CLASSICAL FIELD THEORY IN THE TIME OF COVID-19
11. LECTURE BATCH

The Crittenden connection on an associated bundle

Confronted with the task of constructing a connection on an associated bundle, we would do well to return to the original question that has led us to consider connections on bundles in all generality. Thus, our goal, motivated by a natural physical need, was to define a differentiation of local sections of a fibre bundle E (with a typical fibre F) along vector fields on its base B that would take values in the vertical subbundle $\mathsf{V}E \subset \mathsf{T}E$ and so would, through an appropriate structural diffeomorphism $\mathsf{V}E\restriction_{E_x} \cong \mathsf{T}F$ over an arbitrary point $x \in B$, enable us to functorially (by means of the tangent functor T) transport an arbitrary relevant (*e.g.*, physically) structure from the space of sections $\Gamma(E)$ (induced from the typical fibre F) onto the space of directional derivatives determined by this differentiation. In the setting of associated bundles $\mathsf{P}_{\mathrm{G}} \times_\lambda M$, the relevant structure is that of a manifold with an action $\lambda : \mathrm{G} \times M \longrightarrow M$ of a Lie group G that features in the rôle of the structure group of a principal bundle P_{G} over B and, simultaneously, in the rôle of the typical fibre of the (associated) bundle of groups $\mathrm{Ad}\mathsf{P}_{\mathrm{G}}$, with the group of global sections $\Gamma(\mathrm{Ad}\mathsf{P}_{\mathrm{G}})$ realised $\Gamma(\mathsf{P}_{\mathrm{G}} \times_\lambda M)$ in a manner locally modelled on λ. Our search for a differentiation on $\Gamma(\mathsf{P}_{\mathrm{G}} \times_\lambda M)$ equivariant with respect to the action of $\Gamma(\mathrm{Ad}\mathsf{P}_{\mathrm{G}})$ is complicated by the non-obvious nature of the realisation of that group on $\mathsf{V}(\mathsf{P}_{\mathrm{G}} \times_\lambda M)$. Therefore, instead of contemplating an abstract construction of a convenient model of the bundle $\mathsf{V}(\mathsf{P}_{\mathrm{G}} \times_\lambda M)$ (understood as a choice of a representative of its isomorphism class) that would support such a natural action, we put to work our hitherto observations with regard to principal and associated bundles with view to constructing a differentiation taking values *directly* in the manifold $\mathsf{T}M$. This has an obvious advantage, to wit, it gives us the possibility of postulating a natural smooth action of the group G:

$$\mathsf{T}_2\lambda. : \mathrm{G} \times \mathsf{T}M \longrightarrow \mathsf{T}M : (g, v) \longmapsto \mathsf{T}_{\pi_{\mathsf{T}M}(v)}\lambda_g(v)$$

to be intertwined, in a local trivialisation, with λ by the sought-after covariant derivative on $\Gamma(\mathsf{P}_{\mathrm{G}} \times_\lambda M)$. The ease with which we idetify the action on the model space for the codomain of the derivative comes at a price: the nature of the unavoidable differential transition from sections of the associated bundle $\mathsf{P}_{\mathrm{G}} \times_\lambda M$ to linear mappings between tangent bundles $\mathsf{T}B$ and $\mathsf{T}M$ seems unclear at first. Instrumental in its understanding prove to be the observations made in the context of our structural discussion of associated bundles, and more specifically Prop. 7.3 that enables us to write down a pair of important bijections:

$$\begin{aligned} \Phi_\lambda \quad &: \quad \Gamma(\mathsf{P}_{\mathrm{G}} \times_\lambda M) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{G}}(\mathsf{P}_{\mathrm{G}}, M)\,, \\ \Phi_{\mathsf{T}_2\lambda} \quad &: \quad \Gamma(\mathsf{P}_{\mathrm{G}} \times_{\mathsf{T}_2\lambda} \mathsf{T}M) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{G}}(\mathsf{P}_{\mathrm{G}}, \mathsf{T}M)\,. \end{aligned}$$

These, in conjunction with the construction from Def. 10.2 of a (principal) connection on a principal bundle, and in particular that of the horizontal lift

$$\mathrm{Hor.} : \mathsf{T}B \rightarrowtail \mathsf{T}\mathsf{P}_{\mathrm{G}}\,, \qquad \mathrm{Hor.}(\mathsf{T}B) = \mathsf{H}\mathsf{P}_{\mathrm{G}}$$

of Prop. 9.2, equips us with the requisite tools to reformulate the original question of differentiation of a section $\phi \in \Gamma(\mathsf{P}_{\mathrm{G}} \times_\lambda M)$ of the associated bundle along a vector field $\mathcal{X} \in \Gamma(\mathsf{T}B)$ on its base in terms of differentiation of the corresponding G-equivariant map $\Phi_\lambda[\phi] \in \mathrm{Hom}_{\mathrm{G}}(\mathsf{P}_{\mathrm{G}}, M)$ along the vector field $\mathrm{Hor.}(\mathcal{X}) \in \Gamma(\mathsf{T}\mathsf{P}_{\mathrm{G}})$ on the total space of the principal bundle P_{G}. Such a formal trick turns out to be quite convenient in that it affords us the chance to impose the condition of $\Gamma(\mathrm{Ad}\mathsf{P}_{\mathrm{G}})$-equivariance straightforwardly. Indeed, upon denoting the derivative to be found as

$$\mathscr{D}_{\mathrm{Hor.}(\mathcal{X})}\Phi_\lambda[\phi] : \mathsf{P}_{\mathrm{G}} \longrightarrow \mathsf{T}M\,,$$

we may demand that, for an arbitrary global section $\gamma \in \Gamma(\mathrm{AdP_G})$, the following identity

$$\mathscr{D}_{\mathrm{Hor.}(\mathcal{X})}\Phi_\lambda\big[\Gamma\big[\Gamma[\widetilde{r}]\big]_\gamma(\phi)\big](\cdot) = \mathsf{T}_2\lambda_{\Phi_{\mathrm{Ad}}[\gamma](\cdot)}\big(\mathscr{D}_{\mathrm{Hor.}(\mathcal{X})}\big(\Phi_\lambda[\phi]\big)\big)(\cdot)$$

obtain, and if the mapping derived from such a requirement turns out to be G-equivariant,

$$\forall_{g\in\mathrm{G}} \ : \ \mathscr{D}_{\mathrm{Hor.}(\mathcal{X})}\big(\Phi_\lambda[\phi]\big) \circ r_g = \mathsf{T}_2\lambda_{g^{-1}} \circ \mathscr{D}_{\mathrm{Hor.}(\mathcal{X})}\big(\Phi_\lambda[\phi]\big)\,,$$

then we may define – with rerefence to Prop. 7.3 once more – the covariant derivative

$$\nabla_{\mathcal{X}}\phi \coloneqq \Phi^{-1}_{\mathsf{T}_2\lambda}\big[\mathscr{D}_{\mathrm{Hor.}(\mathcal{X})}\Phi_\lambda[\phi]\big] \in \Gamma(\mathsf{P}_\mathrm{G} \times_{\mathsf{T}_2\lambda} \mathsf{T}M)\,,$$

