CLASSICAL FIELD THEORY IN THE TIME OF COVID-19
10. LECTURE BATCH

Principal connections on principal bundles

Our hitherto cosiderations, ultimately motivated by physics, have led us to study connections on fibre bundles as – in fact – a constructive answer to the question of existence of a natural/useful definition of differentiation of their sections. In the context of the universal gauge principle discussed earlier, it is a study of a connection on a principal bundle that becomes of utmost relevance, and it seems only natural to consider conditions under which this connection is compatible with the action of the structural group on the fibre of the bundle. This is what we turn to next.

Definition 1. A connection on fibres of a principal bundle $(\mathsf{P}_{\rm G}, B, {\rm G}, \pi_{\mathsf{P}_{\rm G}})$ is called **compatible with the action of the structural group** if the diffeomorphisms

$$\mathsf{P}^\gamma_{t_1,t_2} \;:\; \mathsf{P}_{{\rm G}\,\gamma(t_1)} \xrightarrow{\cong} \mathsf{P}_{{\rm G}\,\gamma(t_2)}\,, \qquad t_1, t_2 \in]-\varepsilon, \varepsilon[$$

satisfy the conditions

$$\forall_{g\in{\rm G}} \;:\; \mathsf{P}^\gamma_{t_1,t_2} \circ r_g \restriction_{\mathsf{P}_{{\rm G}\,\gamma(t_1)}} = r_g \circ \mathsf{P}^\gamma_{t_1,t_2}\,, \tag{1}$$

in which case

$$\nabla_{\mathcal{V}}(r_g \circ \sigma)(x) = \mathsf{T}_{\sigma(x)} r_g\big(\nabla_{\mathcal{V}}\sigma(x)\big)\,. \tag{2}$$

Such a connection is also termed a **principal connection on fibres of the bundle** $\mathsf{P}_{\rm G}$.

An equally natural notion of structural compatibility is provided by

Definition 2. An Ehresmann connection on a principal bundle $(\mathsf{P}_{\rm G}, B, {\rm G}, \pi_{\mathsf{P}_{\rm G}})$ is called **compatible with the action of the structural group**, if the maps $r_g,\ g \in {\rm G}$ satisfy the condition

$$\forall_{p\in\mathsf{P}_{\rm G}} \;:\; \mathsf{H}_{r_g(p)}\mathsf{P}_{\rm G} = \mathsf{T}_p r_g(\mathsf{H}_p\mathsf{P}_{\rm G})\,. \tag{3}$$

Such a connection is also termed a **principal Ehresmann connection** on $\mathsf{P}_{\rm G}$.

We also have

Definition 3. A **principal connection form** on a principal bundle $(\mathsf{P}_{\rm G}, B, {\rm G}, \pi_{\mathsf{P}_{\rm G}})$ is a morphism of real vector bundles

$$(\mathcal{A}, {\rm id}_B) \;:\; \mathsf{T}\mathsf{P}_{\rm G} \longrightarrow \mathsf{P}_{\rm G} \times \mathfrak{g}$$

with the properties:

$$\mathcal{A} \circ \widetilde{\rm Vert.} = {\rm id}_{\mathsf{P}_{\rm G}\times\mathfrak{g}}\,. \tag{4}$$

and

$$\forall_{g\in{\rm G}} \;:\; \mathcal{A} \circ \mathsf{T} r_g = (r_g \times \mathsf{T}_e{\rm Ad}_{g^{-1}}) \circ \mathcal{A}\,. \tag{5}$$

The latter naturally induces a **principal connection potential**

$$\underline{\mathcal{A}} := {\rm pr}_2 \circ \mathcal{A} \in \Omega^1(\mathsf{P}_{\rm G}) \otimes_{\mathbb{R}} \mathfrak{g}\,.$$

A proof of equivalence of the above natural definitions of compatibility calls for the ancillary

Proposition 1. Adopt the hitherto notation and let $\mathfrak{g}$ be the Lie algebra of the structural group ${\rm G}$ of a principal bundle $(\mathsf{P}_{\rm G}, B, {\rm G}, \pi_{\mathsf{P}_{\rm G}})$. The vertical subbundle $\mathsf{V}\mathsf{P}_{\rm G}$ of the tangent bundle $\mathsf{T}\mathsf{P}_{\rm G}$ over the total space $\mathsf{P}_{\rm G}$ is trivial in the sense of Example 6.1 and there exists a canonical isomorphism of vector bundles (over $\mathbb{R}$)

$$(\mathsf{V}\mathsf{P}_{\rm G}, \mathsf{P}_{\rm G}, \mathbb{K}^{\times \dim {\rm G}}, \pi_{\mathsf{T}\mathsf{P}_{\rm G}}\restriction_{\mathsf{V}\mathsf{P}_{\rm G}}) \cong (\mathsf{P}_{\rm G} \times \mathfrak{g}, \mathsf{P}_{\rm G}, \mathbb{K}^{\times \dim {\rm G}}, {\rm pr}_1)\,.$$

Proof: Consider the map (manifestly $\mathbb{R}$-linear and smooth)

$$\widetilde{\mathrm{Vert}}. \quad : \quad \mathsf{P}_{\mathrm{G}} \times \mathfrak{g} \xrightarrow{(\mathbf{0}_{\mathsf{TP}_{\mathrm{G}}}, \mathrm{id}_{\mathfrak{g}})} \mathsf{TP}_{\mathrm{G}} \times \mathfrak{g} \equiv \mathsf{T}_{(\cdot,e)}(\mathsf{P}_{\mathrm{G}} \times \mathrm{G}) \xrightarrow{\mathsf{T}_{(\cdot,e)} r.} \mathsf{VP}_{\mathrm{G}} \subset \mathsf{TP}_{\mathrm{G}}$$

$$: \quad (p, X) \longmapsto \big(\mathbf{0}_{\mathsf{TP}_{\mathrm{G}}}(p), X\big) \longmapsto \mathsf{T}_{(p,e)} r.\big(\mathbf{0}_{\mathsf{TP}_{\mathrm{G}}}(p), X\big) \equiv \widetilde{\mathrm{Vert}}_p(X)\,, \tag{7}$$

in whose definition we employ the zero section $\mathbf{0}_{\mathsf{TP}_{\mathrm{G}}}$ of the vector bundle TP_{G} over P_{G}. The codomain of the above map is properly defined. Indeed, owing to the nature of the defining action $r.$, we have

$$\begin{aligned} \mathsf{T}_p \pi_{\mathsf{P}_{\mathrm{G}}}\big(\widetilde{\mathrm{Vert}}_p(X)\big) &\equiv \mathsf{T}_p \pi_{\mathsf{P}_{\mathrm{G}}} \circ \mathsf{T}_{(p,e)} r.\big(\mathbf{0}_{\mathsf{TP}_{\mathrm{G}}}(p), X\big) = \mathsf{T}_{(p,e)}\big(\pi_{\mathsf{P}_{\mathrm{G}}} \circ r.\big)\big(\mathbf{0}_{\mathsf{TP}_{\mathrm{G}}}(p), X\big) \\ &= \mathsf{T}_{(p,e)}\big(\pi_{\mathsf{P}_{\mathrm{G}}} \circ \mathrm{pr}_1\big)\big(\mathbf{0}_{\mathsf{TP}_{\mathrm{G}}}(p), X\big) = \mathsf{T}_{(p,e)}\big(\pi_{\mathsf{P}_{\mathrm{G}}} \circ \mathbf{0}_{\mathsf{TP}_{\mathrm{G}}}(p)\big) \\ &= \mathbf{0}_{\mathsf{T}B} \circ \pi_{\mathsf{P}_{\mathrm{G}}}(p)\,, \end{aligned}$$

and so – indeed – over an arbitrary point $p \in \mathsf{P}_{\mathrm{G}}$, we obtain an inclusion

$$\mathrm{Im}\,\widetilde{\mathrm{Vert}}_p \subset \mathsf{V}_p\mathsf{P}_{\mathrm{G}}\,,$$

and the map $\widetilde{\mathrm{Vert}}.$ covers the identity on the common base of the two bundles,

$$\pi_{\mathsf{P}_{\mathrm{G}}}\big(\widetilde{\mathrm{Vert}}_p(X)\big) = p \equiv \mathrm{pr}_1(p, X)\,,$$

so that we are dealing with a morphism of vector bundles over P_{G}, Since, however, the image of the (constant) field $(\cdot, X)$ is the fundamental field on P_{G} associated with $X \ni \mathfrak{g}$,

$$\mathsf{T}_{(\cdot,e)} r.\big(\mathbf{0}_{\mathsf{TP}_{\mathrm{G}}}(\cdot), X\big) = \mathcal{K}_X \tag{8}$$

as demonstrated by the computation below, carried out for an arbitrary $f \in C^1(\mathsf{P}_{\mathrm{G}}, \mathbb{R})$ at the (arbitrary) point $p \in \mathsf{P}_{\mathrm{G}}$,

$$\begin{aligned} &\mathsf{T}_{\cdot,e} r.\big(\mathbf{0}_{\mathsf{TP}_{\mathrm{G}}}(\cdot), X\big)(f)(p) = \mathsf{T}_{p,e} r.\big(\mathbf{0}_{\mathsf{TP}_{\mathrm{G}}}(p), X\big) \lrcorner\, \mathsf{d}f(p) \\ &= \big(\mathbf{0}_{\mathsf{TP}_{\mathrm{G}}}(p), X\big) \lrcorner\, \mathsf{d}(r.^* f)(p, e) = \tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0} f\big(p \triangleleft \exp(t \triangleright X)\big)\,, \end{aligned}$$

we infer, upon invoking the freeness of the action of G on P_{G}, the equivalence

$$\widetilde{\mathrm{Vert}}_p(X) = 0_{\mathsf{T}_p\mathsf{P}_{\mathrm{G}}} \qquad \Longleftrightarrow \qquad X = 0_{\mathfrak{g}}\,,$$

and hence $\widetilde{\mathrm{Vert}}.$ is a monomorphism. Comparing the ranks of the two bundles,

