrightarrow \mathrm{H} \subset \mathrm{G}\,.$$

Upon invoking Prop. 7.7 (and its proof), define maps

$$H_i \coloneqq \phi_{\pi_{\mathrm{G/H}}} \circ \sigma_i \ :\ \mathcal{O}_i \longrightarrow \mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H} \ :\ x \longmapsto [(\sigma_i(x), \mathrm{H})]\,, \qquad i \in I$$

with the property

$$\pi_{\mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H}} \circ H_i(x) \equiv \pi_{\mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H}}\big([(\sigma_i(x), \mathrm{H})]\big) = \pi_{\mathsf{P}_{\mathrm{G}}} \circ \sigma_i(x) \equiv x\,,$$

rewriting as

$$\pi_{\mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H}} \circ H_i = \mathrm{id}_{\mathcal{O}_i}$$

which identifies the H_i as local sections of the associated bundle $\mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H}$. At an arbitrary point $x \in \mathcal{O}_{ij}$, these satisfy

$$H_j(x) \equiv [(\sigma_j(x), \mathrm{H})] = [(r_{h_{ij}(x)} \circ \sigma_i(x), \mathrm{H})] = [(\sigma_i(x), h_{ij}(x)\,\mathrm{H})] \equiv [(\sigma_i(x), \mathrm{H})] \equiv H_i(x)\,,$$

and so we obtain a global section

$$H \ :\ \Sigma \longrightarrow \mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H}$$

with restrictions

$$H \restriction_{\mathcal{O}_i} \equiv H_i\,.$$

Conversely, let

$$H \ :\ \Sigma \longrightarrow \mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H}$$

be a global section,

$$\pi_{\mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H}} \circ H = \mathrm{id}_\Sigma\,.$$

Cover the image $H(\Sigma) \subset \mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H}$ of the base Σ under the latter section with open sets $\mathcal{U}_i \subset H(\Sigma),\ i \in I$,

$$\bigcup_{i\in I} \mathcal{U}_i \equiv H(\Sigma)\,,$$

over which the principal bundle $(\mathsf{P}_{\mathrm{G}}, \mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H}, \mathrm{H}, \phi_{\pi_{\mathrm{G/H}}})$ of Prop. 7.7 trivialises, *i.e.*, it admits local sections

$$s_i \ :\ \mathcal{U}_i \longrightarrow \mathsf{P}_{\mathrm{G}}$$

related over intersections $\mathcal{U}_{ij} \ni [(p, g\,\mathrm{H})]$ as

$$s_j\big([(p, g\,\mathrm{H})]\big) = r_{h_{ij}\big([(p,g\,\mathrm{H})]\big)} \circ s_i\big([(p, g\,\mathrm{H})]\big)$$

by smooth transition maps

$$h_{ij} \; : \; \mathcal{U}_{ij} \longrightarrow \mathrm{H}\,.$$

Upon covering Σ with the open sets (continuous preimages of open sets)

$$\mathcal{O}_i := H^{-1}(\mathcal{U}_i)\,, \qquad i \in I\,,$$

we define smooth maps

$$\sigma_i := s_i \circ H \; : \; \mathcal{O}_i \longrightarrow \mathsf{P}_{\mathrm{G}}\,, \qquad i \in I$$

with the property, written out for an arbitrary $x \in \mathcal{O}_i$,

$$H(x) \equiv \big(\phi_{\pi_{\mathrm{G/H}}} \circ s_i\big)\big(H(x)\big) = \phi_{\pi_{\mathrm{G/H}}} \circ \sigma_i(x) \equiv [(\sigma_i(x), \mathrm{H})]$$

that implies

$$x \equiv \big(\pi_{\mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H}} \circ H\big)(x) = \pi_{\mathsf{P}_{\mathrm{H}}} \circ \sigma_i(x)\,,$$

or, equivalently,

$$\pi_{\mathsf{P}_{\mathrm{H}}} \circ \sigma_i = \mathrm{id}_{\mathcal{O}_i}\,.$$

Thus, the σ_i are local sections of the principal bundle P_{G} which satisfy, at an arbitrary point $x \in \mathcal{O}_{ij}$,

$$\sigma_j(x) \equiv s_j\big(H(x)\big) = r_{h_{ij}(H(x))}\big(s_i\big(H(x)\big)\big) \equiv r_{h_{ij} \circ H(x)}\big(\sigma_i(x)\big)\,,$$

from which we read off the transition maps for the corresponding local trivialisations $\tau_i \equiv \tau_{\sigma_i}$:

$$g_{ij} := h_{ij} \circ H \; : \; \mathcal{O}_{ij} \longrightarrow \mathrm{H} \subset \mathrm{G}\,.$$

In the light of Prop. 1, this concludes the proof. □

While the statement of the above theorem does not come as a surprise after the discussion preceding it, the precise relation between the reduction that it describes and the formerly constructed reduced bundle remains unclear, not least because the constructive proof of Thm. 1 refers to the alternative notion of reducibility stated in Prop. 1. We pause now to bridge this gap. To this end, we first establish a more direct relation between local sections

$$\underline{\sigma}_i \; : \; \mathcal{O}_i \longrightarrow \mathsf{P}_{\mathrm{G}} \; : \; x \longmapsto \tau_i^{-1}\big(x, \gamma_i(x)\big) \equiv \underline{\tau}_i^{-1}(x, e)\,, \qquad i \in I$$

of the (would-be) reduced bundle[6] $\mathsf{P}_{\mathrm{H}}^{H}$ (over some cover $\mathcal{O} \equiv \{\mathcal{O}_i\}_{i \in I}$ of Σ trivialising for P_{G}) and the Higgs section $H \in \Gamma(\mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H})$, with, for $x \in \mathcal{O}_i$,

$$H(x) = [(\tau_i^{-1}(x, e), \gamma_i(x)\,\mathrm{H})] \equiv [(\underline{\sigma}_i(x), \mathrm{H})] \equiv \phi_{\pi_{\mathrm{G/H}}} \circ \underline{\sigma}_i(x)\,.$$