with, by definition, the desired property

$$\begin{aligned} &\nabla_{\mathcal{X}}\big(\Gamma[\Gamma[\widetilde{r}]]^\lambda_\gamma(\phi)\big) \equiv \Phi^{-1}_{\mathsf{T}_2\lambda}\big[\mathscr{D}_{\mathrm{Hor.}(\mathcal{X})}\Phi_\lambda\big[\Gamma\big[\Gamma[\widetilde{r}]\big]^\lambda_\gamma(\phi)\big]\big] \\ = \ & \Phi^{-1}_{\mathsf{T}_2\lambda}\big[\mathsf{T}_2\lambda_{\Phi_{\mathrm{Ad}}[\gamma]}\big(\mathscr{D}_{\mathrm{Hor.}(\mathcal{X})}\big(\Phi_\lambda[\phi]\big)\big)\big] \equiv \Phi^{-1}_{\mathsf{T}_2\lambda}\big[[\Phi_{\mathrm{Ad}}\mathsf{T}_2\lambda]_\gamma\big(\mathscr{D}_{\mathrm{Hor.}(\mathcal{X})}\big(\Phi_\lambda[\phi]\big)\big)\big] \\ = \ & \Gamma[\Gamma[\widetilde{r}]]^{\mathsf{T}_2\lambda}_\gamma\big(\Phi^{-1}_{\mathsf{T}_2\lambda}\big[\mathscr{D}_{\mathrm{Hor.}(\mathcal{X})}\Phi_\lambda[\phi]\big]\big) \equiv \Gamma[\Gamma[\widetilde{r}]]^{\mathsf{T}_2\lambda}_\gamma\big(\nabla_{\mathcal{X}}\phi\big) \end{aligned}$$

following from Prop. 7.5. After these preliminary explanations, we may now turn directly to the construction of the differentiation $\mathscr{D}_{\mathrm{Hor.}(\mathcal{X})}$.

The point of departure for determining a correction to the ordinary directional derivative, $\mathscr{D}_. - \mathsf{d}$, enforced by the adopted constraints of equivariance of the operator $\mathscr{D}_.$, is an analysis of the transformation properties of that derivative under a gauge transformation $\gamma \in \Gamma(\mathrm{AdP_G})$ applied to the section $\phi \in \Gamma(\mathsf{P}_\mathrm{G} \times_\lambda M)$ being differentiated, in which we invoke Prop. 7.5, the derivation of Eqn. (9.14) and Prop. 5.1,

$$\begin{aligned} \mathsf{d}\Phi_\lambda\big[\Gamma[\Gamma[\widetilde{r}]]^\lambda_\gamma(\phi)\big] = \mathsf{d}\big(\Phi_{\mathrm{Ad}}\lambda_{\Phi_{\mathrm{Ad}}[\gamma]}\big(\Phi_\lambda[\phi]\big)\big) = \big(\mathrm{id}_{\mathsf{T}^*\mathsf{P}_\mathrm{G}} \otimes \mathsf{T}_{\Phi_\lambda[\phi](\cdot)}\Phi_{\mathrm{Ad}}\lambda_{\Phi_{\mathrm{Ad}}[\gamma](\cdot)}\big) \circ \mathsf{d}\Phi_\lambda[\phi] \\ -\big(\mathrm{Inv} \circ \Phi_{\mathrm{Ad}}[\gamma]\big)^*\theta^A_\mathrm{L} \otimes \mathcal{K}_{t_A}\big(\Phi_{\mathrm{Ad}}\lambda_{\Phi_{\mathrm{Ad}}[\gamma](\cdot)}\big(\Phi_\lambda[\phi](\cdot)\big)\big)\,, \end{aligned}$$

where $t_A,\ A \in \overline{1, \dim \mathrm{G}}$ is a basis of the Lie algebra $\mathfrak{g}$ of the Lie group G. Taking into account the transformation rules for the potential $\underline{\mathcal{A}} \equiv \underline{\mathcal{A}}^A \otimes t_A \in \Omega^1(\mathsf{P}_\mathrm{G}) \otimes_\mathbb{R} \mathfrak{g}$ of a principal connection on $\mathsf{P}_\mathrm{G} \ni p$ (introduced in w Def. 10.3),

$$\begin{aligned} {}^\gamma\underline{\mathcal{A}}(p) \ &\equiv \ \mathsf{d}y^\alpha\big(p \triangleleft \Phi_{\mathrm{Ad}}[\gamma](p)^{-1}\big) \otimes \underline{\mathcal{A}}_\alpha\big(p \triangleleft \Phi_{\mathrm{Ad}}[\gamma](p)^{-1}\big) \\ &= \ \big(\mathsf{T}_p r_{\Phi_{\mathrm{Ad}}[\gamma](p)^{-1}}\big)^\alpha_{\ \mu} \triangleright \big(\mathsf{d}x^\mu(p) + \big(\mathrm{Inv} \circ \Phi_{\mathrm{Ad}}[\gamma]\big)^*\theta^A_\mathrm{R}(p)\,\mathcal{K}^\mu_{t_A}(p)\big) \otimes \underline{\mathcal{A}}_\alpha\big(p \triangleleft \Phi_{\mathrm{Ad}}[\gamma](p)^{-1}\big) \\ &= \ \underline{\mathcal{A}}\big(p \triangleleft \Phi_{\mathrm{Ad}}[\gamma](p)^{-1}\big) \circ \mathsf{T}_p r_{\Phi_{\mathrm{Ad}}[\gamma](p)^{-1}} + \big(\mathrm{Inv} \circ \Phi_{\mathrm{Ad}}[\gamma]\big)^*\theta^A_\mathrm{L}(p) \otimes \mathcal{K}^\alpha_{t_A} \triangleright \underline{\mathcal{A}}_\alpha\big(p \triangleleft \Phi_{\mathrm{Ad}}[\gamma](p)^{-1}\big) \\ &= \ \mathsf{T}_e\mathrm{Ad}_{\Phi_{\mathrm{Ad}}[\gamma](p)} \circ \underline{\mathcal{A}}(p) + \big(\mathrm{Inv} \circ \Phi_{\mathrm{Ad}}[\gamma]\big)^*\theta_\mathrm{L}(p)\,, \end{aligned}$$

derived directly from its definition (*cp* Eqn. (10.5)) and Prop. 5.1 (in conjunction with Prop. 4.9) in a notation that uses local coordinates: P_G: $\{y^\alpha\}_{\mu\in\overline{1,\dim \mathsf{P}_\mathrm{G}}}$ on a neighbourhood of $p \triangleleft \Phi_{\mathrm{Ad}}[\gamma](p)$ and $\{x^\mu\}_{\mu\in\overline{1,\dim \mathsf{P}_\mathrm{G}}}$ on a neighbourhood of p, we conclude that

$$\begin{aligned} \mathsf{d}\Phi_\lambda\big[\Gamma[\Gamma[\widetilde{r}]]^\lambda_\gamma(\phi)\big] = \big(\mathrm{id}_{\mathsf{T}^*\mathsf{P}_\mathrm{G}} \otimes \mathsf{T}_{\Phi_\lambda[\phi](\cdot)}\Phi_{\mathrm{Ad}}\lambda_{\Phi_{\mathrm{Ad}}[\gamma](\cdot)}\big) \circ \mathsf{d}\Phi_\lambda[\phi] \\ -\big({}^\gamma\underline{\mathcal{A}} - \big(\mathrm{id}_{\mathsf{T}^*\mathsf{P}_\mathrm{G}} \otimes \mathsf{T}_e\mathrm{Ad}_{\Phi_{\mathrm{Ad}}[\gamma](\cdot)}\big) \circ \underline{\mathcal{A}}\big)^A \otimes \mathcal{K}_{t_A}\big(\Phi_{\mathrm{Ad}}\lambda_{\Phi_{\mathrm{Ad}}[\gamma](\cdot)}\big(\Phi_\lambda[\phi](\cdot)\big)\big)\,, \end{aligned}$$