$$\mathrm{rk}\,(\mathsf{P}_{\mathrm{G}} \times \mathfrak{g}) = \dim_{\mathbb{R}} \mathfrak{g} = \dim \mathrm{G} \equiv \dim \mathsf{P}_{\pi_{\mathsf{P}_{\mathrm{G}}}(p)} = \dim_{\mathbb{R}} \mathsf{T}_p\big(\mathsf{P}_{\pi_{\mathsf{P}_{\mathrm{G}}}(p)}\big) \equiv \mathrm{rk}\,\mathsf{VP}_{\mathrm{G}}\,,$$

we conclude that $\widetilde{\mathrm{Vert}}.$ is the postulated isomorphism. □

Remark 1. The last proposition readily leads to the conclusion that the vertical subbundle is automatically preserved by $\mathsf{T}r_g$. Indeed, for an arbitrary vector $v \equiv\in \mathsf{V}_p\mathsf{P}_{\mathrm{G}}$, $p \in \mathsf{P}_{\mathrm{G}}$ given as the preimage of $X = \mathrm{pr}_2 \circ \widetilde{\mathrm{Vert}}_p^{-1}(v) \in \mathfrak{g}$ and an arbitrary element $g \in \mathrm{G}$, we calculate

$$\begin{aligned} \mathsf{T}_p r_g(v) &= \mathsf{T}_p r_g \circ \widetilde{\mathrm{Vert}}_p\big(X\big) \equiv \mathsf{T}_p r_g\big(\tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0} p \triangleleft \exp(t \triangleright X)\big) \\ &= \tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0} r_g\big(p \triangleleft \exp(t \triangleright X)\big) \equiv \tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0} r_g(p) \triangleleft \mathrm{Ad}_{g^{-1}}\big(\exp(t \triangleright X)\big)\,, \end{aligned}$$

but also – in the light of Prop. 4.7 – the identity

$$\mathrm{Ad}_{g^{-1}}\big(\exp(t \triangleright X)\big) = \exp\big(t \triangleright \mathsf{T}_e\mathrm{Ad}_{g^{-1}}(X)\big)$$

holds true, and so we may rewrite the above equality in the form

$$\begin{aligned} \mathsf{T}_p r_g(v) &= \tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0} r_g(p) \triangleleft \exp\big(t \triangleright \mathsf{T}_e\mathrm{Ad}_{g^{-1}}(X)\big) \\ &\equiv \widetilde{\mathrm{Vert}}. \circ \big(r_g \times \mathsf{T}_e\mathrm{Ad}_{g^{-1}}\big) \circ \widetilde{\mathrm{Vert}}_p^{-1}(v)\,, \end{aligned}$$

whence the conclusion

$$\mathsf{T}_p r_g \restriction_{\mathsf{V}_p\mathsf{P}_{\rm G}} = \widetilde{\rm Vert}_{\cdot} \circ \big(r_g \times \mathsf{T}_e{\rm Ad}_{g^{-1}}\big) \circ \widetilde{\rm Vert}_p^{-1}\,, \tag{9}$$

which – in turn – demonstrates the isomorphic character of the map

$$\mathsf{T}_p r_g \restriction_{\mathsf{V}_p\mathsf{P}_{\rm G}}\ :\ \mathsf{V}_p\mathsf{P}_{\rm G} \longrightarrow \mathsf{V}_{r_g(p)}\mathsf{P}_{\rm G}\,.$$

We are now ready to formulate and prove

Theorem 1. On an arbitrary principal bundle, a principal connection on fibres determines a principal Ehresmann connection and *vice versa.* Furthermore, a principal Ehresmann connection on that bundle determines a principal connection form and *vice versa.*

Proof: Differentiating identity (1), written for $(t_1,t_2) = (0,t)$, along t at $t = 0$, we obtain – for an arbitrary $p \in \mathsf{P}_{{\rm G}\,\gamma(0)}$ –

$$\begin{aligned}{\rm Hor}_{r_g(p)}\big(\dot\gamma(0)\big) &= \tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0} \mathsf{P}^\gamma_{0,t}\big(r_g(p)\big) = \tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0} r_g \circ \mathsf{P}^\gamma_{0,t}(p) = \mathsf{T}_p r_g\big(\tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0} \mathsf{P}^\gamma_{0,t}(p)\big) \\ &= \mathsf{T}_p r_g \circ {\rm Hor}_p\big(\dot\gamma(0)\big)\,,\end{aligned}$$

which, due to the arbitrariness of γ and bijectivity of ${\rm Hor}_p$, enables us to conclude that condition (3) is satisfied.

Conversely, having fixed a path $\gamma\ :\ ]-\varepsilon,\varepsilon[\longrightarrow B$, $\varepsilon > 0$ subject to the constraints $\gamma(0) = x$ and $\dot\gamma(0) = X \in \mathsf{T}_x B$, and subsequently also points: $p \in \mathsf{P}_x$ and $g \in {\rm G}$, we lift γ horizontally to $\mathsf{P}_{\rm G}$ whereby the path

$$\widetilde\gamma_p\ :\ ]-\varepsilon,\varepsilon[\longrightarrow \mathsf{P}_{\rm G}$$

is obtained that integrates the initial condition

$$\tfrac{\mathsf{d}}{\mathsf{d}t}\widetilde\gamma_p(t) = {\rm Hor}_{\widetilde\gamma_p(t)}\big(\dot\gamma(t)\big)\,, \qquad \widetilde\gamma_p(0) = p\,,$$

where – just to recall – ${\rm Hor}_q = (\mathsf{T}_q\pi_{\mathsf{P}_{\rm G}}\restriction_{\mathsf{H}_q\mathsf{P}_{\rm G}})^{-1}$. Let, moreover, $\widetilde\gamma_{r_g(p)}$ be the horizontal lift of γ to $\mathsf{P}_{\rm G}$ through $\widetilde\gamma_{r_g(p)}(0) = r_g(p)$. We compute

$$\tfrac{\mathsf{d}}{\mathsf{d}t}\, r_g \circ \widetilde\gamma_p(t) = \mathsf{T}_{\widetilde\gamma_p(t)} r_g\big(\tfrac{\mathsf{d}}{\mathsf{d}t}\,\widetilde\gamma_p(t)\big) = \mathsf{T}_{\widetilde\gamma_p(t)} r_g \circ {\rm Hor}_{\widetilde\gamma_p(t)}\big(\dot\gamma(t)\big) = {\rm Hor}_{r_g\circ\widetilde\gamma_p(t)}\big(\dot\gamma(t)\big)\,,$$

and the latter equality, reflecting the assumed compatibility of the Ehresmann connection with the group action shows that the pushforward of the lift (to p at $t = 0$) of the vector field tangent to γ, *i.e.*, $\mathsf{T}_{\widetilde\gamma_p(t)} r_g \circ {\rm Hor}_{\widetilde\gamma_p(t)}(\dot\gamma(t))$, is also a horizontal vector field, that is some horizontal lift of $\dot\gamma(t)$ (to $r_g \circ \widetilde\gamma_p(0) = r_g(p)$ at $t = 0$). However, the integral curve (locally unique) of the horizontal lift $\dot\gamma$ to $\mathsf{T}\mathsf{P}_{\rm G}$ through $r_g(p)$ is – by definition – given by the path $\widetilde\gamma_{r_g(p)}$, and so, necessarily,

$$\widetilde\gamma_{r_g(p)} = r_g \circ \widetilde\gamma_p\,,$$

which – in the light of the construction of the diffeomorphism $\mathsf{P}^\gamma_{t_1,t_2}$ – implies the desired G-equivariance of the latter, (1).

Passing to the second part of the statement of the theorem, we consider the principal connection $\mathsf{T}\mathsf{P}_{\rm G} = \mathsf{V}\mathsf{P}_{\rm G} \oplus_{\mathbb{R},\mathsf{P}_{\rm G}} \mathsf{H}\mathsf{P}_{\rm G}$, $\mathsf{H}_{r_g(p)}\mathsf{P}_{\rm G} = \mathsf{T}_p r_g(\mathsf{H}_p\mathsf{P}_{\rm G})$ that allows – in virtue of Prop. 1 – to define a morphism of vector bundles (over $\mathbb{R}$)

$$(\mathcal{A}, {\rm id}_B) := \big(\widetilde{\rm Vert}_{\cdot}^{-1} \circ P^{(\mathsf{H}\mathsf{P}_{\rm G})}_{\mathsf{V}\mathsf{P}_{\rm G}}, {\rm id}_B\big)\ :\ \mathsf{T}\mathsf{P}_{\rm G} \twoheadrightarrow \mathsf{V}\mathsf{P}_{\rm G} \xrightarrow{\ \cong\ } \mathsf{P}_{\rm G} \times \mathfrak{g}\,, \tag{10}$$

in whose definition $P^{(\mathsf{H}\mathsf{P}_{\rm G})}_{\mathsf{V}\mathsf{P}_{\rm G}}$ is a smooth (of class C^∞) family of projections onto the subspace of vertical vectors along the subspace of horizontal vectors. This morphism has the desired property

$$\mathcal{A} \circ \widetilde{\rm Vert}_{\cdot} = \widetilde{\rm Vert}_{\cdot}^{-1} \circ P^{(\mathsf{H}\mathsf{P}_{\rm G})}_{\mathsf{V}\mathsf{P}_{\rm G}} \circ \widetilde{\rm Vert}_{\cdot} \equiv \widetilde{\rm Vert}_{\cdot}^{-1} \circ \widetilde{\rm Vert}_{\cdot} = {\rm id}_{\mathsf{P}_{\rm G}\times\mathfrak{g}}\,.$$

Moreover, in virtue of Eq. (9) and of the assumed G-invariance of the decomposition $\mathsf{T}\mathsf{P}_{\rm G} = \mathsf{V}\mathsf{P}_{\rm G} \oplus_{\mathbb{R},\mathsf{P}_{\rm G}} \mathsf{H}\mathsf{P}_{\rm G}$, we obtain, in conformity with our expectations,

$$\begin{aligned}\mathcal{A} \circ \mathsf{T}r_g &\equiv \widetilde{\rm Vert}_{\cdot}^{-1} \circ P^{(\mathsf{H}\mathsf{P}_{\rm G})}_{\mathsf{V}\mathsf{P}_{\rm G}} \circ \mathsf{T}r_g = \widetilde{\rm Vert}_{\cdot}^{-1} \circ \mathsf{T}r_g \circ P^{(\mathsf{H}\mathsf{P}_{\rm G})}_{\mathsf{V}\mathsf{P}_{\rm G}} \\ &= \widetilde{\rm Vert}_{\cdot}^{-1} \circ \big(\widetilde{\rm Vert}_{\cdot} \circ \big(r_g \times \mathsf{T}_e{\rm Ad}_{g^{-1}}\big) \circ \widetilde{\rm Vert}_{\cdot}^{-1}\big) \circ P^{(\mathsf{H}\mathsf{P}_{\rm G})}_{\mathsf{V}\mathsf{P}_{\rm G}}\end{aligned}$$

$$\equiv \quad \big(r_g \times \mathsf{T}_e \mathrm{Ad}_{g^{-1}}\big) \circ \mathcal{A} \,.$$