Thus, we conclude that H is precisely *the* global section of the associated bundle $\mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H}$ determined by the distinguished local sections of P_{G} related by transition maps with values in $\mathrm{H} \subset \mathrm{G}$ from the reconstruction of the Higgs field in the first part of the proof. Upon noting, once more, a trace of the structure of the principal bundle

$$\begin{array}{ccc} \mathrm{H} & \longrightarrow & \mathsf{P}_{\mathrm{G}} \\ & & \big\downarrow {\scriptstyle \phi_{\pi_{\mathrm{G/H}}}} \\ & & \mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G/H} \end{array} \tag{4}$$

of Prop. 7.7, we may, next, definitively confirm the status of $\mathsf{P}_{\mathrm{H}}^{H}$ by exploring the anatomy of the above principal bundle.

[6]Using the freedom of choice at our disposal, we fix once and for all $f_* \equiv f_0$ with $\mathrm{H}_* \equiv \mathrm{H}$, and consider the model case $F_0 \equiv \mathrm{G/H}$.

This we recover by judiciously combining local trivialisations of $\mathsf{P}_{\rm G}$,

$$\tau_i \;:\; \pi_{\mathsf{P}_{\rm G}}^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times {\rm G}\,, \qquad i \in I \tag{5}$$

(related by transition maps $g_{ij} \;:\; \mathcal{O}_{ij} \longrightarrow {\rm G}$), over the open cover $\mathcal{O}$ of the common base of $\mathsf{P}_{\rm G}$ and $\mathsf{P}_{\rm G} \times_{[\ell]} {\rm G}/{\rm H}$, and those of the principal bundle ${\rm G} \xrightarrow{\pi_{\rm G/H}} {\rm G}/{\rm H}$ of Cor. 6.1,

$$g_\alpha \;:\; \mathcal{U}_\alpha \longrightarrow {\rm G}\,, \qquad \alpha \in \mathscr{A}$$

(related by transition maps $\gamma_{\alpha\beta} \;:\; \mathcal{U}_{\alpha\beta} \longrightarrow {\rm H}$), associated with an open cover $\{\mathcal{U}_\alpha\}_{\alpha\in\mathscr{A}}$ of G/H, the former inducing local trivialisations

$$\widetilde{\tau}_i \;:\; \pi_{\mathsf{P}_{\rm G}\times_{[\ell]}{\rm G/H}}^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times {\rm G}/{\rm H} \;:\; [(p, g\,{\rm H})] \longmapsto \big(\pi_{\mathsf{P}_{\rm G}}(p), \big({\rm pr}_2 \circ \tau_i(p) \cdot g\big)\,{\rm H}\big)\,, \qquad i \in I$$

of the associated bundle

$$\begin{array}{ccc} {\rm G}/{\rm H} & \longrightarrow & \mathsf{P}_{\rm G} \times_{[\ell]} {\rm G}/{\rm H} \\ & & \Big\downarrow {\scriptstyle \pi_{\mathsf{P}_{\rm G}\times_{[\ell]}{\rm G/H}}} \\ & & \Sigma \end{array} .$$

As the open sets

$$\mathcal{W}_{i\alpha} := \mathcal{O}_i \times \mathcal{U}_\alpha\,, \qquad \alpha \in \mathscr{A}$$

compose an open cover of $\mathcal{O}_i \times {\rm G}/{\rm H}$,

$$\bigcup_{\alpha\in\mathscr{A}} \mathcal{W}_{i\alpha} \equiv \mathcal{O}_i \times {\rm G}/{\rm H}\,,$$

their diffeomorphic preimages

$$\widetilde{\mathcal{W}}_{i\alpha} := \widetilde{\tau}_i^{-1}(\mathcal{W}_{i\alpha})\,, \qquad (i,\alpha) \in I \times \mathscr{A}$$

cover the total space $\mathsf{P}_{\rm G} \times_{[\ell]} {\rm G}/{\rm H}$ of the associated bundle,

$$\bigcup_{(i,\alpha)\in I\times\mathscr{A}} \widetilde{\mathcal{W}}_{i\alpha} \equiv \mathsf{P}_{\rm G} \times_{[\ell]} {\rm G}/{\rm H}\,.$$

Define smooth maps

$$\widetilde{\sigma}_{i\alpha} \;:\; \widetilde{\mathcal{W}}_{i\alpha} \longrightarrow \mathsf{P}_{\rm G} \;:\; [(\tau_i^{-1}(x,e), g\,{\rm H})] \longmapsto r_{g_\alpha \circ [\tau_i^{-1}(x,e)]_{[\ell]}^{-1}([(\tau_i^{-1}(x,e), g\,{\rm H})])}\big(\tau_i^{-1}(x,e)\big)$$

(here, the $[\tau_i^{-1}(x,e)]_{[\ell]}$ are the fibre-modelling isomorphisms of Def. 7.1), with

$$\begin{aligned} r_{g_\alpha \circ [\tau_i^{-1}(x,e)]_{[\ell]}^{-1}([(\tau_i^{-1}(x,e), g\,{\rm H})])}\big(\tau_i^{-1}(x,e)\big) &\equiv r_{g_\alpha([\ell]_{\phi_{\mathsf{P}_{\rm G}}(\tau_i^{-1}(x,e), \tau_i^{-1}(x,e))}(g\,{\rm H}))}\big(\tau_i^{-1}(x,e)\big) \\ &= r_{g_\alpha(g\,{\rm H})}\big(\tau_i^{-1}(x,e)\big) = \tau_i^{-1}\big(x, g_\alpha(g\,{\rm H})\big)\,. \end{aligned}$$