that is, for

$${}^\gamma\phi \coloneqq \Gamma[\Gamma[\widetilde{r}]]^\lambda_\gamma(\phi)\,,$$

we obtain, upon invoking Props. 5.1 i 7.5, the structural identity

$$\begin{aligned} \mathsf{d}\Phi_\lambda[{}^\gamma\phi] + {}^\gamma\underline{\mathcal{A}}^A \otimes \mathcal{K}_{t_A}\big(\Phi_\Lambda[{}^\gamma\phi]\big) \ &= \ \big(\mathrm{id}_{\mathsf{T}^*\mathsf{P}_\mathrm{G}} \otimes \mathsf{T}_{\Phi_\lambda[\phi](\cdot)}\Phi_{\mathrm{Ad}}\lambda_{\Phi_{\mathrm{Ad}}[\gamma](\cdot)}\big) \circ \big(\mathsf{d}\Phi_\lambda[\phi] + \underline{\mathcal{A}}^A \otimes \mathcal{K}_{t_A}\big(\Phi_\lambda[\phi]\big)\big) \\ &\equiv \ \big(\mathrm{id}_{\mathsf{T}^*\mathsf{P}_\mathrm{G}} \otimes \mathsf{T}_2\lambda_{\Phi_{\mathrm{Ad}}[\gamma](\cdot)}\big) \circ \big(\mathsf{d}\Phi_\lambda[\phi] + \underline{\mathcal{A}}^A \otimes \mathcal{K}_{t_A}\big(\Phi_\lambda[\phi]\big)\big)\,. \end{aligned}$$

Given that for any element $g \in \mathrm{G}$ we have the identity

$$\begin{aligned} & \mathsf{d}\Phi_\lambda[\phi](p \triangleleft g) + \underline{\mathcal{A}}^A(p \triangleleft g) \otimes \mathcal{K}_{t_A}\big(\Phi_\lambda[\phi](p \triangleleft g)\big) \\ = \quad & \mathsf{d}\big(g^{-1} \triangleright \Phi_\lambda[\phi](\cdot)\big)(p) + \underline{\mathcal{A}}^B(p) \otimes \big(\mathsf{T}_e \mathrm{Ad}_{g^{-1}}\big)^A_{\ B} \triangleright \mathcal{K}_{t_A}\big(\Phi_\lambda[\phi](p \triangleleft g)\big) \\ = \quad & \big(\mathrm{id}_{\mathsf{T}^*\mathsf{P}_\mathrm{G}} \otimes \mathsf{T}_2\lambda_{g^{-1}}\big) \circ \big(\mathsf{d}\Phi_\lambda[\phi] + \underline{\mathcal{A}}^A \otimes \mathcal{K}_{t_A}\big(\Phi_\lambda[\phi]\big)\big)(p)\,, \end{aligned}$$

we abstract from the above

Definition 1. Adopt the hitherto notation and let G be a Lie group with the Lie algebra $\mathfrak{g}$ in which we choose a basis $\{t_A\}_{A \in \overline{1, \dim \mathrm{G}}}$, and let M be a manifold with a smooth action $\lambda. : \mathrm{G} \times M \longrightarrow M$ of G, inducing the action

$$\mathsf{T}_2\lambda. \ : \ \mathrm{G} \times \mathsf{T}M \longrightarrow \mathsf{T}M \ : \ (g, v) \longmapsto \mathsf{T}_{\pi_{\mathsf{T}M}(v)}\lambda_g(v)\,.$$

Finally, let $(\mathsf{P}_\mathrm{G}, B, \mathrm{G}, \pi_{\mathsf{P}_\mathrm{G}})$ be a principal bundle over the base B, with the principal connection potential $\underline{\mathcal{A}} \equiv \underline{\mathcal{A}}^A \otimes t_A \in \Omega^1(\mathsf{P}_\mathrm{G}) \otimes_\mathbb{R} \mathfrak{g}$. For an arbitrary (global) section $\phi \in \Gamma(\mathsf{P}_\mathrm{G} \times_\lambda M)$, we define its $\Gamma(\mathrm{Ad}\mathsf{P}_\mathrm{G})$**-covariant derivative**

$$\nabla^{\underline{\mathcal{A}}}\phi := \big(\mathrm{id}_{\mathsf{T}^*\mathsf{P}_\mathrm{G}} \otimes \Phi^{-1}_{\mathsf{T}_2\lambda}\big) \circ \big(\mathsf{d}\Phi_\lambda[\phi] + \underline{\mathcal{A}}^A \otimes_\mathbb{R} \mathcal{K}_{t_A}\big(\Phi_\lambda[\phi]\big)\big)\,.$$

We have the all-important (from the physical vantage point)

Proposition 1. Adopt the hitherto notation, let $(\phi, \gamma) \in \Gamma(\mathsf{P}_\mathrm{G} \times_\lambda M) \times \Gamma(\mathrm{Ad}\mathsf{P}_\mathrm{G})$, and denote

$${}^\gamma\underline{\mathcal{A}}(\cdot) := \mathsf{T}_e \mathrm{Ad}_{\Phi_{\mathrm{Ad}}[\gamma](\cdot)} \circ \big(\underline{\mathcal{A}} - \Phi_{\mathrm{Ad}}[\gamma]^*\theta_\mathrm{L}\big)(\cdot)$$

and

$${}^\gamma\phi := \Gamma[\Gamma[\widetilde{r}]]^\lambda_\gamma(\phi)\,.$$

The following idetity holds true:

$$\nabla^{{}^\gamma\underline{\mathcal{A}}}{}^\gamma\phi = \big(\mathrm{id}_{\mathsf{T}^*\mathsf{P}_\mathrm{G}} \otimes \Phi^{-1}_{\mathsf{T}_2\lambda}\big) \circ \big(\mathrm{id}_{\mathsf{T}^*\mathsf{P}_\mathrm{G}} \otimes \mathsf{T}_2\lambda_{\Phi_{\mathrm{Ad}}[\gamma](\cdot)}\big) \circ \big(\mathrm{id}_{\mathsf{T}^*\mathsf{P}_\mathrm{G}} \otimes \Phi_{\mathsf{T}_2\lambda}\big) \circ \nabla^{\underline{\mathcal{A}}}\phi\,.$$

Proof: *Cp*: Eqn. (1). □

The price to be paid for the naturalness of the postulates on which we base our derivation of the above differentiation of sections of an associated bundle is the structural complexity and non-obviousness of the result. Approaching the problem in hand from the 'other' side, we may postulate a natural *form* of such a differentiation instead, which we do in