Conversely, a principal connection form defines – according to Thm. 9.2 – a vector subbundle

$$\mathsf{HP}_{\mathrm{G}} := \mathrm{Ker}\, \mathcal{A} \subset \mathsf{TP}_{\mathrm{G}} \,. \tag{11}$$

Furthermore, for an arbitrary $v \in \mathrm{Ker}\,(\mathsf{T}\pi_{\mathsf{P}_{\mathrm{G}}} \restriction_{\mathsf{H}_p\mathsf{P}_{\mathrm{G}}}) \equiv \mathrm{Ker}\,(\mathcal{A}\restriction_{\mathsf{T}_p\mathsf{P}_{\mathrm{G}}}) \cap \mathrm{Ker}\,(\mathsf{T}\pi_{\mathsf{P}_{\mathrm{G}}} \restriction_{\mathsf{T}_p\mathsf{P}_{\mathrm{G}}})$, we establish

$$\begin{aligned} v \quad &= \quad \widetilde{\mathrm{Vert}}. \circ \widetilde{\mathrm{Vert}}_p^{-1}(v) \equiv \widetilde{\mathrm{Vert}}. \circ \mathrm{id}_{\mathsf{P}_{\mathrm{G}}\times\mathfrak{g}} \circ \widetilde{\mathrm{Vert}}_p^{-1}(v) = \widetilde{\mathrm{Vert}}. \circ (\mathcal{A} \circ \widetilde{\mathrm{Vert}}.) \circ \widetilde{\mathrm{Vert}}_p^{-1}(v) \\ &= \quad \widetilde{\mathrm{Vert}}. \circ \mathcal{A}(v) = 0_{\mathsf{T}_p\mathsf{P}_{\mathrm{G}}} \,, \end{aligned}$$

which implies injectivity of $\mathsf{T}\pi_{\mathsf{P}_{\mathrm{G}}} \restriction_{\mathsf{H}_p\mathsf{P}_{\mathrm{G}}}$ and, in so doing, proves existence of the isomorphism

$$\mathrm{Im}\big(\mathsf{T}\pi_{\mathsf{P}_{\mathrm{G}}} \restriction_{\mathsf{H}_p\mathsf{P}_{\mathrm{G}}}\big) \cong \mathsf{H}_p\mathsf{P}_{\mathrm{G}} \,.$$

At the same time, we have, for $\mathcal{A}\restriction_{\mathsf{T}_p\mathsf{P}_{\mathrm{G}}} \in \mathrm{Hom}_{\mathbb{R}}(\mathsf{T}_p\mathsf{P}_{\mathrm{G}}, \mathfrak{g})$, the equality

$$\begin{aligned} \dim_{\mathbb{R}} \mathrm{Ker}\big(\mathcal{A}\restriction_{\mathsf{T}_p\mathsf{P}_{\mathrm{G}}}\big) \quad &= \quad \dim_{\mathbb{R}} \mathsf{T}_p\mathsf{P}_{\mathrm{G}} - \dim_{\mathbb{R}} \mathrm{Im}\big(\mathcal{A}\restriction_{\mathsf{T}_p\mathsf{P}_{\mathrm{G}}}\big) \\ &= \quad \dim_{\mathbb{R}} \mathsf{V}_p\mathsf{P}_{\mathrm{G}} + \dim_{\mathbb{R}} \mathsf{T}_{\pi_{\mathsf{P}_{\mathrm{G}}}(p)}B - \dim_{\mathbb{R}} \mathfrak{g} = \dim_{\mathbb{R}} \mathsf{T}_{\pi_{\mathsf{P}_{\mathrm{G}}}(p)}B \,, \end{aligned}$$

which infers that $\mathsf{T}\pi_{\mathsf{P}_{\mathrm{G}}} \restriction_{\mathsf{H}_p\mathsf{P}_{\mathrm{G}}}$ is, in fact, an isomorphism. Finally, we convince ourselves of the G-invariance of the thus defined horizontal subbundle. Let $\xi \in \mathsf{H}_p\mathsf{P}_{\mathrm{G}} \equiv \mathrm{Ker}\,(\mathcal{A}\restriction_{\mathsf{T}_p\mathsf{P}_{\mathrm{G}}})$, so that

$$\mathcal{A} \circ \mathsf{T}_p r_g(\xi) = \big(r_g \times \mathsf{T}_e \mathrm{Ad}_{g^{-1}}\big) \circ \mathcal{A}(\xi) = \big(r_g(p), \mathsf{T}_e \mathrm{Ad}_{g^{-1}}(0_{\mathfrak{g}})\big) = \big(r_g(p), 0_{\mathfrak{g}}\big) \,,$$

whence the inclusion

$$\mathsf{T}_p r_g(\mathsf{H}_p\mathsf{P}_{\mathrm{G}}) \subset \mathsf{H}_{r_g(p)}\mathsf{P}_{\mathrm{G}} \,,$$

but then also – in consequence of the invertibility of $\mathsf{T}_p r_g$ –

$$\mathsf{T}_{r_g(p)} r_{g^{-1}}(\mathsf{H}_{r_g(p)}\mathsf{P}_{\mathrm{G}}) \subset \mathsf{H}_{r_g^{-1}\circ r_g(p)}\mathsf{P}_{\mathrm{G}} = \mathsf{H}_p\mathsf{P}_{\mathrm{G}} \,,$$

and so

$$\mathsf{H}_{r_g(p)}\mathsf{P}_{\mathrm{G}} \equiv \mathsf{T}_p r_g \circ \mathsf{T}_{r_g(p)} r_{g^{-1}}(\mathsf{H}_{r_g(p)}\mathsf{P}_{\mathrm{G}}) \subset \mathsf{T}_p r_g\big(\mathsf{H}_p\mathsf{P}_{\mathrm{G}}\big) \,,$$

which, in the end, gives us the desired equality

$$\mathsf{T}_p r_g(\mathsf{H}_p\mathsf{P}_{\mathrm{G}}) = \mathsf{H}_{r_g(p)}\mathsf{P}_{\mathrm{G}} \,.$$

□

An answer to the fundamental question of existence of a compatible connection is given in

Theorem 2. There exists a principal connection on an arbitrary principal bundle.

Proof: Let $(\mathsf{P}_{\mathrm{G}}, B, \mathrm{G}, \pi_{\mathsf{P}_{\mathrm{G}}})$ be a principal bundle with local trivialisations $\tau_i \;:\; \pi_{\mathsf{P}_{\mathrm{G}}}^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times \mathrm{G}$, $i \in I$. Using the relations

$$\mathsf{T}_{(x,g)}(\mathcal{O}_i \times \mathrm{G}) \equiv \mathsf{T}_x\mathcal{O}_i \oplus \mathsf{T}_g\mathrm{G} \equiv \mathsf{T}_x\mathcal{O}_i \oplus \mathsf{T}_e\ell_g(\mathfrak{g}) \,,$$

we define, over each element $\mathcal{O}_i$ of the trivialising cover, a mapping

$$\mathcal{A}_i \;:\; \mathsf{T}\pi_{\mathsf{P}_{\mathrm{G}}}^{-1}(\mathcal{O}_i) \longrightarrow \pi_{\mathsf{P}_{\mathrm{G}}}^{-1}(\mathcal{O}_i) \times \mathfrak{g} \;:\; \mathsf{T}_{\tau_i^{-1}(x,g)}\tau_i^{-1}(v, V) \longmapsto \big(\tau_i^{-1}(x,g), \mathsf{T}_g\ell_{g^{-1}}(V)\big) \,,$$

manifestly $\mathbb{R}$-linear and fibre-preserving. We readily check that these mappings have the properties listed in Def. 3. Thus, first of all, taking into account commutativity of

$$\begin{array}{ccc} \pi_{\mathsf{P}_{\mathrm{G}}}^{-1}(\mathcal{O}_i) \times \mathrm{G} & \xrightarrow{\ r.\ } & \pi_{\mathsf{P}_{\mathrm{G}}}^{-1}(\mathcal{O}_i) \\ \Big\downarrow{\scriptstyle \tau_i \times \mathrm{id}_{\mathrm{G}}} & & \Big\downarrow{\scriptstyle \tau_i} \\ \mathcal{O}_i \times \mathrm{G} \times \mathrm{G} & \xrightarrow[\mathrm{id}_B \times m]{} & \mathcal{O}_i \times \mathrm{G} \end{array}$$

alongside Eqn. (3.4) (p. 21), we compute – for an arbitrary vector $X \in \mathfrak{g}$ –