We obtain

$$\begin{aligned} \phi_{\pi_{\rm G/H}} \circ \widetilde{\sigma}_{i\alpha}\big([(\tau_i^{-1}(x,e), g\,{\rm H})]\big) &= \phi_{\pi_{\rm G/H}}\big(\tau_i^{-1}\big(x, g_\alpha(g\,{\rm H})\big)\big) \equiv [(\tau_i^{-1}(x, g_\alpha(g\,{\rm H})), {\rm H})] \\ &= [(\tau_i^{-1}(x,e), g_\alpha(g\,{\rm H})\,{\rm H})]\,, \end{aligned}$$

but – by definition –

$$g\,{\rm H} \equiv \pi_{\rm G/H} \circ g_\alpha(g\,{\rm H}) = g_\alpha(g\,{\rm H})\,{\rm H}\,,$$

whence

$$\phi_{\pi_{\rm G/H}} \circ \widetilde{\sigma}_{i\alpha}\big([(\tau_i^{-1}(x,e), g\,{\rm H})]\big) = [(\tau_i^{-1}(x,e), g\,{\rm H})]\,.$$

We infer that the $\widetilde{\sigma}_{i\alpha}$ are local sections of (4),

$$\phi_{\pi_{\rm G/H}} \circ \widetilde{\sigma}_{i\alpha} = {\rm id}_{\widetilde{\mathcal{W}}_{i\alpha}}\,.$$

The corresponding local trivialisations can be obtained by inverting the maps

$$\widetilde{\tau}_{i\alpha}^{-1} \quad : \quad \widetilde{\mathcal{W}}_{i\alpha} \times {\rm H} \xrightarrow{\cong} \phi_{\pi_{\rm G/H}}^{-1}\big(\widetilde{\mathcal{W}}_{i\alpha}\big)$$

$$: \quad \big([(\tau_i^{-1}(x,e),g\,\mathrm{H})],h\big) \longmapsto r_h \circ \widetilde{\sigma}_{i\alpha}\big([(\tau_i^{-1}(x,e),g\,\mathrm{H})]\big) \equiv \tau_i^{-1}\big(x,g_\alpha(g\,\mathrm{H})\cdot h\big)\,,$$

and so they take the explicit form

$$\widetilde{\tau}_{i\alpha} \;:\; \phi^{-1}_{\pi_{\mathrm{G/H}}}\big(\widetilde{\mathcal{W}}_{i\alpha}\big) \longrightarrow \widetilde{\mathcal{W}}_{i\alpha} \times \mathrm{H} \;:\; p \longmapsto \Big([(p,\mathrm{H})],\big(g_\alpha \circ \pi_{\mathrm{G/H}} \circ \mathrm{pr}_2 \circ \tau_i(p)\big)^{-1} \cdot \big(\mathrm{pr}_2 \circ \tau_i(p)\big)\Big)\,.$$

Indeed, for $p = \tau_i^{-1}(x,g)$, we obtain the identity

$$\begin{aligned}
&\widetilde{\tau}_{i\alpha}^{-1} \circ \widetilde{\tau}_{i\alpha}(p) \equiv \widetilde{\tau}_{i\alpha}^{-1}\Big([(p,\mathrm{H})],\big(g_\alpha \circ \pi_{\mathrm{G/H}} \circ \mathrm{pr}_2 \circ \tau_i(p)\big)^{-1} \cdot \big(\mathrm{pr}_2 \circ \tau_i(p)\big)\Big) \\
\equiv\;& \widetilde{\tau}_{i\alpha}^{-1}\big([(\tau_i^{-1}(x,e),g\,\mathrm{H})],g_\alpha(g\,\mathrm{H})^{-1}\cdot g\big) = \tau_i^{-1}\big(x,g_\alpha(g\,\mathrm{H})\cdot g_\alpha(g\,\mathrm{H})^{-1}\cdot g\big) = \tau_i^{-1}(x,g) \equiv p
\end{aligned}$$

and

$$\begin{aligned}
&\widetilde{\tau}_{i\alpha} \circ \widetilde{\tau}_{i\alpha}^{-1}\big([(\tau_i^{-1}(x,e),g\,\mathrm{H})],h\big) \equiv \widetilde{\tau}_{i\alpha}\big(\tau_i^{-1}\big(x,g_\alpha(g\,\mathrm{H})\cdot h\big)\big) \\
\equiv\;& \Big([(\tau_i^{-1}(x,g_\alpha(g\,\mathrm{H})\cdot h),\mathrm{H})],\big(g_\alpha \circ \pi_{\mathrm{G/H}} \circ \mathrm{pr}_2 \circ \tau_i \circ \tau_i^{-1}\big(x,g_\alpha(g\,\mathrm{H})\cdot h\big)\big)^{-1} \\
&\cdot\big(\mathrm{pr}_2 \circ \tau_i \circ \tau_i^{-1}\big(x,g_\alpha(g\,\mathrm{H})\cdot h\big)\big)\Big) \\
=\;& \Big([(\tau_i^{-1}(x,e),(g_\alpha(g\,\mathrm{H})\cdot h)\,\mathrm{H})],\Big(g_\alpha\big(\big(g_\alpha(g\,\mathrm{H})\cdot h\big)\,\mathrm{H}\big)^{-1}\cdot g_\alpha(g\,\mathrm{H})\cdot h\Big)\Big) \\
=\;& \Big([(\tau_i^{-1}(x,e),g_\alpha(g\,\mathrm{H})\,\mathrm{H})],\Big(g_\alpha\big(g_\alpha(g\,\mathrm{H})\,\mathrm{H}\big)^{-1}\cdot g_\alpha(g\,\mathrm{H})\cdot h\Big)\Big) \\
=\;& \Big([(\tau_i^{-1}(x,e),g\,\mathrm{H})],\Big(g_\alpha\big(g\,\mathrm{H}\big)^{-1}\cdot g_\alpha(g\,\mathrm{H})\cdot h\Big)\Big) = \big([(\tau_i^{-1}(x,e),g\,\mathrm{H})],h\big)\,.
\end{aligned}$$