Definition 2. Adopt the notation of Def. 1. The **Crittenden derivative on the associated bundle** $\mathsf{P}_\mathrm{G} \times_\lambda M$ over B is the mapping

$$\begin{aligned} \nabla^\mathrm{C} \quad : \quad & \Gamma(\mathsf{P}_\mathrm{G} \times_\lambda M) \times \Gamma(\mathsf{T}B) \longrightarrow \Gamma(\mathsf{P}_\mathrm{G} \times_{\mathsf{T}_2\lambda} \mathsf{T}M) \\ : \quad & (\phi, \mathcal{X}) \longmapsto \Phi^{-1}_{\mathsf{T}_2\lambda}\big[\mathsf{T}\big(\Phi_\lambda[\phi]\big) \circ \mathrm{Hor.}(\mathcal{X})\big]\,. \end{aligned}$$

A statement of a simple relation between the two constructs, and, at the same time, an *a posteriori* proof of well-definedness of the latter one is given in

Proposition 2. Adopt the notation of Def. 2. For an arbitrary (global) section $\phi \in \Gamma(\mathsf{P}_\mathrm{G} \times_\lambda M)$ and an arbitrary vector field $\mathcal{X} \in \Gamma(\mathsf{T}B)$, the following identity obtains:

$$\nabla^\mathrm{C}_\mathcal{X}\phi = \nabla^{\underline{\mathcal{A}}}_{\mathrm{Hor.}(\mathcal{X})}\phi\,.$$

Proof: Equality of the two expressions, in which there appears the vector field $\mathcal{X}$ on the base of the principal bundle P_G (and of the associated bundle $\mathsf{P}_\mathrm{G} \times_\lambda M$), follows straightforwardly from the definition $\mathsf{H}\mathsf{P}_\mathrm{G} \equiv \mathrm{Ker}\,\mathcal{A}$ as the latter implies a reduction of the expression for the $\Gamma(\mathrm{Ad}\mathsf{P}_\mathrm{G})$-covariant derivative of the form

$$\nabla^{\underline{\mathcal{A}}}_{\mathrm{Hor.}(\mathcal{X})}\phi \quad = \quad \Phi^{-1}_{\mathsf{T}_2\lambda}\big[\mathrm{Hor.}(\mathcal{X}) \lrcorner \big(\mathsf{d}\Phi_\lambda[\phi] + \underline{\mathcal{A}}^A \otimes_\mathbb{R} \mathcal{K}_{t_A}\big(\Phi_\lambda[\phi]\big)\big)\big]$$

$$= \Phi^{-1}_{\mathsf{T}_2\lambda}\big[\mathrm{Hor.}(\mathcal{X}) \lrcorner\, \mathsf{d}\Phi_\lambda[\phi] + \mathrm{Hor.}(\mathcal{X}) \lrcorner\, \underline{\mathcal{A}}^A \triangleright \mathcal{K}_{t_A}\big(\Phi_\lambda[\phi]\big)\big]$$

$$= \Phi^{-1}_{\mathsf{T}_2\lambda}\big[\mathrm{Hor.}(\mathcal{X}) \lrcorner\, \mathsf{d}\Phi_\lambda[\phi] + \mathcal{K}_{\mathrm{Hor.}(\mathcal{X})\lrcorner\underline{\mathcal{A}}^A \triangleright t_A}\big(\Phi_\lambda[\phi]\big)\big]$$

$$\equiv \Phi^{-1}_{\mathsf{T}_2\lambda}\big[\mathrm{Hor.}(\mathcal{X}) \lrcorner\, \mathsf{d}\Phi_\lambda[\phi] + \mathcal{K}_{\mathrm{pr}_2\circ\mathcal{A}(\mathrm{Hor.}(\mathcal{X}))}\big(\Phi_\lambda[\phi]\big)\big]$$

$$= \Phi^{-1}_{\mathsf{T}_2\lambda}\big[\mathrm{Hor.}(\mathcal{X}) \lrcorner\, \mathsf{d}\Phi_\lambda[\phi]\big] \equiv \nabla^{\mathrm{C}}_{\mathcal{X}}\phi\,.$$

□

The constructions and conclusions from our hitherto considerations concerning differentiation of sections of associated bundles generalise without any significant alterations to the class of product associated bundles $\mathsf{P}^\Lambda_{\mathrm{G}}E$ described in Def. 8.3. Thus, we have

Definition 3. Adopt the hitherto notation and let G be a Lie group with the Lie algebra $\mathfrak{g}$ in which a basis $\{t_A\}_{A\in\overline{1,\dim\mathrm{G}}}$ has been chosen, and let $\mathcal{E} \equiv (E, B, F, \pi_E)$ be a fibre bundle over the base B of dimension $n \in \mathbb{N}^\times$, with local charts $\kappa_i \,:\, \mathcal{O}_i \longrightarrow \mathcal{U}_i \subset \mathbb{R}^{\times n}$, $i \in I$ associated with the cover $\{\mathcal{O}_i\}_{i\in I}$ over which $\mathcal{E}$ trivialises, with the corresponding local trivialisations $\tau_i \,:\, \pi_E^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times F$. Let also $\Lambda. \,:\, \mathrm{G}\times E \longrightarrow E$ be a smooth action of the group G described in Def. 8.3 and inducing an action

$$\mathsf{T}_2\Lambda. \,:\, \mathrm{G}\times \mathsf{T}E \longrightarrow \mathsf{T}E \,:\, (g, v) \longmapsto \mathsf{T}_{\pi_{\mathsf{T}E}(v)}\Lambda_g(v)\,,$$

giving rise to the family $\{(\mathsf{T}_2\Lambda_g, \mathrm{id}_B)\}_{g\in\mathrm{G}}$ of automorphisms of the bundle $(\mathsf{T}E, B, \mathbb{R}^{\times n}\times\mathsf{T}F, \pi_E\circ\pi_{\mathsf{T}E})$ with local trivialisations $(\kappa_i^{-1}\times\mathrm{id}_{\mathbb{R}^{\times n}\times\mathsf{T}F})\circ(\mathsf{T}\kappa_i\times\mathrm{id}_{\mathsf{T}F})\circ\mathsf{T}\tau_i$. Let, furthermore, $(\mathsf{P}_{\mathrm{G}}, B, \mathrm{G}, \pi_{\mathsf{P}_{\mathrm{G}}})$ be a principal bundle over the base B, with a principal connection potential $\underline{\mathcal{A}} \equiv \underline{\mathcal{A}}^A \otimes_{\mathbb{R}} t_A \in \Omega^1(\mathsf{P}_{\mathrm{G}})\otimes_{\mathbb{R}}\mathfrak{g}$. For an arbitrary (global) section $\phi \in \Gamma(\mathsf{P}^\Lambda_{\mathrm{G}}E)$, we define its $\Gamma(\mathrm{Ad}\mathsf{P}_{\mathrm{G}})$**-covariant derivative**

$$\nabla^{\times\underline{\mathcal{A}}}\phi \coloneqq \big(\mathrm{id}_{\mathsf{T}^*\mathsf{P}_{\mathrm{G}}}\otimes\Phi^{\times\,-1}_{\mathsf{T}_2\Lambda}\big)\circ\big(\mathsf{d}\Phi^\times_\Lambda[\phi] + \underline{\mathcal{A}}^A \otimes_{C^k(\mathsf{P}_{\mathrm{G}},\mathbb{R})} \mathcal{K}_{t_A}\big(\Phi^\times_\Lambda[\phi]\big)\big)\,.$$