$$\begin{aligned} & \mathcal{A}_i \circ \widetilde{\mathrm{Vert}}_{\tau_i^{-1}(x,g)}(X) \equiv \mathcal{A}_i \circ \mathsf{T}_{(\tau_i^{-1}(x,g),e)} r.\big(\mathbf{0}_{\mathsf{TP}_{\mathrm{G}}} \circ \tau_i^{-1}(x,g), X\big) \\ = \ & \mathcal{A}_i \circ \mathsf{T}_{\tau_i^{-1}(x,g)} \tau_i^{-1} \circ \mathsf{T}_{(\tau_i^{-1}(x,g),e)}(\tau_i \circ r.)\big(\mathbf{0}_{\mathsf{TP}_{\mathrm{G}}} \circ \tau_i^{-1}(x,g), X\big) \\ = \ & \mathcal{A}_i \circ \mathsf{T}_{\tau_i^{-1}(x,g)} \tau_i^{-1} \circ \mathsf{T}_{(x,g,e)}(\mathrm{id}_B \times m) \circ \mathsf{T}_{(\tau_i^{-1}(x,g),e)}(\tau_i \times \mathrm{id}_{\mathrm{G}})\big(\mathbf{0}_{\mathsf{TP}_{\mathrm{G}}} \circ \tau_i^{-1}(x,g), X\big) \\ = \ & \mathcal{A}_i \circ \mathsf{T}_{\tau_i^{-1}(x,g)} \tau_i^{-1} \circ \big(\mathsf{T}_x \mathrm{id}_B \oplus \mathsf{T}_{(g,e)} m\big)\big(\mathsf{T}_{\tau_i^{-1}(x,g)} \tau_i \circ \mathbf{0}_{\mathsf{TP}_{\mathrm{G}}} \circ \tau_i^{-1}(x,g), \mathsf{T}_e \mathrm{id}_{\mathrm{G}}(X)\big) \\ = \ & \mathcal{A}_i \circ \mathsf{T}_{\tau_i^{-1}(x,g)} \tau_i^{-1} \circ \big(\mathrm{id}_{\mathsf{T}_x B} \oplus \mathsf{T}_{(g,e)} m\big)\big(0_{\mathsf{T}_x B}, 0_{\mathsf{T}_g \mathrm{G}}, \mathrm{id}_{\mathsf{T}_e \mathrm{G}}(X)\big) \\ \equiv \ & \big(\tau_i^{-1}(x,g), \mathsf{T}_g \ell_{g^{-1}} \circ \mathsf{T}_{(g,e)} m(0_{\mathsf{T}_g \mathrm{G}}, X)\big) = \big(\tau_i^{-1}(x,g), \mathsf{T}_g \ell_{g^{-1}} \circ \mathsf{T}_e \ell_g(X)\big) \\ = \ & \big(\tau_i^{-1}(x,g), X\big) \equiv \mathrm{id}_{\mathsf{P}_{\mathrm{G}} \times \mathfrak{g}}\big(\tau_i^{-1}(x,g), X\big) . \end{aligned}$$

Secondly, in the hitherto notation and for an arbitrary element $h \in \mathrm{G}$, taking into account the commutative diagram

$$\begin{array}{ccc} \pi_{\mathsf{P}_{\mathrm{G}}}^{-1}(\mathcal{O}_i) & \xrightarrow{\ r_h\ } & \pi_{\mathsf{P}_{\mathrm{G}}}^{-1}(\mathcal{O}_i) \\ \Big\uparrow{\scriptstyle \tau_i^{-1}} & & \Big\uparrow{\scriptstyle \tau_i^{-1}} \\ \mathcal{O}_i \times \mathrm{G} & \xrightarrow[\mathrm{id}_B \times \wp_h]{} & \mathcal{O}_i \times \mathrm{G} \end{array} ,$$

we check the condition of G-equivariance of $\mathcal{A}_i$,

$$\begin{aligned} & \mathcal{A}_i \circ \mathsf{T} r_h \circ \mathsf{T}_{\tau_i^{-1}(x,g)} \tau_i^{-1}(v,V) = \mathcal{A}_i \circ \mathsf{T}_{\tau_i^{-1}(x,gh)} \tau_i^{-1} \circ \mathsf{T}_{(x,g)}(\mathrm{id}_B \times \wp_h)(v,V) \\ = \ & \mathcal{A}_i \circ \mathsf{T}_{\tau_i^{-1}(x,gh)} \tau_i^{-1} \circ \big(\mathsf{T}_x \mathrm{id}_B \oplus \mathsf{T}_g \wp_h\big)(v,V) \\ = \ & \mathcal{A}_i \circ \mathsf{T}_{\tau_i^{-1}(x,gh)} \tau_i^{-1}\big(\mathrm{id}_{\mathsf{T}_x B}(v), \mathsf{T}_g \wp_h(V)\big) \equiv \big(\tau_i^{-1}(x,gh), \mathsf{T}_{gh} \ell_{(gh)^{-1}} \circ \mathsf{T}_g \wp_h(V)\big) \\ = \ & \big(r_h \circ \tau_i^{-1}(x,g), \mathsf{T}_e \mathrm{Ad}_{h^{-1}} \circ \mathsf{T}_g \ell_{g^{-1}}(V)\big) \\ \equiv \ & \big(r_h \times \mathsf{T}_e \mathrm{Ad}_{h^{-1}}\big) \circ \mathcal{A}_i \circ \mathsf{T}_{\tau_i^{-1}(x,g)} \tau_i^{-1}(v,V) . \end{aligned}$$

In the light of the above results, the $\mathcal{A}_i$ compose a family of local principal connection forms. Upon fixing an arbitrary partition of unity $\{\lambda_i\}_{i \in I}$ (of class C^∞) associated with the trivialising cover $\{\mathcal{O}_i\}_{i \in I}$ of the base B, we use them to induce a globally defined (and smooth) form

$$\mathcal{A}(\cdot) := \sum_{i \in I} \lambda_i \circ \pi_{\mathsf{P}_{\mathrm{G}}} \circ \pi_{\mathsf{TP}_{\mathrm{G}}}(\cdot) \triangleright \mathcal{A}_i(\cdot) ,$$

with all the desired properties. □

Having stated several (equivalent) definitions of structural compatibility of a connection on a principal bundle and studied the issue of existence of a compatible connection, we may, now, augment the hitherto discussion with an indication of a subclass of morphisms that preserve the latter.

Theorem 3. Adopt the hitherto notation and let

$$\begin{array}{ccc} \mathsf{P}^1_{\mathrm{G}} & \xrightarrow{\ \Phi\ } & \mathsf{P}^2_{\mathrm{G}} \\ \Big\downarrow{\scriptstyle \pi_{\mathsf{P}^1_{\mathrm{G}}}} & & \Big\downarrow{\scriptstyle \pi_{\mathsf{P}^2_{\mathrm{G}}}} \\ B_1 & \xrightarrow[\ f\ ]{} & B_2 \end{array}$$

be a morphism of principal bundles (with a common structural group) with a principal connection (covering a diffeomorphism between the bases). Each of the (mutually equivalent) conditions (FCM1), (FCM2) and (FCM3) of Def. 9.6, augmented with the requirement (6.1) of G-equivariance in the case of a principal bundle with a principal connection, is equivalent to the condition

(PFCM4) the morphism Φ preserves the principal connection form, as expressed by the identity

$$\mathcal{A}_2 \circ \mathsf{T}\Phi = (\Phi \times \mathrm{id}_{\mathfrak{g}}) \circ \mathcal{A}_1\,.$$

Proof: Basing on Thm. 9.5, we may restrict to verifying the following implications.

$(FCM2) \Rightarrow (PFCM4)$ Taking into account Def. (7) and condition (6.1), we establish the identity

$$\begin{aligned} &\widetilde{\mathrm{Vert}}^{2\,-1}_{\Phi(p)} \circ \mathsf{T}_p\Phi \circ \widetilde{\mathrm{Vert}}^{1}_{\cdot}(p,X) \equiv \widetilde{\mathrm{Vert}}^{2\,-1}_{\Phi(p)} \circ \mathsf{T}_p\Phi \circ \mathsf{T}_{(p,e)} r^1_{\cdot}\big(\mathbf{0}_{\mathsf{TP}^1_{\mathrm{G}}}(p), X\big) \\ =\ &\widetilde{\mathrm{Vert}}^{2\,-1}_{\Phi(p)} \circ \mathsf{T}_{(p,e)}(\Phi \circ r^1_{\cdot})\big(\mathbf{0}_{\mathsf{TP}^1_{\mathrm{G}}}(p), X\big) \\ =\ &\widetilde{\mathrm{Vert}}^{2\,-1}_{\Phi(p)} \circ \mathsf{T}_{(p,e)}\big(r^2_{\cdot} \circ (\Phi \times \mathrm{id}_{\mathrm{G}})\big)\big(\mathbf{0}_{\mathsf{TP}^1_{\mathrm{G}}}(p), X\big) \\ =\ &\widetilde{\mathrm{Vert}}^{2\,-1}_{\Phi(p)} \circ \mathsf{T}_{(\Phi(p),e)} r^2_{\cdot} \circ (\mathsf{T}_p\Phi \oplus \mathsf{T}_e\mathrm{id}_{\mathrm{G}})\big(\mathbf{0}_{\mathsf{TP}^1_{\mathrm{G}}}(p), X\big) \\ =\ &\widetilde{\mathrm{Vert}}^{2\,-1}_{\Phi(p)} \circ \mathsf{T}_{(\Phi(p),e)} r^2_{\cdot}\big(\mathsf{T}_p\Phi \circ \mathbf{0}_{\mathsf{TP}^1_{\mathrm{G}}}(p), \mathrm{id}_{\mathsf{T}_e\mathrm{G}}(X)\big) \\ =\ &\widetilde{\mathrm{Vert}}^{2\,-1}_{\Phi(p)} \circ \mathsf{T}_{(\Phi(p),e)} r^2_{\cdot}\big(\mathbf{0}_{\mathsf{TP}^2_{\mathrm{G}}} \circ \Phi(p), X\big) = \big(\Phi(p), X\big) \equiv (\Phi \times \mathrm{id}_{\mathfrak{g}})(p,X)\,, \end{aligned}$$

written for arbitrary $(p,X) \in \mathsf{P}^1_{\mathrm{G}} \times \mathfrak{g}$. In view of the above and of Eqn. (10), we obtain

$$\begin{aligned} \mathcal{A}_2 \circ \mathsf{T}\Phi \ &\equiv\ \widetilde{\mathrm{Vert}}^{2\,-1}_{\Phi(p)} \circ P^{(\mathsf{HP}^2_{\mathrm{G}})}_{\mathsf{VP}^2_{\mathrm{G}}} \circ \mathsf{T}\Phi = \widetilde{\mathrm{Vert}}^{2\,-1}_{\Phi(p)} \circ \mathsf{T}\Phi \circ P^{(\mathsf{HP}^1_{\mathrm{G}})}_{\mathsf{VP}^1_{\mathrm{G}}} \\ &\equiv\ \widetilde{\mathrm{Vert}}^{2\,-1}_{\Phi(p)} \circ \mathsf{T}\Phi \circ \widetilde{\mathrm{Vert}}^{1}_{\cdot} \circ \mathcal{A}_1 = (\Phi \times \mathrm{id}_{\mathfrak{g}}) \circ \mathcal{A}_1\,. \end{aligned}$$