Having identified the local trivialisations, we readily derive the corresponding transition maps (computed at $([(\tau_j^{-1}(x,e),g\,\mathrm{H})],h) \in \widetilde{\mathcal{W}}_{i\alpha j\beta}$),

$$\begin{aligned}
&\widetilde{\tau}_{i\alpha} \circ \widetilde{\tau}_{j\beta}^{-1}\big([(\tau_j^{-1}(x,e),g\,\mathrm{H})],h\big) \equiv \widetilde{\tau}_{i\alpha}\big(\tau_j^{-1}\big(x,g_\beta(g\,\mathrm{H})\cdot h\big)\big) \\
\equiv\;& \Big([(\tau_j^{-1}(x,g_\beta(g\,\mathrm{H})\cdot h),\mathrm{H})],\big(g_\alpha \circ \pi_{\mathrm{G/H}} \circ \mathrm{pr}_2 \circ \tau_i \circ \tau_j^{-1}\big(x,g_\beta(g\,\mathrm{H})\cdot h\big)\big)^{-1} \\
&\cdot\big(\mathrm{pr}_2 \circ \tau_i \circ \tau_j^{-1}\big(x,g_\beta(g\,\mathrm{H})\cdot h\big)\big)\Big) \\
=\;& \Big([(\tau_j^{-1}(x,e),g\,\mathrm{H})],g_\alpha\big(\big(g_{ij}(x)\cdot g\big)\,\mathrm{H}\big)^{-1}\cdot g_{ij}(x)\cdot g_\alpha(g\,\mathrm{H})\cdot\gamma_{\alpha\beta}(g\,\mathrm{H})\cdot h\Big)\,.
\end{aligned}$$

Clearly, if we could assume the γ_α G-equivariant, we would have the $\gamma_{\alpha\beta} \circ \mathrm{pr}_2 \circ \widetilde{\tau}_j$ as the ensuing transition maps. In any event,

$$g_\alpha\big(\big(g_{ij}(x)\cdot g\big)\,\mathrm{H}\big)^{-1}\cdot g_{ij}(x)\cdot g_\alpha(g\,\mathrm{H})\cdot\gamma_{\alpha\beta}(g\,\mathrm{H}) \in \mathrm{H}\,. \tag{6}$$

This can be seen as follows. The g_α satisfy the identities

$$\pi_{\mathrm{G/H}} \circ g_\alpha = \mathrm{id}_{\mathcal{U}_\alpha}\,,$$

and so there exist smooth maps

$$h_\alpha \;:\; \mathcal{U}_\alpha \longrightarrow \mathrm{H}$$

such that, for an arbitrary point $g\,\mathrm{H} \in \mathcal{U}_\alpha$,

$$g_\alpha(g\,\mathrm{H}) = g\cdot h_\alpha(g\,\mathrm{H})\,. \tag{7}$$

Therefore,

$$g_\alpha\big(\big(g_{ij}(x)\cdot g\big)\,\mathrm{H}\big)^{-1}\cdot g_{ij}(x)\cdot g_\alpha(g\,\mathrm{H})\cdot\gamma_{\alpha\beta}(g\,\mathrm{H}) = h_\alpha\big(\big(g_{ij}(x)\cdot g\big)\,\mathrm{H}\big)\cdot h_\alpha(g\,\mathrm{H})\cdot\gamma_{\alpha\beta}(g\,\mathrm{H}) \in \mathrm{H}\,,$$

as claimed.

Returning to the discussion of the reduced bundle, we choose as the open cover of Σ trivialising for P_{G} the one comprised of the sets

$$\mathcal{O}_{i\alpha} := H^{-1}\big(\widetilde{\mathcal{W}}_{i\alpha}\big) \subset \mathcal{O}_i\,, \qquad (i,\alpha) \in I \times \mathscr{A}$$

and take the smooth maps

$$\sigma_{i\alpha} := \widetilde{\sigma}_{i\alpha} \circ H \;:\; \mathcal{O}_{i\alpha} \longrightarrow \mathsf{P}_{\mathrm{G}} \;:\; x \longmapsto \tau_i^{-1}\big(x, g_\alpha\big(\gamma_i(x)\,\mathrm{H}\big)\big)$$

in the rôle of local sections. Over double intersections $\mathcal{O}_{i\alpha j\beta} \ni x$, these are related as

$$\begin{aligned}
\sigma_{j\beta}(x) &\equiv \tau_j^{-1}\big(x, g_\beta\big(\gamma_j(x)\,\mathrm{H}\big)\big) \\
&= r_{g_\alpha\left(\gamma_i(x)\,\mathrm{H}\right)^{-1}\cdot g_{ij}(x)\cdot g_\beta\left(\gamma_j(x)\,\mathrm{H}\right)}\big(\tau_i^{-1}\big(x, g_\alpha\big(\gamma_i(x)\,\mathrm{H}\big)\big)\big) \\
&= r_{g_\alpha((g_{ij}(x)\cdot\gamma_j(x)\cdot h_{ij}(x)^{-1})\,\mathrm{H})^{-1}\cdot g_{ij}(x)\cdot g_\beta(\gamma_j(x)\,\mathrm{H})}\big(\tau_i^{-1}\big(x, g_\alpha\big(\gamma_i(x)\,\mathrm{H}\big)\big)\big) \\
&= r_{g_\alpha((g_{ij}(x)\cdot\gamma_j(x))\,\mathrm{H})^{-1}\cdot g_{ij}(x)\cdot g_\beta(\gamma_j(x)\,\mathrm{H})}\big(\tau_i^{-1}\big(x, g_\alpha\big(\gamma_i(x)\,\mathrm{H}\big)\big)\big)
\end{aligned}$$