The **Crittenden derivative on the product associated bundle** $\mathsf{P}^\Lambda_{\mathrm{G}}E$ over B is the mapping

$$\nabla^{\times\,\mathrm{C}} \,:\, \Gamma(\mathsf{P}^\Lambda_{\mathrm{G}}E)\times\Gamma(\mathsf{T}B) \longrightarrow \Gamma(\mathsf{P}^{\mathsf{T}_2\Lambda}_{\mathrm{G}}\mathsf{T}E)$$

$$: (\phi, \mathcal{X}) \longmapsto \Phi^{\times\,-1}_{\mathsf{T}_2\Lambda}\big[\mathsf{T}\big(\Phi^\times_\Lambda[\phi]\big)\circ\mathrm{Hor.}(\mathcal{X})\big]\,,$$

in whose definition

$$\mathsf{P}^{\mathsf{T}_2\Lambda}_{\mathrm{G}}\mathsf{T}E \equiv \big(\mathsf{P}_{\mathrm{G}}\,{}_{\pi_{\mathsf{P}_{\mathrm{G}}}}\!\times_{\pi_E\circ\pi_{\mathsf{T}E}}\mathsf{T}E\big)/\mathrm{G}$$

is the product associated bundle obtained – in the procedure determined in Def. 8.3 – from the fibred product $\mathsf{P}_{\mathrm{G}}\times_B\mathsf{T}E \equiv \mathsf{P}_{\mathrm{G}}\,{}_{\pi_{\mathsf{P}_{\mathrm{G}}}}\!\times_{\pi_E\circ\pi_{\mathsf{T}E}}\mathsf{T}E$.

Remark 1. The proof of G-equivariance of the expression $\mathsf{d}\Phi^\times_\Lambda[\phi] + \underline{\mathcal{A}}^A\otimes_{\mathbb{R}}\mathcal{K}_{t_A}(\Phi^\times_\Lambda[\phi])$ develops identically as in the case of the $\Gamma(\mathrm{Ad}\mathsf{P}_{\mathrm{G}})$-equivariant derivative on the associated bundle, *cp* the discussion preceding Def. 1, and ensures that the derivative $\nabla^{\times\underline{\mathcal{A}}}\phi$ is well-defined. A similar equivariance of the expression $\mathsf{T}\big(\Phi^\times_\Lambda[\phi]\big)\circ\mathrm{Hor.}(\mathcal{X})$ is granted by zcompatibility of the connection on P_{G} with the action of the structure group together with the G-equivariant character of the map $\Phi^\times_\Lambda[\phi]$. Indeed, for any $g\in\mathrm{G}$, we obtain

$$\mathsf{T}\big(\Phi^\times_\Lambda[\phi]\big)\circ\mathrm{Hor.}(\mathcal{X})\circ r_g = \mathsf{T}\big(\Phi^\times_\Lambda[\phi]\big)\circ\mathsf{T}r_g\circ\mathrm{Hor.}(\mathcal{X}) = \mathsf{T}\big(\Phi^\times_\Lambda[\phi]\circ r_g\big)\circ\mathrm{Hor.}(\mathcal{X})$$

$$= \mathsf{T}\big(\Lambda_{g^{-1}}\circ\Phi^\times_\Lambda[\phi]\big)\circ\mathrm{Hor.}(\mathcal{X}) = \mathsf{T}\Lambda_{g^{-1}}\circ\mathsf{T}\big(\Phi^\times_\Lambda[\phi]\big)\circ\mathrm{Hor.}(\mathcal{X})\,.$$

In the light of commutativity of the diagrams

$$\begin{tikzcd} \mathsf{T}\mathsf{P}_{\mathrm{G}} \arrow[r, "\mathsf{T}(\Phi^\times_\Lambda[\phi])"] \arrow[d, "\pi_{\mathsf{T}\mathsf{P}_{\mathrm{G}}}"'] & \mathsf{T}E \arrow[d, "\pi_{\mathsf{T}E}"] \\ \mathsf{P}_{\mathrm{G}} \arrow[r, "\Phi^\times_\Lambda[\phi]"'] & E \end{tikzcd}\ , \qquad \begin{tikzcd} \mathsf{P}_{\mathrm{G}} \arrow[r, "\Phi^\times_\Lambda[\phi]"] \arrow[d, "\pi_{\mathsf{P}_{\mathrm{G}}}"'] & E \arrow[d, "\pi_E"] \\ B \arrow[r, equal, "\mathrm{id}_B"'] & B \end{tikzcd}$$

the thus determined map between total spaces of the bundles P_{G} and $\mathsf{T}E$ over the common base B preserves the fibres,

$$\begin{aligned} (\pi_E \circ \pi_{\mathsf{T}E}) \circ \mathsf{T}\big(\Phi^\times_\Lambda[\phi]\big) \circ \mathrm{Hor.}(\mathcal{X}) &= \pi_E \circ \Phi^\times_\Lambda[\phi] \circ \pi_{\mathsf{T}\mathsf{P}_{\mathrm{G}}} \circ \mathrm{Hor.}(\mathcal{X}) \\ &= \pi_{\mathsf{P}_{\mathrm{G}}} \circ \pi_{\mathsf{T}\mathsf{P}_{\mathrm{G}}} \circ \mathrm{Hor.}(\mathcal{X}) = \pi_{\mathsf{P}_{\mathrm{G}}}\,, \end{aligned}$$

the last equality following from the fact that $\mathrm{Hor.}(\mathcal{X})$ is a section of the horizontal subbundle $\mathsf{H}\mathsf{P}_{\mathrm{G}} \subset \mathsf{T}\mathsf{P}_{\mathrm{G}}$. The map is, therefore, a G-equivariant fibre-bundle morphism. Finally, $\Gamma(\mathrm{Ad}\mathsf{P}_{\mathrm{G}})$-covariance of the former of the derivatives defined above is verified analogously as in the former setting.

As previously, we establish

Proposition 3. Adopt the notation of Def. 3. For an arbitrary (global) section $\phi \in \Gamma(\mathsf{P}^\Lambda_{\mathrm{G}}E)$ and an arbitrary vector field $\mathcal{X} \in \Gamma(\mathsf{T}B)$, the following idetity holds true:

$$\nabla^{\times\mathrm{C}}_{\mathcal{X}}\phi = \nabla^{\times\underline{\mathcal{A}}}_{\mathrm{Hor.}(\mathcal{X})}\phi\,.$$

Proof: Analogous to the one for Prop. 2. □

Thus equipped, we may, now, pass to the discussion of physical applications of the connection induced on an associated bundle.

Cechowanie symetrii globalnych – aspekt dynamiczny w schemacie minimalnym

As was argued amply and in detail in Lecture 8, an associated bundle provides us with a natural model of the geometry of a covariant configuration bundle (a field bundle) of a field theory with a global-symmetry group G gauged. This raises an obvious question as to the possibilty of employing the Crittenden connection on that associated bundle, induced from the principal connection on the underlying principal bundle P_{G}, in a transcription of the original action functional into a form manifestly invariant under gauge transformations from $\Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}})$. Thus posed, the question turns out too general to be answered in a universal manner – an effective scheme of gauging of a global symmetry in the dynamical sector depends on the structure of the theory, a fact convincingly demonstrated by the results obtained by the Author *et al.* in Refs. [GSW10, Sus12, GSW13]. Below, we examine in considerable detail a situation in which that answer is both simple and natural, and the gauging procedure admits algorithmisation.