$(PFCM4) \Rightarrow (FCM2)$ In the light of Eqn. (11), it suffices to convince oneself that there exists, over an arbitrary point $p \in \mathsf{P}^1_{\mathrm{G}}$, an inclusion

$$\mathsf{T}_p\Phi\big(\mathrm{Ker}\,(\mathcal{A}_1 \restriction_{\mathsf{T}_p\mathsf{P}^1_{\mathrm{G}}})\big) \subset \mathrm{Ker}\,(\mathcal{A}_2 \restriction_{\mathsf{T}_{\Phi(p)}\mathsf{P}^2_{\mathrm{G}}})\,,$$

which we do through a direct calculation

$$\begin{aligned} \mathcal{A}_2 \circ \mathsf{T}_p\Phi\big(\mathrm{Ker}\,(\mathcal{A}_1 \restriction_{\mathsf{T}_p\mathsf{P}^1_{\mathrm{G}}})\big) \ &=\ (\Phi \times \mathrm{id}_{\mathfrak{g}}) \circ \mathcal{A}_1\big(\mathrm{Ker}\,(\mathcal{A}_1 \restriction_{\mathsf{T}_p\mathsf{P}^1_{\mathrm{G}}})\big) \\ &=\ (\Phi \times \mathrm{id}_{\mathfrak{g}})\big(\{(p, 0_{\mathfrak{g}})\}\big) = \big\{\big(\Phi(p), 0_{\mathfrak{g}}\big)\big\}\,. \end{aligned}$$

□

In the next step, we discuss the physically much relevant local description of a compatible connection. We commence with the ancillary

Proposition 2. Adopt the hitherto notation and let ∇ be a covariant derivative on a principal bundle $(\mathsf{P}_{\rm G}, B, {\rm G}, \pi_{\mathsf{P}_{\rm G}})$ associated with a choice of a principal connection. The mappings $\alpha_i,\ i \in I$ defined in Eqn. (9.13) are $C^\infty(B, {\rm G})$-equivariant in the second argument, *i.e.*, for arbitrary: map $g \in C^\infty(B, {\rm G})$ and section $\sigma \in \Gamma_{\rm loc}(\mathsf{P}_{\rm G})$ of the local presentation $\tau_i \circ \sigma(\cdot) = (\cdot, \sigma_i(\cdot))$ under a local trivialisation $\tau_i : \pi_{\mathsf{P}_{\rm G}}^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times {\rm G}$, there obtains

$$\forall_{x \in \mathcal{O}_i} : \alpha_i\big(x, r_{g(x)}(\sigma(x))\big) = \big({\rm id}_{\mathsf{T}^*B} \otimes \mathsf{T}_{\sigma_i(x)} \wp_{g(x)}\big) \circ \alpha_i\big(x, \sigma(x)\big)\,.$$

<u>*Proof:*</u> Instrumental in our considerations is the relation between the covariant derivative and the principal connection form. In the light of the identity

$$P^{(\mathsf{HP}_{\rm G})}_{\mathsf{VP}_{\rm G}} \restriction_{\mathsf{T}_p \mathsf{P}_{\rm G}} \equiv {\rm id}_{\mathsf{T}_p \mathsf{P}_{\rm G}} - {\rm Hor}_p \circ \mathsf{T}_p \pi_{\mathsf{P}_{\rm G}}$$

and of Eqn. (9.12), we establish – for an arbitrary section σ as in the statement of the proposition and an arbitrary vector field $\mathcal{V} \in \Gamma_{({\rm loc})}(\mathsf{T}B)$ – a relation

$$P^{(\mathsf{HP}_{\rm G})}_{\mathsf{VP}_{\rm G}} \circ \mathsf{T}.\sigma(\mathcal{V}) = \mathsf{T}_x \sigma(\mathcal{V}) - {\rm Hor}_{\sigma(\cdot)} \circ \mathsf{T}_{\sigma(\cdot)} \pi_{\mathsf{P}_{\rm G}} \circ \mathsf{T}.\sigma(\mathcal{V})$$

$$= \mathsf{T}.\sigma(\mathcal{V}) - {\rm Hor}_{\sigma(\cdot)} \circ \mathsf{T}.(\pi_{\mathsf{P}_{\rm G}} \circ \sigma)(\mathcal{V}) = \mathsf{T}.\sigma(\mathcal{V}) - {\rm Hor}_{\sigma(\cdot)} \circ \mathsf{T}.{\rm id}_B(\mathcal{V})$$

$$= \mathsf{T}.\sigma(\mathcal{V}) - {\rm Hor}_{\sigma(\cdot)}(\mathcal{V}) \equiv \nabla_{\mathcal{V}} \sigma(\cdot)$$

When put in conjunction with Eqn. (10), the latter gives us a useful identity

$$\nabla_{\mathcal{V}} \sigma(\cdot) = \widetilde{\rm Vert}_{\sigma(\cdot)} \circ \mathcal{A} \circ \mathsf{T}.\sigma(\mathcal{V})\,. \tag{12}$$

This, in turn, enables us to write – with direct reference to the detailed computation presented in the body of Remark 9.3 –

$$\mathsf{T}_x\big(\wp_{g(\cdot)}(\sigma_i(\cdot))\big)(V) + V \lrcorner\, \alpha_i\big(x, r_{g(x)}(\sigma(x))\big)$$

$$\equiv \varpi_i \circ \mathsf{T}_{r_{g(x)}(\sigma(x))} \tau_i\big(\nabla_{\mathcal{V}} r_{(\cdot)}(\sigma(\cdot))(x)\big)$$

$$= \varpi_i \circ \mathsf{T}_{r_{g(x)}(\sigma(x))} \tau_i \circ \widetilde{\rm Vert}_{r_{g(x)}(\sigma(x))} \circ \mathcal{A} \circ \mathsf{T}_x\big(r_{(\cdot)}(\sigma(\cdot))\big)(V)$$

$$= \varpi_i \circ \mathsf{T}_{r_{g(x)}(\sigma(x))} \tau_i \circ \widetilde{\rm Vert}_{r_{g(x)}(\sigma(x))} \circ \mathcal{A}\big(\mathsf{T}_{\sigma(x)} r_{g(x)} \circ \mathsf{T}_x \sigma(V)$$

$$+ (\mathcal{V} \lrcorner\, g^* \theta^A_{\rm R})(x)\, R_A\big(r.(\sigma(x))\big)(g(x))\big)\,,$$

where in the last line we are dealing with the derivative of the map $r.(\sigma(x)) : {\rm G} \longrightarrow \mathsf{P}_{\rm G}$ in the direction of the right-invariant vector field R_A. We calculate the derivative directly,

$$R_A\big(r.(\sigma(x))\big)(g(x)) \equiv \tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0} r.(\sigma(x))\big(\exp(t \triangleright t_A) \cdot g(x)\big)$$

$$= \tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0} r_{g(x)} \circ r_{\exp(t \triangleright t_A)}(\sigma(x)) = \mathsf{T}_{\sigma(x)} r_{g(x)}\big(\tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0} r_{\exp(t \triangleright t_A)}(\sigma(x))\big)$$

$$\equiv \mathsf{T}_{\sigma(x)} r_{g(x)}\big(\mathcal{K}_{t_A}(\sigma(x))\big)\,,$$

denoting by $\mathcal{K}_{t_A}$, as before, the (right) fundamental vector field on $\mathsf{P}_{\rm G}$ associated with the generator t_A of the Lie algebra $\mathfrak{g}$. Upon substituting the above result in our former computation, and subsequently using identities (5) and (9), we obtain

$$\mathsf{T}_x\big(\wp_{g(\cdot)}(\sigma_i(\cdot))\big)(V) + V \lrcorner\, \alpha_i\big(x, r_{g(x)}(\sigma(x))\big)$$

$$\equiv \varpi_i \circ \mathsf{T}_{r_{g(x)}(\sigma(x))} \tau_i\big(\nabla_{\mathcal{V}} r_{(\cdot)}(\sigma(\cdot))(x)\big)$$

$$= \varpi_i \circ \mathsf{T}_{r_{g(x)}(\sigma(x))} \tau_i \circ \widetilde{\rm Vert}_{r_{g(x)}(\sigma(x))} \circ \mathcal{A}\big(\mathsf{T}_{\sigma(x)} r_{g(x)} \circ \mathsf{T}_x \sigma(V)$$

$$+ (\mathcal{V} \lrcorner\, g^* \theta^A_{\rm R})(x)\, \mathsf{T}_{\sigma(x)} r_{g(x)}\big(\mathcal{K}_{t_A}(\sigma(x))\big)\big)$$