by the transition maps

$$h_{i\alpha j\beta}(\cdot) := h_\alpha\big(\big(g_{ij}(\cdot)\cdot\gamma_j(\cdot)\big)\,\mathrm{H}\big)^{-1}\cdot h_\alpha\big(\gamma_j(\cdot)\,\mathrm{H}\big)\cdot\gamma_{\alpha\beta}\big(\gamma_j(\cdot)\,\mathrm{H}\big) \;:\; \mathcal{O}_{i\alpha j\beta} \longrightarrow \mathrm{H} \subset \mathrm{G}\,,$$

cp Eqs. (6) and (7). We may, now, invoke (the proof of) the Clutching Theorem of Lecture 1 (pp. 29–31) to reconstruct a principal bundle with the structure group H and the above-extracted transition maps,

$$\mathsf{P}_{\mathrm{H}} \equiv \Big(\bigsqcup_{(i,\alpha)\in I\times\mathscr{A}} \big(\mathcal{O}_{i\alpha} \times \mathrm{H}\big)\Big)/_{\sim_{h_{..}}}\,.$$

Its faithful local image within P_{G} along the mapping $\Phi_{i\alpha}$ of Eq. (1) is given by (*cp* Eq. (7))

$$\begin{aligned}
\Phi_{i\alpha}\big(\mathsf{P}_{\mathrm{H}}\restriction_{\mathcal{O}_{i\alpha}}\big) &\equiv \big\{\; \tau_{\sigma_{i\alpha}}^{-1}(x,h) \equiv r_h \circ \sigma_{i\alpha}(x) \;\mid\; (x,h) \in \mathcal{O}_{i\alpha} \times \mathrm{H} \;\big\} \\
&\equiv \big\{\; \tau_i^{-1}\big(x, g_\alpha\big(\gamma_i(x)\,\mathrm{H}\big)\cdot h\big) \;\mid\; (x,h) \in \mathcal{O}_{i\alpha} \times \mathrm{H} \;\big\} \\
&= \big\{\; \tau_i^{-1}\big(x, \gamma_i(x)\cdot h_\alpha\big(\gamma_i(x)\,\mathrm{H}\big)\cdot h\big) \;\mid\; (x,h) \in \mathcal{O}_{i\alpha} \times \mathrm{H} \;\big\} \\
&= \big\{\; \tau_i^{-1}\big(x, \gamma_i(x)\cdot h\big) \;\mid\; (x,h) \in \mathcal{O}_{i\alpha} \times \mathrm{H} \;\big\} \equiv \mathsf{P}_{\mathrm{H}}^{H}\restriction_{\mathcal{O}_{i\alpha}}\,.
\end{aligned}$$

Thus, the subspace

$$\mathsf{P}_{\mathrm{H}}^{H} \equiv \bigcup_{(i,\alpha)\in I\times\mathscr{A}} \mathsf{P}_{\mathrm{H}}^{H}\restriction_{\mathcal{O}_{i\alpha}} \subset \mathsf{P}_{\mathrm{G}}$$

identified previously is, indeed, the reduced bundle, as claimed before. Accordingly, we may carry out all local considerations with regard to the reductiom mechanism on the hands-on model $\mathsf{P}_{\mathrm{H}}^{H}$. This approach enables us to straightforwardly prove the following important uniqueness result.

Proposition 2. Adopt the hitherto notation, and in particular that of Prop. 8.5 and of the proof of Prop. 7.6. Let $H \in \Gamma(\mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G}/\mathrm{H})$ be a Higgs field. For any $\Gamma \in \Gamma(\mathrm{Ad}\mathsf{P}_{\mathrm{G}})$, there exists an isomorphism $(\Phi_\Gamma, \mathrm{id}_\Sigma, \mathrm{id}_{\mathrm{H}})$ of principal bundles with the structure group H between the reduced bundle $\mathsf{P}_{\mathrm{H}}^{{}^\Gamma H} \subset \mathsf{P}_{\mathrm{G}}$ associated with the Γ-transform ${}^\Gamma H$ of H and the automorphic image $\alpha_\Gamma(\mathsf{P}_{\mathrm{H}}^{H}) \subset \alpha_\Gamma(\mathsf{P}_{\mathrm{G}}) \equiv \mathsf{P}_{\mathrm{G}}$ of the reduced bundle $\mathsf{P}_{\mathrm{H}}^{H} \subset \mathsf{P}_{\mathrm{G}}$,

$$(\Phi_\Gamma, \mathrm{id}_\Sigma, \mathrm{id}_{\mathrm{H}}) \;:\; \mathsf{P}_{\mathrm{H}}^{{}^\Gamma H} \xrightarrow{\;\cong\;} \alpha_\Gamma\big(\mathsf{P}_{\mathrm{H}}^{H}\big)\,.$$