The situation alluded to above is one in which the field bundle is trivial, $\mathcal{F} \equiv \Sigma \times F \xrightarrow{\mathrm{pr}_1} \Sigma$, and the action functional is expressed in terms of covariant tensors on its fibre,

$$\mathscr{T} \ :\ \Gamma\big(\mathsf{T}F^{\otimes_{F,\mathbb{R}} N}\big) \longrightarrow C^\infty(F,\mathbb{R})\,, \tag{2}$$

evaluated on tangents of the theory's fields, *i.e.*,

$$\mathcal{L}_{\mathscr{T}} \ :\ \Gamma(\mathcal{F}) \equiv C^\infty(\Sigma, F) \ni \varphi \longmapsto \mathscr{T}_{(2)}\big(\mathsf{T}\varphi(\cdot) \otimes \mathsf{T}\varphi(\cdot) \otimes \cdots \otimes \mathsf{T}\varphi(\cdot)\big)\,, \tag{3}$$

and subsequently integrated over the spacetime Σ with a suitable measure, *e.g.*,

- the standard „kinetic" term of the Klein–Gordon type:

  $$C^\infty(F,\mathbb{R}) \ni \varphi \longmapsto \int_\Sigma \mathscr{G}_{(2)}\big(\varphi(\cdot)\big)\big(\mathsf{T}\varphi \otimes (\star \otimes \mathrm{id}_{\mathsf{T}F})\mathsf{T}\varphi\big) + 2\pi\mathbb{Z} \in \mathbb{R}/2\pi\mathbb{Z}$$

  in which $\mathsf{T}\varphi$ jis naturally interpreted as a vector from the space $\mathsf{T}^*\Sigma \otimes \mathsf{T}\varphi(\Sigma)$, $\mathscr{G}$ is a metric structure on $\mathsf{T}F$, and $\star$ is the Hodge operator on $\Omega^\bullet(\Sigma)$,

- a current term

  $$C^\infty(F,\mathbb{R}) \ni \varphi \longmapsto \int_\Sigma \mathscr{A}_{(2)}\big(\varphi(\cdot)\big)\big(\mathsf{T}\varphi(\cdot)\wedge\mathsf{T}\varphi(\cdot)\wedge\cdots\wedge\mathsf{T}\varphi(\cdot)\big) + 2\pi\mathbb{Z} \in \mathbb{R}/2\pi\mathbb{Z}$$

  in which $\mathscr{A} \in \Omega^d(\mathcal{F})$, $d = \dim\Sigma$ is a (global) potential of an external charge field coupling to the charge current defined by the evolution of a field configuration φ in the fibre F

(in all the above cases, the index (2) indicates that the corresponding tensor field on F is to be evaluated on the second tensor factor of its arguments), in which we assume that tensors $\mathscr{T}$ appearing in the definition of the action functional are invariant under the (natural) action

$$\lambda_\cdot \;:\; \mathrm{G}\times F \longrightarrow F$$

of some distinguished subgroup $\mathrm{G}\subset \mathrm{Aut}\,F$ of the group of automorphisms $\mathrm{Aut}\,F \equiv \mathrm{Diff}\,(F,F)$ of the fibre F, *i.e.*,

$$\forall_{g\in\mathrm{G}} \;:\; \mathscr{T}\circ\big(\mathsf{T}_2\lambda_g\otimes\mathsf{T}_2\lambda_g\otimes\cdots\otimes\mathsf{T}_2\lambda_g\big) = \mathscr{T}\,,$$

where

$$\mathsf{T}_2\lambda_\cdot \;:\; \mathrm{G}\times\mathsf{T}F \longrightarrow \mathsf{T}F \;:\; (g,v)\longmapsto \mathsf{T}_{\pi_{\mathsf{T}F}(v)}\lambda_g(v)$$

is a tangent action induced according to the familiar scheme. It is perfectly clear that in the curcumstances described, in which computing the action integral requires that each copy of the tangent mapping $\mathsf{T}\varphi$ be evaluated on an element of a local basis $\{\partial_\mu\}_{\mu\in\overline{0,d-1}}$ of the space of sections $\Gamma_{\mathrm{loc}}(\mathsf{T}\Sigma)$, a 'minimal' reformulation of the original model should boil down to replacing ordinary directional derivatives $\mathsf{T}\varphi(\partial_\mu)$ of fields $\varphi\in C^\infty(\Sigma,F)$ with covariant derivatives $\nabla^{\mathrm{C}}_{\partial_\mu}\phi$ of global sections $\phi\in\Gamma(\mathsf{P}_{\mathrm{G}}\times_\lambda F)$ of a bundle associated with some gauge bundle P_{G} through the action $\lambda_\cdot$. The only problem that remains at this stage is... the codomain of the Crittenden derivative – the result of the derivation $\nabla^{\mathrm{C}}_{\partial_\mu}\phi$ is a section of the associated bundle $\mathsf{P}_{\mathrm{G}}\times_{\mathsf{T}_2\lambda}\mathsf{T}F$, not that of the bundle $\mathsf{T}F$, as we should certainly prefer. Luckily, that, too, can be resolved as the map $\Phi_{\mathsf{T}_2\lambda}$ turns the said section into a G-equivariant map $\mathsf{P}_{\mathrm{G}}\longrightarrow\mathsf{T}F$, which we may eventually precompose with an arbitrary reference section σ_* (along the lines of Def. 8.5) to obtain the deisred result. *Global* smoothness of the map $\Sigma\longrightarrow\mathsf{T}F$ thus constructed requires, of course, existence of a *global* reference section $\sigma_*\in\Gamma(\mathsf{P}_{\mathrm{G}})$, which – in the light of Prop. 6.5 – is tantamount to triviality of the gauge bundle P_{G}, which we shall assume henceforth in our physical considerations.

The point of departure is

Definition 4. Adopt the hitherto notation and let $(\mathsf{P}_{\mathrm{G}},\Sigma,\mathrm{G},\mathrm{pr}_1)$ be a principal bundle over the base Σ, inducing the Crittenden connection ∇^{C} on the bundle $\mathsf{P}_{\mathrm{G}}\times_\lambda F$ associated with it through an action $\lambda_\cdot \;:\; \mathrm{G}\times F\longrightarrow F$ on its typical fibre F. A **local presentation of the covariant derivative of a field** $\phi\in\Gamma(\mathsf{P}_{\mathrm{G}}\times_\lambda F)$ **in the gauge** $\sigma_* \;:\; \mathcal{O}\longrightarrow\mathsf{P}_{\mathrm{G}}$, $\mathcal{O}\subset\Sigma$ is the map

$$D^{\sigma_*}\phi \equiv \Phi_{\mathsf{T}_2\lambda}[\nabla^{\mathrm{C}}_\cdot\phi]_{\sigma_*} \;:\; \Gamma(\mathsf{T}\Sigma)\longrightarrow C^\infty(\mathcal{O},\mathsf{T}F) \;:\; \mathcal{X}\longmapsto \Phi_{\mathsf{T}_2\lambda}[\nabla^{\mathrm{C}}_{\mathcal{X}}\phi]\circ\sigma_*\,.$$