$$
\begin{aligned}
&= \varpi_i \circ \mathsf{T}_{r_{g(x)}(\sigma(x))}\tau_i \circ \widetilde{\mathrm{Vert}}_{r_{g(x)}(\sigma(x))} \circ \mathcal{A} \circ \mathsf{T}_{\sigma(x)} r_{g(x)}\big(\mathsf{T}_x\sigma(V) \\
&\quad + \big(\mathcal{V} \lrcorner\, g^*\theta_{\mathrm{R}}^A\big)(x) \triangleright \mathcal{K}_{t_A}\big(\sigma(x)\big)\big) \\
&= \varpi_i \circ \mathsf{T}_{r_{g(x)}(\sigma(x))}\tau_i \circ \widetilde{\mathrm{Vert}}_{r_{g(x)}(\sigma(x))} \circ \big(r_{g(x)} \times \mathsf{T}_e\mathrm{Ad}_{g(x)^{-1}}\big) \circ \mathcal{A}\big(\mathsf{T}_x\sigma(V) \\
&\quad + \big(\mathcal{V} \lrcorner\, g^*\theta_{\mathrm{R}}^A\big)(x) \triangleright \mathcal{K}_{t_A}\big(\sigma(x)\big)\big) \\
&= \varpi_i \circ \mathsf{T}_{r_{g(x)}(\sigma(x))}\tau_i \circ \mathsf{T}_{\sigma(x)} r_{g(x)} \circ \widetilde{\mathrm{Vert}}_{\sigma(x)} \circ \mathcal{A}\big(\mathsf{T}_x\sigma(V) \\
&\quad + \big(\mathcal{V} \lrcorner\, g^*\theta_{\mathrm{R}}^A\big)(x) \triangleright \mathcal{K}_{t_A}\big(\sigma(x)\big)\big) \\
&= \varpi_i \circ \mathsf{T}_{\tau_i\circ\sigma(x)}\big(\mathrm{id}_B \times \wp_{g(x)}\big) \circ \mathsf{T}_{\sigma(x)}\tau_i \circ \widetilde{\mathrm{Vert}}_{\sigma(x)} \circ \mathcal{A}\big(\mathsf{T}_x\sigma(V) \\
&\quad + \big(\mathcal{V} \lrcorner\, g^*\theta_{\mathrm{R}}^A\big)(x) \triangleright \mathcal{K}_{t_A}\big(\sigma(x)\big)\big) \\
&= \mathsf{T}_{\sigma_i(x)}\wp_{g(x)} \circ \varpi_i \circ \mathsf{T}_{\sigma(x)}\tau_i \circ \widetilde{\mathrm{Vert}}_{\sigma(x)} \circ \mathcal{A}\big(\mathsf{T}_x\sigma(V) \\
&\quad + \big(\mathcal{V} \lrcorner\, g^*\theta_{\mathrm{R}}^A\big)(x) \triangleright \mathcal{K}_{t_A}\big(\sigma(x)\big)\big) \\
&\equiv \mathsf{T}_{\sigma_i(x)}\wp_{g(x)}\big(\mathsf{T}_x\sigma_i(V) + V \lrcorner\, \alpha_i\big(x,\sigma(x)\big) \\
&\quad + \big(\mathcal{V} \lrcorner\, g^*\theta_{\mathrm{R}}^A\big)(x) \triangleright \varpi_i \circ \mathsf{T}_{\sigma(x)}\tau_i \circ \widetilde{\mathrm{Vert}}_{\sigma(x)} \circ \mathcal{A} \circ \mathcal{K}_{t_A}\big(\sigma(x)\big)\big)\big)\,.
\end{aligned}
$$

If, now, we take into account the vrtical nature of the fundamental vector fields $\mathcal{K}_A$ (a straightforward consequence of the very nature of the defining action $r_\cdot$), we may rewrite the above as

$$
\begin{aligned}
&\mathsf{T}_x\big(\wp_{g(\cdot)}\big(\sigma_i(\cdot)\big)\big)(V) + V \lrcorner\, \alpha_i\big(x, r_{g(x)}\big(\sigma(x)\big)\big) \\
&= \mathsf{T}_{\sigma_i(x)}\wp_{g(x)}\big(\mathsf{T}_x\sigma_i(V) + V \lrcorner\, \alpha_i\big(x,\sigma(x)\big) \\
&\quad + \big(\mathcal{V} \lrcorner\, g^*\theta_{\mathrm{R}}^A\big)(x) \triangleright \varpi_i \circ \mathsf{T}_{\sigma(x)}\tau_i \circ \widetilde{\mathrm{Vert}}_{\sigma(x)} \circ \big(\mathcal{A} \circ \widetilde{\mathrm{Vert}}_{\sigma(x)}\big) \circ \widetilde{\mathrm{Vert}}_{\sigma(x)}^{-1}\big(\mathcal{K}_{t_A}\big(\sigma(x)\big)\big)\big)\big) \\
&= \mathsf{T}_{\sigma_i(x)}\wp_{g(x)}\big(\mathsf{T}_x\sigma_i(V) + V \lrcorner\, \alpha_i\big(x,\sigma(x)\big) \\
&\quad + \big(\mathcal{V} \lrcorner\, g^*\theta_{\mathrm{R}}^A\big)(x) \triangleright \varpi_i \circ \mathsf{T}_{\sigma(x)}\tau_i\big(\mathcal{K}_{t_A}\big(\sigma(x)\big)\big)\big)\big)\,,
\end{aligned}
$$

whence the desired identity ensues,

$$
V \lrcorner\, \alpha_i\big(x, r_{g(x)}\big(\sigma(x)\big)\big) = \mathsf{T}_{\sigma_i(x)}\wp_{g(x)}\big(V \lrcorner\, \alpha_i\big(x,\sigma(x)\big)\big)\,.
$$

□

The above leads us directly to

Definition 4. Adpot the notation of Def. 3 and let $\tau_i \;:\; \pi_{\mathsf{P}_{\mathrm{G}}}^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times \mathrm{G},\; i \in I$ be local trivialisations of the principal bundle $(\mathsf{P}_{\mathrm{G}}, B, \mathrm{G}, \pi_{\mathsf{P}_{\mathrm{G}}})$. Define

$$
s_{(i)} \;:\; \mathcal{O}_i \longrightarrow \pi_{\mathsf{P}_{\mathrm{G}}}^{-1}(\mathcal{O}_i) \;:\; x \longmapsto \tau_i^{-1}(x,e)\,.
$$

A **local potential of the principal connection** $\mathcal{A}$ on a principal bundle P_{G} over $\mathcal{O}_i$ (associated with the sections $s_{(i)}$) is a mapping (of class C^∞)

$$
\mathsf{A}_i \in \Omega^1(\mathcal{O}_i) \otimes_{\mathbb{R}} \mathfrak{g}
$$

taking, at an arbitrary point $x \in \mathcal{O}_i$, the form[1]

$$
\mathsf{A}_i(x) := \big(\mathrm{id}_{\mathsf{T}^*B} \otimes \underline{\mathcal{A}}\big) \circ \mathsf{T}_x s_{(i)}\,. \tag{13}
$$

[1]The tangent map $\mathsf{T}_x s_{(i)} \;:\; \mathsf{T}_x\mathcal{O}_i \longrightarrow \mathsf{T}_{s_{(i)}(x)}\pi_{\mathsf{P}_{\mathrm{G}}}^{-1}(\mathcal{O}_i)$ is treated here as an element of the vector space $\mathsf{T}_x^*\mathcal{O}_i \otimes_{\mathbb{R}} \mathsf{T}_{s_{(i)}(x)}\pi_{\mathsf{P}_{\mathrm{G}}}^{-1}(\mathcal{O}_i)$.

Remark 2. Based on relation (12) between the covariant derivative and the principal connection form as well as the statement of Prop. 2, and with direct reference to an observation similar to that made in the proof of Prop. 2 and to Eqn. (8), we derive (in the formerly adopted notation), for the section

$$\sigma(x) = \tau_i^{-1}\big(x, \sigma_i(x)\big) = r_{\sigma_i(x)}\big(\tau_i^{-1}(x,e)\big) \equiv r_{\sigma_i(x)}\big(s_{(i)}(x)\big),$$

a relation

$$\begin{aligned}
&\big(\mathrm{id}_{\mathsf{T}^*B} \otimes \mathrm{pr}_2 \circ \widetilde{\mathrm{Vert}}_{\sigma(x)}^{-1}\big)\big(\nabla.\sigma(x)\big) = \big(\mathrm{id}_{\mathsf{T}^*B} \otimes \underline{\mathcal{A}}\big) \circ \mathsf{T}_x\sigma \\
\equiv\ &\big(\mathrm{id}_{\mathsf{T}^*B} \otimes \underline{\mathcal{A}}\big) \circ \mathsf{T}_x\big(r_{\sigma_i(\cdot)}\big(s_{(i)}(\cdot)\big)\big) \\
=\ &\big(\mathrm{id}_{\mathsf{T}^*B} \otimes \underline{\mathcal{A}}\big) \circ \big(\mathrm{id}_{\mathsf{T}^*B} \otimes \mathsf{T}_{s_{(i)}(x)} r_{\sigma_i(x)}\big) \circ \big(\mathsf{T}_x s_{(i)} + \sigma_i^*\theta_{\mathrm{R}}^A(x) \otimes_{\mathbb{R}} \mathcal{K}_{t_A}\big(s_{(i)}(x)\big)\big) \\
=\ &\big(\mathrm{id}_{\mathsf{T}^*B} \otimes \mathsf{T}_e\mathrm{Ad}_{\sigma_i(x)^{-1}}\big) \circ \big(\mathrm{id}_{\mathsf{T}^*B} \otimes \underline{\mathcal{A}}\big) \circ \big(\mathsf{T}_x s_{(i)} + \sigma_i^*\theta_{\mathrm{R}}^A(x) \otimes_{\mathbb{R}} \mathcal{K}_{t_A}\big(s_{(i)}(x)\big)\big) \\
=\ &\big(\mathrm{id}_{\mathsf{T}^*B} \otimes \mathsf{T}_e\mathrm{Ad}_{\sigma_i(x)^{-1}}\big) \circ \big(\mathsf{A}_i(x) \\
&+\sigma_i^*\theta_{\mathrm{R}}^A(x) \otimes_{\mathbb{R}} \mathrm{pr}_2 \circ \big(\mathcal{A} \circ \widetilde{\mathrm{Vert}}_{s_i(x)}\big) \circ \widetilde{\mathrm{Vert}}_{s_i(x)}^{-1}\big(\mathcal{K}_{t_A}\big(s_{(i)}(x)\big)\big)\big) \\
=\ &\big(\mathrm{id}_{\mathsf{T}^*B} \otimes \mathsf{T}_e\mathrm{Ad}_{\sigma_i(x)^{-1}}\big) \circ \big(\mathsf{A}_i(x) + \sigma_i^*\theta_{\mathrm{R}}^A(x) \otimes_{\mathbb{R}} \mathrm{pr}_2\big(s_{(i)}(x), t_A\big)\big) \\
\equiv\ &\big(\mathrm{id}_{\mathsf{T}^*B} \otimes \mathsf{T}_e\mathrm{Ad}_{\sigma_i(x)^{-1}}\big) \circ \big(\mathsf{A}_i(x) + (\sigma_i^* \otimes \mathrm{id}_{\mathfrak{g}})\theta_{\mathrm{R}}(x)\big) \\
=\ &\big(\mathrm{id}_{\mathsf{T}^*B} \otimes \mathsf{T}_e\mathrm{Ad}_{\sigma_i(x)^{-1}}\big) \circ \mathsf{A}_i(x) + (\sigma_i^* \otimes \mathrm{id}_{\mathfrak{g}})\theta_{\mathrm{L}}(x),
\end{aligned}$$

using Prop. 4.9 in the last step.