<u>*Proof:*</u> Consider the reduced bundles defined by a pair of Higgs fields: $H \in \Gamma(\mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G}/\mathrm{H})$ and its gauge-transform ${}^\Gamma H$. Upon choosing an open cover $\{\mathcal{O}_i\}_{i\in I}$ of Σ trivialising for P_{G}, with the corresponding local trivialisations

$$\tau_i \;:\; \pi_{\mathsf{P}_{\mathrm{G}}}^{-1}(\mathcal{O}_i) \xrightarrow{\;\cong\;} \mathcal{O}_i \times \mathrm{G}$$

with the associated transition maps

$$g_{ij} \;:\; \mathcal{O}_{ij} \longrightarrow \mathrm{G}\,,$$

we may always write γ as

$$\Gamma\restriction_{\mathcal{O}_i} = [(\tau_i^{-1}(\cdot, e), \Gamma_i(\cdot))]$$

for some smooth maps

$$\Gamma_i \;:\; \mathcal{O}_i \longrightarrow \mathrm{G}$$

subject to the relation

$$\Gamma_j\restriction_{\mathcal{O}_{ij}} = \mathrm{Ad}_{g_{ij}}\big(\Gamma_i\restriction_{\mathcal{O}_{ij}}\big)$$

implied by the global smoothness of γ. Assume, furthermore, that the Higgs field H has, as before, the local presentation

$$H\restriction_{\mathcal{O}_i} = [(\tau_i^{-1}(\cdot,e),\gamma_i(\cdot)\,\mathrm{H})]\,,$$

so that

$$^{\Gamma}H\restriction_{\mathcal{O}_i} = [(\tau_i^{-1}(\cdot,e),(\Gamma_i\cdot\gamma_i)(\cdot)\,\mathrm{H})]\,.$$

When identifying the local profiles

$$^{\Gamma}\gamma_i \;:\; \mathcal{O}_i \longrightarrow \mathrm{G}$$

for $^{\Gamma}H$, we must account for the irremovable ambiguity implicit in the identity

$$^{\Gamma}\gamma_i(\cdot)\,\mathrm{H} = (\Gamma_i\cdot\gamma_i)(\cdot)\,\mathrm{H}\,,$$

that is we allow an arbitrary correction

$$^{\Gamma}\gamma_i = \Gamma_i\cdot\gamma_i\cdot\mathrm{Inv}\circ h_i$$

quantified by a smooth map

$$h_i \;:\; \mathcal{O}_i \longrightarrow \mathrm{H}\,.$$

If, now, the transition maps of the reduced bundle defined by H are given by

$$h_{ij} \;:\; \mathcal{O}_{ij} \longrightarrow \mathrm{H}\,,$$

or – in other words –

$$g_{ij}\cdot\gamma_j\restriction_{\mathcal{O}_{ij}} = \gamma_i\restriction_{\mathcal{O}_{ij}}\cdot h_{ij}\,,$$

cp Eq. (2), then those of the reduced bundle defined by $^{\Gamma}H$,

$$^{\Gamma}h_{ij} \;:\; \mathcal{O}_{ij} \longrightarrow \mathrm{H}\,,$$

are fixed by the identity

$$\begin{aligned} {}^{\Gamma}\gamma_i\restriction_{\mathcal{O}_{ij}}\cdot{}^{\Gamma}h_{ij} &\equiv g_{ij}\cdot{}^{\Gamma}\gamma_j\restriction_{\mathcal{O}_{ij}} \equiv g_{ij}\cdot\big(\Gamma_j\cdot\gamma_j\cdot\mathrm{Inv}\circ h_j\big)\restriction_{\mathcal{O}_{ij}} = \Gamma_i\restriction_{\mathcal{O}_{ij}}\cdot g_{ij}\cdot\big(\gamma_j\cdot\mathrm{Inv}\circ h_j\big)\restriction_{\mathcal{O}_{ij}} \\ &= \big(\Gamma_i\cdot\gamma_i\cdot\mathrm{Inv}\circ h_i\big)\restriction_{\mathcal{O}_{ij}}\cdot h_i\restriction_{\mathcal{O}_{ij}}\cdot h_{ij}\cdot\mathrm{Inv}\circ h_j\restriction_{\mathcal{O}_{ij}} \\ &\equiv {}^{\Gamma}\gamma_i\restriction_{\mathcal{O}_{ij}}\cdot\big(h_i\restriction_{\mathcal{O}_{ij}}\cdot h_{ij}\cdot\mathrm{Inv}\circ h_j\restriction_{\mathcal{O}_{ij}}\big) \end{aligned}$$

in the form

$$^{\Gamma}h_{ij} = h_i\restriction_{\mathcal{O}_{ij}}\cdot h_{ij}\cdot\mathrm{Inv}\circ h_j\restriction_{\mathcal{O}_{ij}}\,. \tag{8}$$

Keeping that in mind, we write out inverses of the respective distinguished local trivialisations of the reduced bundles introduced in Eq. (3), that is

$$\underline{\tau}_i^{-1} \;:\; \mathcal{O}_i\times\mathrm{H} \xrightarrow{\cong} \big(\pi_{\mathsf{P}_{\mathrm{G}}}\restriction_{\mathsf{P}_{\mathrm{H}}^{H}}\big)^{-1}(\mathcal{O}_i) \;:\; (x,h) \longmapsto \tau_i^{-1}\big(x,\gamma_i(x)\cdot h\big) \tag{9}$$

for $\mathsf{P}_{\mathrm{H}}^{H}$ and

$$^{\Gamma}\underline{\tau}_i^{-1} \;:\; \mathcal{O}_i\times\mathrm{H} \xrightarrow{\cong} \big(\pi_{\mathsf{P}_{\mathrm{G}}}\restriction_{\mathsf{P}_{\mathrm{H}}^{\Gamma H}}\big)^{-1}(\mathcal{O}_i) \;:\; (x,h) \longmapsto \tau_i^{-1}\big(x,\Gamma_i(x)\cdot\gamma_i(x)\cdot h_i(x)^{-1}\cdot h\big)$$

for $\mathsf{P}_{\mathrm{H}}^{\Gamma H}$.