The physical meaning of the object introduced above is summarised in

Proposition 4. Adopt the notation of Def. 4, under the extra assumption of triviality of the bundle P_{G}, which ensures existence of a *global* section σ_*, and let $\mathscr{T}$ be a ($\mathsf{T}_2\lambda$-invariant) tensor as in Eqn. (2). The mapping

$$\mathcal{L}^{\mathrm{min}}_{\mathscr{T}} \;:\; \Gamma(\mathsf{P}_{\mathrm{G}}\times_\lambda F)\ni\phi\longmapsto \mathscr{T}_{(2)}\big(D^{\sigma_*}\phi(\cdot)\otimes D^{\sigma_*}\phi(\cdot)\otimes\cdots\otimes D^{\sigma_*}\phi(\cdot)\big)\,,$$

obtained from that defined in Eqn. (3) in terms of $\mathscr{T}$ through the substitution

$$\mathsf{T}\varphi\longmapsto D^{\sigma_*}\phi\,, \tag{4}$$

is **invariant under gauge transformations** in the sense expressed by the identity

$$\mathcal{L}^{\mathrm{min}}_{\mathscr{T}}({}^\gamma\phi) = \mathcal{L}^{\mathrm{min}}_{\mathscr{T}}(\phi)\,.$$

Proof: Obvious. □

We have succeeded – in the aforementioned special circumstances – in attaining the primary physical goal which was to reformulate the original field theory in such a manner that the ensuing description of dynamics of a field of the same type is not only covariant with rspect to the action of the gauge group but also *structurally* identical with the original one, *i.e.*, that it is determined by the same $\mathsf{T}_2\lambda$-invariant tensors on the space F of internal degrees of freedom of the field, albeit in terms of adapted derivations. This does not exhaust the physical meaning of the substitution (4), though, which we are about to discover below.

To this end, we shall now reverse the reasoning presented in the proof of Prop. 2 that has taken us from the $\Gamma(\mathrm{Ad}\mathsf{P}_{\mathrm{G}})$-covariant derivative of a section of the associated bundle along (the horizontal lift of) a vector field on the base of that bundle to the Crittenden derivative of that section. Thus, consider a vector field $\widetilde{\mathcal{X}} \in \Gamma(\mathsf{TP}_{\mathrm{G}})$ with the horizontal component $\mathrm{Hor.}(\mathcal{X})$ determined by some field $\mathcal{X} \in \Gamma(\mathsf{T}\Sigma)$ on the base and with the vertical component $\mathcal{V} \in \Gamma(\mathsf{VP}_{\mathrm{G}})$,

$$\widetilde{\mathcal{X}} := \mathrm{Hor.}(\mathcal{X}) + \mathcal{V} \in \Gamma(\mathsf{HP}_{\mathrm{G}} \oplus_{\mathsf{P}_{\mathrm{G}},\mathbb{R}} \mathsf{VP}_{\mathrm{G}}) \equiv \Gamma(\mathsf{TP}_{\mathrm{G}})\,.$$

Note also that in the light of Prop. 10.1, we have the decomposition

$$\mathcal{V} = \mathcal{V}^A \triangleright \mathcal{K}^{(\mathsf{P}_{\mathrm{G}})}_{t_A}\,,$$

and so we calculate – with reference to Def. 2, Def. 5.2 and the remark on p. 6 of Lecture 4, as well as the defining property of the map $\Phi_\lambda[\phi]$ (which ensures that the image of a fundamental field along the tangent of $\Phi_\lambda[\phi]$ is the fundamental field associated with the same element of the Lie algebra $\mathfrak{g}$, the replacement of a right action with a left action resulting in a sign flip) –

$$\begin{aligned}
\Phi_{\mathsf{T}_2\lambda}[\nabla^{\mathrm{C}}_{\mathcal{X}}\phi] &\equiv \mathsf{T}\big(\Phi_\lambda[\phi]\big)\circ \mathrm{Hor.}(\mathcal{X}) = \mathsf{T}\big(\Phi_\lambda[\phi]\big)\circ\big(\widetilde{\mathcal{X}} - \mathcal{V}\big) = \mathsf{T}\big(\Phi_\lambda[\phi]\big)\circ\widetilde{\mathcal{X}} + \mathcal{V}^A \triangleright \mathcal{K}^{(F)}_{t_A}\big(\Phi_\lambda[\phi]\big)\\
&\equiv \mathsf{T}\big(\Phi_\lambda[\phi]\big)\circ\widetilde{\mathcal{X}} + \mathcal{V}^A \triangleright \mathsf{T}_{(e,\Phi_\lambda[\phi])}\lambda\big(t_A, \mathbf{0}_{\mathsf{T}_{\Phi_\lambda[\phi]}F}\big)\\
&= \mathsf{T}\big(\Phi_\lambda[\phi]\big)\circ\widetilde{\mathcal{X}} + \mathcal{V}^A \triangleright \mathsf{T}_{(e,\Phi_\lambda[\phi])}\lambda\big(\mathcal{K}^{(\mathsf{P}_{\mathrm{G}})}_{t_A} \lrcorner \underline{\mathcal{A}}, \mathbf{0}_{\mathsf{T}_{\Phi_\lambda[\phi]}F}\big)\\
&= \mathsf{T}\big(\Phi_\lambda[\phi]\big)\circ\widetilde{\mathcal{X}} + \mathsf{T}_{(e,\Phi_\lambda[\phi])}\lambda\big(\mathcal{V} \lrcorner \underline{\mathcal{A}}, \mathbf{0}_{\mathsf{T}_{\Phi_\lambda[\phi]}F}\big)\\
&\equiv \mathsf{T}\big(\Phi_\lambda[\phi]\big)\circ\widetilde{\mathcal{X}} + \mathsf{T}_{(e,\Phi_\lambda[\phi])}\lambda\big(\widetilde{\mathcal{X}} \lrcorner \underline{\mathcal{A}}, \mathbf{0}_{\mathsf{T}_{\Phi_\lambda[\phi]}F}\big)\\
&\equiv \widetilde{\mathcal{X}} \lrcorner \Big(\mathsf{d}\Phi_\lambda[\phi] + \big(\mathrm{id}_{\mathsf{T}^*\mathsf{P}_{\mathrm{G}}} \otimes \mathsf{T}_{(e,\Phi_\lambda[\phi])}\lambda\big)\circ\big(\underline{\mathcal{A}}, 0\big)\Big)\,.
\end{aligned}$$

Restricting our discussion, once more, to the case of a trivial gauge bundle $\mathsf{P}_{\mathrm{G}} = \Sigma \times \mathrm{G}$ of prime interest to us (resp. to some open subset $\mathcal{O} \subset B$ over which the gauge bundle trivialises), we may choose $\widetilde{\mathcal{X}}$ given by the pushforward of a field $\mathcal{X}$ along an arbitrary (local) section $\sigma_* \in \Gamma_{(\mathrm{loc})}(\mathsf{P}_{\mathrm{G}})$ (over $\mathcal{O}$) – indeed, the horizontal component of the field

$$\widetilde{\mathcal{X}} \equiv \mathsf{T}\sigma_*(\mathcal{X})$$

is given by $\mathrm{Hor.}(\mathcal{X})$, which follows by uniqueness of the decomposition of vector fields on P_{G} into their horizontal and vertical components and from the identity

$$\mathsf{T}\pi_{\mathsf{P}_{\mathrm{G}}}(\widetilde{\mathcal{X}}) = \mathsf{T}(\pi_{\mathsf{P}_{\mathrm{G}}}\circ\sigma_*)(\mathcal{X}) = \mathsf{T}\mathrm{id}_\Sigma(\mathcal{X}) = \mathrm{id}_{\mathsf{T}\Sigma}(\mathcal{X}) = \mathcal{X}\,.$$