The relation between local potentials is determined in

Proposition 3. Adopt the hitherto notation. At an arbitrary point $x \in \mathcal{O}_{ij}$ from the intersection of domains of local trivialisations τ_i and τ_j of the principal bundle $(\mathsf{P}_{\mathrm{G}}, B, \mathrm{G}, \pi_{\mathsf{P}_{\mathrm{G}}})$ with transition maps $g_{ij} : \mathcal{O}_{ij} \longrightarrow \mathrm{G}$, the following identity obtains:

$$\mathsf{A}_j(x) = \big(\mathrm{id}_{\mathsf{T}^*B} \otimes \mathsf{T}_e\mathrm{Ad}_{g_{ji}(x)}\big)\mathsf{A}_i(x) + (g_{ij}^* \otimes \mathrm{id}_{\mathfrak{g}})\theta_L(x).$$

Proof: It suffices to note that

$$s_{(j)}(x) \equiv \tau_j^{-1}(x,e) = \tau_i^{-1}\big(x, g_{ij}(x)\big) = r_{g_{ij}(x)}\big(\tau_i^{-1}(x,e)\big) \equiv r_{g_{ij}(x)}\big(s_{(i)}(x)\big),$$

and subsequently carry out a calculation analogous to that from Remark 2. □

We are, now, ready to finally discuss in detail the presentation of the principal connection form under a local trivialisation.

Proposition 4. Adopt the notation of Def. 4. In the image of a local trivialisation $\tau_i : \pi_{\mathsf{P}_{\mathrm{G}}}^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times \mathrm{G}$, a principal connection form can be expressed in terms of a potential of the connection as

$$\underline{\mathcal{A}} \circ \mathsf{T}_{\tau_i^{-1}(x,g)}\tau_i^{-1} = \big(\mathrm{id}_{\mathsf{T}^*B} \otimes \mathsf{T}_e\mathrm{Ad}_{g^{-1}}\big) \circ \mathsf{A}_i(x) + \theta_{\mathrm{L}}(g),$$

where the object on the right-hand side of the equality sign ought to be viewed as a vector from the space $\mathsf{T}^*_{(x,g)}(\mathcal{O}_i \times \mathrm{G}) \otimes_{\mathbb{R}} \mathfrak{g} \equiv (\mathsf{T}^*_x B \oplus \mathsf{T}^*_g \mathrm{G}) \otimes_{\mathbb{R}} \mathfrak{g}$ at an arbitrary point $(x,g) \in \mathcal{O}_i \times \mathrm{G}$.

Proof: Taking into account the above decomposition of the space $\mathsf{T}^*_{(x,g)}(\mathcal{O}_i \times \mathrm{G})$, we may always write

$$\underline{\mathcal{A}} \circ \mathsf{T}_{\tau_i^{-1}(x,g)}\tau_i^{-1} = a_i(x;g) + \vartheta_i(g;x), \tag{14}$$

where $a_i(x;g) \in \mathsf{T}^*_x B \otimes_{\mathbb{R}} \mathfrak{g}$ and $\vartheta_i(g;x) \in \mathsf{T}^*_g \mathrm{G} \otimes_{\mathbb{R}} \mathfrak{g}$ are 1-forms with the properties

$$\forall_{(v,V) \in \mathsf{T}_x B \oplus \mathsf{T}_g \mathrm{G}} : V \lrcorner\, a_i(x;g) = 0_{\mathfrak{g}} = v \lrcorner\, \vartheta_i(g;x).$$

Upon decomposing the 1-form $\vartheta_i(g;x)$ in the basis of left-invariant forms on G,

$$\vartheta_i(g;x) =: \vartheta_{i\,A}^{B}(x,g) \triangleright \theta_{\mathrm{L}}^{A}(g) \otimes_{\mathbb{R}} t_B\,,$$

we first evaluate both sides of Eqn. (14) on a vertical vector $(0_{\mathsf{T}_xB}, L_A(g))$, whereby we obtain – invoking Def. (7) and Eqn. (8), as well as Eqn. (4) –

$$\begin{aligned}&\vartheta_{i\,A}^{B}(x,g) \triangleright t_A = L_A(g) \lrcorner\, \vartheta_i(g;x) = \big(0_{\mathsf{T}_xB}, L_A(g)\big) \lrcorner \big(a_i(x;g) + \vartheta_i(g;x)\big)\\ = \;& \underline{\mathcal{A}} \circ \mathsf{T}_{\tau_i^{-1}(x,g)}\tau_i^{-1}\big(0_{\mathsf{T}_xB}, L_A(g)\big) = \underline{\mathcal{A}}\big(\mathcal{K}_{t_A}\big(\tau_i^{-1}(x,g)\big)\big)\\ = \;& \underline{\mathcal{A}} \circ \widetilde{\mathrm{Vert}}_{\tau_i^{-1}(x,g)}(t_A) = \mathrm{pr}_2\big(\tau_i^{-1}(x,g), t_A\big) = t_A\,.\end{aligned}$$

From that, we infer

$$\vartheta_i(g;x) \equiv \theta_{\mathrm{L}}(g)\,.$$

In the next step, we replace the vertical vector with $(v, 0_{\mathsf{T}_g\mathrm{G}})$ and use the identity

$$\begin{aligned}&\mathsf{T}_{\tau_i^{-1}(x,g)}\tau_i^{-1}(v, 0_{\mathsf{T}_g\mathrm{G}}) = \mathsf{T}_{\tau_i^{-1}(x,g)}\tau_i^{-1} \circ \mathsf{T}_{(x,e)}(\mathrm{id}_B \times \wp_g)(v, 0_{\mathsf{T}_e\mathrm{G}})\\ = \;& \mathsf{T}_{(x,e)}\big(\tau_i^{-1} \circ (\mathrm{id}_B \times \wp_g)\big)(v, 0_{\mathsf{T}_e\mathrm{G}}) = \mathsf{T}_{(x,e)}\big(r_g \circ \tau_i^{-1}\big)(v, 0_{\mathsf{T}_e\mathrm{G}})\\ = \;& \mathsf{T}_{\tau_i^{-1}(x,e)}r_g \circ \mathsf{T}_{\tau_i^{-1}(x,g)}\tau_i^{-1}(v, 0_{\mathsf{T}_e\mathrm{G}}) \equiv \mathsf{T}_{\tau_i^{-1}(x,e)}r_g \circ \mathsf{T}_{\tau_i^{-1}(x,g)}\tau_i^{-1} \circ \mathsf{T}_x(\cdot, e)(v)\\ = \;& \mathsf{T}_{\tau_i^{-1}(x,e)}r_g \circ \mathsf{T}_x\big(\tau_i^{-1} \circ (\cdot, e)\big)(v) = \mathsf{T}_{\tau_i^{-1}(x,e)}r_g \circ \mathsf{T}_x s_{(i)}(v)\end{aligned}$$

that allows (owing to Eqn. (5)) to directly apply definition (13) of the potential of a principal connection and thus obtain

$$\begin{aligned}&v \lrcorner\, a_i(x;g) = (v, 0_{\mathsf{T}_g\mathrm{G}}) \lrcorner \big(a_i(x;g) + \vartheta_i(g;x)\big) = \underline{\mathcal{A}} \circ \mathsf{T}_{\tau_i^{-1}(x,g)}\tau_i^{-1}(v, 0_{\mathsf{T}_g\mathrm{G}})\\ = \;& = \underline{\mathcal{A}} \circ \mathsf{T}_{\tau_i^{-1}(x,e)}r_g \circ \mathsf{T}_x s_{(i)}(v) = \mathrm{pr}_2 \circ \big(r_r \times \mathsf{T}_e\mathrm{Ad}_{g^{-1}}\big) \circ \mathcal{A} \circ \mathsf{T}_x s_{(i)}(v)\\ = \;& \mathsf{T}_e\mathrm{Ad}_{g^{-1}} \circ \underline{\mathcal{A}} \circ \mathsf{T}_x s_{(i)}(v) \equiv \mathsf{T}_e\mathrm{Ad}_{g^{-1}} \circ \big(v \lrcorner\, \mathsf{A}_i(x)\big)\,,\end{aligned}$$

whence also – in view of the arbitrariness of v –

$$a_i(x;g) = \big(\mathrm{id}_{\mathsf{T}^*B} \otimes \mathsf{T}_e\mathrm{Ad}_{g^{-1}}\big) \circ \mathsf{A}_i(x)\,.$$

Thus, in the end, we recover the postulated local presentation of the principal connection form. □

After a long walk, we arrive at the fundamental result of our analysis, which is a generalisation of the clutching/reconstruction theorem for (principal) bundles to the setting of a principal bundle with a compatible connection.

Theorem 4 (The clutching theorem for a principal bundle with connection)**.** Adopt the hitherto notation. Every principal bundle $(\mathsf{P}_{\mathrm{G}}, B, \mathrm{G}, \pi_{\mathsf{P}_{\mathrm{G}}})$ with a principal connection, understood according to any one of the definitions 1, 2 i 3, determines, over its trivialising cover[2] $\{\mathcal{O}_i\}_{i\in I}$,

- a family $\{g_{ij}\}_{(i,j)\in\langle I^{\times 2}\rangle_{\mathcal{O}}}$ of locally smooth maps
$$g_{ij} \;:\; \mathcal{O}_{ij} \longrightarrow \mathrm{G}$$
satisfying the 1-cocle condition (1.12);
- a family $\{\mathsf{A}_i\}_{i\in I}$ of locally smooth 1-forms with values in the Lie algebra $\mathfrak{g}$ of the Lie group G
$$\mathsf{A}_i \in \Omega^1(\mathcal{O}_i) \otimes_{\mathbb{R}} \mathfrak{g}$$
satisfying the conditions
$$\forall_{(i,j)\in\langle I^{\times 2}\rangle_{\mathcal{O}},\ x\in\mathcal{O}_{ij}} \;:\; \mathsf{A}_j(x) = \big(\mathrm{id}_{\mathsf{T}^*B} \otimes \mathsf{T}_e\mathrm{Ad}_{g_{ji}(x)}\big) \circ \mathsf{A}_i(x) + (g_{ij}^* \otimes \mathrm{id}_{\mathfrak{g}})\theta_L(x)\,.$$

[2]We are *not* assuming the cover to be good.