Following the prescription from the constructive proof of Prop. 8.5, we obtain

$$\begin{aligned} \alpha_{\Gamma}\big(\underline{\tau}_i^{-1}(x,h)\big) &= \tau_i^{-1}\big(x,\Gamma_i(x)\cdot\gamma_i(x)\cdot h\big) \equiv \tau_i^{-1}\big(x,\big(\Gamma_i(x)\cdot\gamma_i(x)\cdot h_i(x)^{-1}\big)\cdot h_i(x)\cdot h\big) \\ &\equiv {}^{\Gamma}\underline{\tau}_i^{-1}\big(x,h_i(x)\cdot h\big)\,, \end{aligned} \tag{10}$$

which identifies the $h_i : \mathcal{O}_i \longrightarrow \mathrm{H}$ as candidates for the local data of Φ_Γ. We refer to them as **local data of the effective gauge transformation** in what follows. Their status is confirmed by identity (8) on the basis of Thm. 6.1 (note that the transition maps of the reduced bundle associated with the local profiles $\Gamma_i \cdot \gamma_i$ are identical with those of the one associated with the γ_i). □

Remark 1. The last proposition demonstrates that realising the effective (gauge) symmetry modelled on the vacuum isotropy group H in the original setting[7] associated with the mother gauge bundle P_{G} requires a simultaneous gauge transformation of the gauge bundle (for which it is simply an automorphism in virtue of Prop. 7.6) *and* of the reference Higgs section, as it is only then that we obtain

$$\alpha_{\mathrm{Inv}\circ\Gamma}\big(\mathsf{P}_{\mathrm{H}}^{\Gamma H}\big) \cong \mathsf{P}_{\mathrm{H}}^{H}\,.$$

That the reduced gauge symmetry cannot be realised in terms of the original one is best seen, at this stage, by trying to identify those gauge automorphisms $\Gamma \in \Gamma(\mathrm{Ad}\mathsf{P}_{\mathrm{G}})$ of the mother gauge bundle P_{G} which preserve (up to an isomorphism in the category of principal bundles with the structure group H) the reduced subbundle $\mathsf{P}_{\mathrm{H}}^{H}$ associated with a *fixed* Higgs field $H \in \Gamma(\mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G}/\mathrm{H})$. To this end, we consider, once more, as in Eq. (10), the effect of the gauge automorphism on a point in $\mathsf{P}_{\mathrm{H}}^{H}$,

$$\begin{aligned}\alpha_\Gamma\big(\underline{\tau}_i^{-1}(x,h)\big) &= \tau_i^{-1}\big(x,\Gamma_i(x)\cdot\gamma_i(x)\cdot h\big) = \tau_i^{-1}\big(x,\gamma_i(x)\cdot \mathrm{Ad}_{\gamma_i(x)^{-1}}\big(\Gamma_i(x)\big)\cdot h\big)\\ &\equiv \underline{\tau}_i^{-1}\big(x,\mathrm{Ad}_{\gamma_i(x)^{-1}}\big(\Gamma_i(x)\big)\cdot h\big)\,,\end{aligned}$$

and conclude that a generic gauge transformation does *not* preserve (the isomorphism class of) the reduced bundle as

$$\mathrm{Ad}_{\gamma_i(x)^{-1}}\big(\Gamma_i(x)\big) \notin \mathrm{H}\,,$$

and even restricting to those with $\Gamma_i : \mathcal{O}_i \longrightarrow \mathrm{H}$ (which seems only natural) does *not* save the day, unless the vacuum isotropy group H is a normal subgroup in G. What does work, on the other hand, is the radical restriction of the codomain of local data of admisible gauge automorphisms to the subgroup

$$\mathrm{H}_{\mathrm{centr}} \equiv \mathrm{H} \cap Z(\mathrm{G})\,,$$

where $Z(\mathrm{G})$ is the centre of G.

Having discussed the existence and uniqueness of a reduction of the structure group of a gauge bundle in the presence of a Higgs field, we may, finally, proceed with a study of its physical implications.

The fundamental configurational consequence of the reduction is stated in

Proposition 3. Adopt the hitherto notation and let M be a manifold with a smooth action

$$\lambda : \mathrm{G} \times M \longrightarrow M$$

of the Lie group G. Whenever there exists a Higgs field $H \in \Gamma(\mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G}/\mathrm{H})$, to every global section of the associated bundle $\mathsf{P}_{\mathrm{G}} \times_\lambda M$ there corresponds a global section of the associated bundle $\mathsf{P}_{\mathrm{H}}^{H} \times_{\underline{\lambda}} M$, where

$$\underline{\lambda} : \mathrm{H} \times M \longrightarrow M : (h,m) \longmapsto \lambda_h(m)$$

is the one-sided restriction of λ to the vacuum-isotropy subgroup.