Upon evaluating the former expression on this vector field, and subsequently computing the end result for the section σ_*, we arrive at

$$\begin{aligned}
D^{\sigma_*}_{\mathcal{X}}\phi &\equiv \Phi_{\mathsf{T}_2\lambda}[\nabla^{\mathrm{C}}_{\mathcal{X}}\phi]\circ\sigma_* = \mathsf{T}\sigma_*(\mathcal{X}) \lrcorner \Big(\mathsf{d}\Phi_\lambda[\phi] + \big(\mathrm{id}_{\mathsf{T}^*\mathsf{P}_{\mathrm{G}}} \otimes \mathsf{T}_{(e,\Phi_\lambda[\phi])}\lambda\big)\circ\big(\underline{\mathcal{A}}, 0\big)\Big)\circ\sigma_*\\
&= \mathcal{X} \lrcorner \big(\sigma_*^* \otimes \mathrm{id}_{\mathsf{T}F}\big)\Big(\mathsf{d}\Phi_\lambda[\phi] + \big(\mathrm{id}_{\mathsf{T}^*\mathsf{P}_{\mathrm{G}}} \otimes \mathsf{T}_{(e,\Phi_\lambda[\phi])}\lambda\big)\circ\big(\underline{\mathcal{A}}, 0\big)\Big)\\
&= \mathcal{X} \lrcorner \Big(\mathsf{d}\big(\Phi_\lambda[\phi]\circ\sigma_*\big) + \big(\sigma_*^* \otimes \mathsf{T}_{(e,\Phi_\lambda[\phi]\circ\sigma_*)}\lambda\big)\circ\big(\underline{\mathcal{A}}, 0\big)\Big)\\
&\equiv \mathcal{X} \lrcorner \Big(\mathsf{d}\phi_{\sigma_*} + \big(\sigma_*^* \otimes \mathsf{T}_{(e,\phi_{\sigma_*})}\lambda\big)\circ\big(\underline{\mathcal{A}}, 0\big)\Big)\,,
\end{aligned}$$

that is ultimately the local presentation of the covariant derivative in the gauge σ_* takes the form

$$D^{\sigma_*}\phi = \mathsf{d}\phi_{\sigma_*} + \big(\sigma_*^* \otimes \mathsf{T}_{(e,\phi_{\sigma_*})}\lambda\big)\circ\big(\underline{\mathcal{A}}, 0\big)\,.$$

Denote

$$\underline{\mathcal{A}} =: \underline{\mathcal{A}}^A \otimes t_A$$

and

$$\mathrm{A}^A_{\sigma_*} := \sigma_*^* \underline{\mathcal{A}}^A \,,$$

to cast the above formula in the form

$$D^{\sigma_*}\phi = \mathsf{d}\phi_{\sigma_*} + \mathrm{A}^A_{\sigma_*} \otimes \mathcal{K}^{(F)}_{t_A}\big(\phi_{\sigma_*}\big)$$

in which it appears customarily (if at all!) in the physics literature. Our hitherto considerations legitimise

Definition 5. Adopt the notation of Prop. 4 and let σ_* be a (local) section of the principal bundle $\big(\mathsf{P}_{\mathrm{G}}, \Sigma, \mathrm{G}, \pi_{\mathsf{P}_{\mathrm{G}}}\big)$ with the principal connection potential $\underline{\mathcal{A}}$. The 1-form with values in the Lie algebra $\mathfrak{g}$ of the structure group G given by the formula

$$\mathrm{A}_{\sigma_*} := \sigma_*^* \underline{\mathcal{A}}$$

is called the (**local**) **gauge field in the gauge** σ_*. Under the substitution of Eqn. (4) that realises the gauging of a global symmetry modelled on the Lie group G in the dynamical sector of the field theory of type F, the gauge field **couples** to matter fields $\phi \in \Gamma(\mathsf{P}_{\mathrm{G}} \times_\lambda F)$ (*i.a.*, through the dependence of $\mathcal{K}^{(F)}_{t_A}$ on ϕ_{σ_*}) in the local presentation of the covariant derivative

$$D^{\sigma_*}\phi = \mathsf{d}\phi_{\sigma_*} + \mathrm{A}^A_{\sigma_*} \otimes \mathcal{K}^{(F)}_{t_A}\big(\phi_{\sigma_*}\big)\,,$$

a fact customarily reflected in the term **minimal coupling** used to refer to the transcription

$$\big(\Gamma(\Sigma \times F), \mathsf{T}\varphi\big) \longmapsto \big(\Gamma(\mathsf{P}_{\mathrm{G}} \times_\lambda F), D^{\sigma_*}\phi\big)\,.$$

The appearance of a new physical field – the (local) gauge field – in a field theory with a global symmetry gauged (in the rôle of a nondynamical background[1]) prompts us to enquire as to the possible dynamics of that field. As usual, answers are provided by the study of invariants of the natural action of the gauge group $\Gamma(\mathrm{Ad}\mathsf{P}_{\mathrm{G}}) \cong \mathrm{Aut}_{\mathbf{GrpBun}_{\mathrm{G}}(B)}(\mathsf{P}_{\mathrm{G}}\,|\,B)$ (Prop. 7.6) on the connection form and objects derived from it. The one most frequently encountered in physical modelling is given in

Definition 6. Adopt the notation of Def. 10.5 and let $\big(\mathsf{P}_{\mathrm{G}}, \Sigma, \mathrm{G}, \pi_{\mathsf{P}_{\mathrm{G}}}\big)$ be a principal bundle with a principal connection with the connection form $\mathcal{A} \in \Omega^1(\mathsf{P}_{\mathrm{G}}) \otimes_{\mathbb{R}} \mathfrak{g}$, and with the trivialising cover $\{\mathcal{O}_i\}_{i \in I}$. Denote as

$$\|\mathrm{F}\|_{\Omega^2(\Sigma),\mathfrak{g}} \in \Omega^d(\Sigma)\,, \qquad d \equiv \dim \Sigma$$

the d-form with local presentations

$$\|\mathrm{F}\|_{\Omega^2(\Sigma),\mathfrak{g}} \restriction_{\mathcal{O}_i} = \big(\mathrm{id}_{\Omega^d(\Sigma)} \otimes \mathrm{tr}_{\mathfrak{g}}\big)\big(\mathrm{F}_i \wedge \star_\Sigma \mathrm{F}_i\big)\,, \qquad i \in I\,,$$

written in terms of the Hodge isomorphism $\star_\Sigma$ on $\Omega^\bullet(\Sigma)$. The **Yang–Mills functional** is the functional

$$\mathrm{A}_i \longmapsto -\tfrac{1}{4} \int_\Sigma \|\mathrm{F}\|_{\Omega^2(\Sigma),\mathfrak{g}} \equiv S_{\mathrm{YM}}[\mathrm{A}_i]\,.$$

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[1]Practically speaking, the nondynamical nature of the gauge field is reflected by the purely algebraic (*i.e.*, non-differential) character of its appearances in the action functional.