Proof: Consider a global section $\varphi \in \Gamma(\mathsf{P}_{\mathrm{G}} \times_\lambda M)$ and use the local trivialisations (5) to write it as

$$\varphi\restriction_{\mathcal{O}_i} = \big[\big(\tau_i^{-1}(\cdot,e),\varphi_i(\cdot)\big)\big]$$

[7] Of course, the reduced bundle, as any principal bundle, has its proper automorphism group $\Gamma(\mathrm{Ad}\mathsf{P}_{\mathrm{H}}^{H})$.

in terms of some smooth maps

$$\varphi_i \,:\, \mathcal{O}_i \longrightarrow M$$

with properties

$$\varphi_i \restriction_{\mathcal{O}_{ij}} = \lambda_{g_{ij}(\cdot)}\big(\varphi_j \restriction_{\mathcal{O}_{ij}}\big)\,.$$

Upon invoking the definition (9) of the associated local sections of the reduced bundle, we trivially rewrite the above as

$$\varphi\restriction_{\mathcal{O}_i} \equiv \big[\big(\underline{\tau}_i^{-1}(\cdot,e),\lambda_{\gamma_i(\cdot)^{-1}}(\varphi_i(\cdot))\big)\big]\,,$$

and the smooth maps

$$\underline{\varphi}_i := \lambda_{\gamma_i(\cdot)^{-1}}\big(\varphi_i(\cdot)\big) \,:\, \mathcal{O}_i \longrightarrow M$$

are readily found to satisfy, at an arbitrary point $x \in \mathcal{O}_{ij}$,

$$\begin{aligned} \underline{\varphi}_i(x) &\equiv \lambda_{\gamma_i(x)^{-1}}\big(\varphi_i(x)\big) = \lambda_{(g_{ji}(x)^{-1}\cdot\gamma_j(x)\cdot h_{ji}(x))^{-1}}\big(\varphi_i(x)\big) = \lambda_{h_{ij}(x)}\circ\lambda_{\gamma_j(x)^{-1}}\Big(\lambda_{g_{ij}(x)}\big(\varphi_i(x)\big)\Big) \\ &\equiv \lambda_{h_{ij}(x)}\big(\underline{\varphi}_j(x)\big)\,, \end{aligned}$$

as desired. □

In order to appreciate the 'ontological-status' transition effected by the above seemingly trivial rewriting, we should recall that it is not the section φ itself but, instead, its local presentation enters the lagrangean model of gauge-symmetric field dynamics, *cp* Def. 8.5. This requires fixing the (local) gauge, and it is precisely at this stage that the difference between the two interpretations of φ becomes apparent: While in the absence of the Higgs field, we have solely the trivialising sections $\sigma_i(\cdot) \equiv \tau_i^{-1}(\cdot,e)$ of P_{G} at our disposal, giving rise to

$$\varphi_{\sigma_i}(\cdot) \equiv \Phi_\lambda[\varphi]\circ\sigma_i(\cdot) \equiv \lambda_{\phi_{\mathsf{P}_{\mathrm{G}}}(\sigma_i(\cdot),\tau_i^{-1}(\pi_{\mathsf{P}_{\mathrm{G}}}\circ\sigma_i(\cdot),e))}\big(\varphi_i(\cdot)\big) = \varphi_i(\cdot)\,,$$

the appearance of H provides us with the alternative choice

$${}^H\varphi_i(\cdot) := \varphi_{\underline{\sigma}_i}(\cdot) \equiv \Phi_{\underline{\lambda}}[\varphi]\circ\underline{\sigma}_i(\cdot) \equiv \lambda_{\phi_{\mathsf{P}^H_{\mathrm{H}}}(\underline{\sigma}_i(\cdot),\underline{\tau}_i^{-1}(\pi_{\mathsf{P}_{\mathrm{G}}}\circ\underline{\sigma}_i(\cdot),e))}\big(\underline{\varphi}_i(\cdot)\big) = \underline{\varphi}_i(\cdot)\,,$$

with the gluing properties over double intersections controlled by the isotropy group H, as derived in the proof of the last proposition. The physical significance of this transition is reflected in

Definition 3. Adopt the hitherto notation. The local presentation ${}^H\varphi_i = \varphi_{\underline{\sigma}_i}$ of the (matter) field $\varphi \in \Gamma(\mathsf{P}_{\mathrm{G}} \times_\lambda M)$ of type M associated with the local section $\underline{\sigma}_i \equiv \tau_i^{-1}(\cdot,e)$ of the reduced (gauge) bundle P_{G} in the presence of a Higgs field $H \in \Gamma(\mathsf{P}_{\mathrm{G}} \times_{[\ell]} \mathrm{G}/\mathrm{H})$ is called the **reduced local (matter) field of type** M.

In particular,

Corollary 1. Adopt the hitherto notation. The reduced Higgs field is globally constant,

$${}^H H_i \equiv \mathrm{H}\,, \qquad i \in I\,.$$

Proof: Obvious. □

We conclude the present discussion of the configurational aspect of the Higgs phenomenon by establishing the nature of the effective gauge symmetry of a field theory formulated in terms of the reduced fields, in conformity with our former findings. This we do in

Proposition 4. Adopt the hitherto notation. An arbitrary gauge transformation $\Gamma \in \Gamma(\mathrm{Ad}\mathsf{P}_{\mathrm{G}})$ of a (matter) field $\varphi \in \Gamma(\mathsf{P}_{\mathrm{G}} \times_\lambda M)$ of type M,

$$(\Gamma,\varphi) \longmapsto {}^\Gamma\phi\,,$$

is realised on the corresponding reduced local fields $\{{}^H\varphi_i\}_{i\in I}$ by the local data $\{h_i\}_{i\in I}$ of the effective gauge transformation as *per*

$${}^{{}^\Gamma H}\big({}^\Gamma\varphi\big)_i(\cdot) = \lambda_{h_i(\cdot)}\big({}^H\varphi_i\big)\,, \qquad i \in I\,.$$

Proof: A simple exercise. □

Remark 2. The simlutaneous gauge transformation of both: the matter field *and* the Higgs field in the statement of the proposition takes into account our former findings, stated in Rem. 1. Were we to disregerd them, a derivation of the local presentation of a gauge transformation of the reduced fields with the Higgs field kept *fixed* would lead to precisely the same conclusions (constraints on the admissible gauge transformations G) as before – a verification of this claim is left to the Reader.

The dynamical aspect. To be continued...