Conversely, let $\mathscr{O} = \{\mathcal{O}_i\}_{i\in I}$ be an open cover of a smooth manifold B. Every pair of families of locally smooth maps

$$\big(\{g_{ij}\}_{(i,j)\in\langle I^{\times 2}\rangle_{\mathscr{O}}}, \{\mathsf{A}_k\}_{k\in I}\big)$$

associated with $\mathscr{O}$ and satisfying the above conditions determines – along the lines of (the constructive proof of) The Clutching Theorem of Lecture 1 (pp. 29–31) – a principal bundle $\mathsf{P}_{\rm G} = (\bigsqcup_{i\in I}(\mathcal{O}_i\times{\rm G}))/_{g_{..}}$ with the structural group ${\rm G}$ and with transition maps associated with $\mathcal{O}_{ij}$ given by g_{ij}, $(i,j)\in\langle I^{\times 2}\rangle_{\mathscr{O}}$ and with a principal connection form given by the formula

$$\mathcal{A}(v,V) := \big(x,g,v\lrcorner\big({\rm id}_{\mathsf{T}^*B}\otimes\mathsf{T}_e{\rm Ad}_{g^{-1}}\big)\circ\mathsf{A}_i(x)+V\lrcorner\,\theta_L(g)\big)\,,\qquad (x,g)\in\mathcal{O}_i\times{\rm G}\,, \tag{15}$$

written for an arbitrary vector $(v,V)\in\mathsf{T}_x\mathcal{O}_i\oplus\mathsf{T}_g{\rm G}\equiv\mathsf{T}_{(x,g)}(\mathcal{O}_i\times{\rm G})$. Whenever the local maps are local data of some principal bundle over B with a typical fibre ${\rm G}$, the latter bundle is isomorphic with the bundle determined by the g_{ij} and A_i.

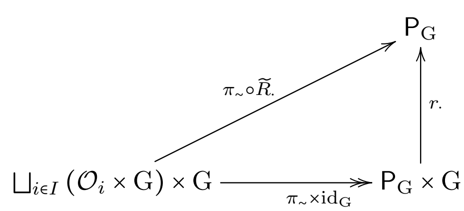
Proof: The first part of the statement follows directly from the previous analysis and from The Clutching Theorem. The only missing element is the action of the structural group ${\rm G}$ on the fibres of the bundle reconstructed according to the scheme presented in the proof of The Clutching Theorem,

$$\mathsf{P}_{\rm G} = \big(\bigsqcup_{i\in I}(\mathcal{O}_i\times{\rm G})\big)/_{g_{..}}\,.$$

Define a map

$$r_. \quad:\quad \mathsf{P}_{\rm G}\times{\rm G}\longrightarrow\mathsf{P}_{\rm G}\ :\ \big([(x,g,i)],h\big)\longmapsto[(x,g\cdot h,i)]\,,$$

whose smoothness is a consequence of surjective submersivity of the map $\pi_\sim$ defined analogously to the one from the proof of The Clutching Theorem, and of Prop. Niezb.10 – indeed, $r_.$ is the (unique) map closing the commutative diagram



, ,

in which

$$\widetilde{R}_.\ :\ \bigsqcup_{i\in I}(\mathcal{O}_i\times{\rm G})\times{\rm G}\longrightarrow\bigsqcup_{i\in I}(\mathcal{O}_i\times{\rm G})\ :\ \big((x,g,i),h\big)\longmapsto\big(x,g\cdot h,i\big)$$

is manifestly smooth.

At this stage, it remains to check that the connection form reconstructed from the local data is a globally smooth object (of class C^∞) with properties enumerated in Def. 3. Such a conclusion could, in principle, be drawn directly from Prop. 4, but, instead, we verify meticulously all of its properties. Thus, we should compare, at an arbitrary point $x\in\mathcal{O}_{ij}$, the result of the evaluation of the 1-form $\mathcal{A}$ expressed in terms of the local potential A_i on an arbitrary vector $(v,V)\in\mathsf{T}_x\mathcal{O}_i\oplus\mathsf{T}_g{\rm G}$ with the result of the evaluation of the same 1-form $\mathcal{A}$ expressed in terms of the local potential A_j on the image of that vector along the tangent of the transition mapping $(x,g)\longmapsto(x,g_{ji}(x)\cdot g)$. In so doing, instead of pushing forward (v,V) along the transition mapping, we may, equivalently, pull back the 1-form A_j along the same mapping, and subsequently evaluate it on (v,V). Thus, it suffices to compare the result of pulling back the 1-form $\mathcal{A}$ expressed in terms of the local potential A_j with the same 1-form expressed in terms of the local potential A_i, whereby, upon invoking Props. 3, 4.13 and 4.9, we obtain the desired result

$$\begin{aligned}&\big({\rm id}_{\mathsf{T}^*B}\otimes\mathsf{T}_e{\rm Ad}_{(g_{ji}(x)\cdot g)^{-1}}\big)\circ\mathsf{A}_j(x)+\theta_{\rm L}\big(g_{ji}(x)\cdot g\big)\\ =\ &\big({\rm id}_{\mathsf{T}^*B}\otimes\mathsf{T}_e{\rm Ad}_{g^{-1}}\big)\circ\big({\rm id}_{\mathsf{T}^*B}\otimes\mathsf{T}_e{\rm Ad}_{g_{ji}(x)^{-1}}\big)\circ\big(\big({\rm id}_{\mathsf{T}^*B}\otimes\mathsf{T}_e{\rm Ad}_{g_{ji}(x)}\big)\circ\mathsf{A}_i(x)\end{aligned}$$

$$
\begin{aligned}
&+(g_{ij}^*\otimes \mathrm{id}_{\mathfrak{g}})\theta_{\mathrm{L}}(x)\big)+\theta_{\mathrm{L}}(g)+\big(\mathrm{id}_{\mathsf{T}^*\mathrm{G}}\otimes \mathsf{T}_e\mathrm{Ad}_{g^{-1}}\big)\circ(g_{ji}^*\otimes \mathrm{id}_{\mathfrak{g}})\theta_{\mathrm{L}}(x)\\
=\ &\big(\mathrm{id}_{\mathsf{T}^*B}\otimes \mathsf{T}_e\mathrm{Ad}_{g^{-1}}\big)\circ \mathsf{A}_i(x)+\theta_{\mathrm{L}}(g)+\big(\mathrm{id}_{\mathsf{T}^*\mathrm{G}}\otimes \mathsf{T}_e\mathrm{Ad}_{g^{-1}}\big)\circ(g_{ji}^*\otimes \mathrm{id}_{\mathfrak{g}})\theta_{\mathrm{L}}(x)\\
&+\big(\mathrm{id}_{\mathsf{T}^*B}\otimes \mathsf{T}_e\mathrm{Ad}_{g^{-1}}\big)\circ\big(\mathrm{id}_{\mathsf{T}^*B}\otimes \mathsf{T}_e\mathrm{Ad}_{g_{ji}(x)^{-1}}\big)\circ(g_{ji}^*\otimes \mathrm{id}_{\mathfrak{g}})\circ(\mathrm{Inv}^*\otimes \mathrm{id}_{\mathfrak{g}})\theta_L(x)\\
=\ &\big(\mathrm{id}_{\mathsf{T}^*B}\otimes \mathsf{T}_e\mathrm{Ad}_{g^{-1}}\big)\circ \mathsf{A}_i(x)+\theta_{\mathrm{L}}(g)+\big(\mathrm{id}_{\mathsf{T}^*\mathrm{G}}\otimes \mathsf{T}_e\mathrm{Ad}_{g^{-1}}\big)\circ(g_{ji}^*\otimes \mathrm{id}_{\mathfrak{g}})\theta_{\mathrm{L}}(x)\\
&-\big(\mathrm{id}_{\mathsf{T}^*B}\otimes \mathsf{T}_e\mathrm{Ad}_{g^{-1}}\big)\circ\big(\mathrm{id}_{\mathsf{T}^*B}\otimes \mathsf{T}_e\mathrm{Ad}_{g_{ji}(x)^{-1}}\big)\circ(g_{ji}^*\otimes \mathrm{id}_{\mathfrak{g}})\theta_{\mathrm{R}}(x)\\
=\ &\big(\mathrm{id}_{\mathsf{T}^*B}\otimes \mathsf{T}_e\mathrm{Ad}_{g^{-1}}\big)\circ \mathsf{A}_i(x)+\theta_{\mathrm{L}}(g)+\big(\mathrm{id}_{\mathsf{T}^*\mathrm{G}}\otimes \mathsf{T}_e\mathrm{Ad}_{g^{-1}}\big)\circ(g_{ji}^*\otimes \mathrm{id}_{\mathfrak{g}})\theta_{\mathrm{L}}(x)\\
&-\big(\mathrm{id}_{\mathsf{T}^*B}\otimes \mathsf{T}_e\mathrm{Ad}_{g^{-1}}\big)\circ\big(\mathrm{id}_{\mathsf{T}^*B}\otimes \mathsf{T}_e\mathrm{Ad}_{g_{ji}(x)^{-1}}\big)\circ\big(\mathrm{id}_{\mathsf{T}^*\mathrm{G}}\otimes \mathsf{T}_e\mathrm{Ad}_{g_{ji}(x)}\big)\\
&\circ(g_{ji}^*\otimes \mathrm{id}_{\mathfrak{g}})\circ\theta_{\mathrm{L}}(x)=\big(\mathrm{id}_{\mathsf{T}^*B}\otimes \mathsf{T}_e\mathrm{Ad}_{g^{-1}}\big)\circ \mathsf{A}_i(x)+\theta_{\mathrm{L}}(g)\,.
\end{aligned}
$$

The other one of the desired properties, Eqn. (4), is checked *via* direct reference to Eqn. (8) upon noting, first, that the structure of (the fibre of) the bundle reconstructed from the local data in the constructive proof of The Clutching Theorem leads to the identification $\mathcal{K}_X(x,g)\equiv\big(0_{\mathsf{T}_xB},L_X(g)\big)$ in the domain $\pi_{\mathsf{P}_{\mathrm{G}}}^{-1}(\mathcal{O}_i)\ni(x,g)$ of a local trivialisation of the reconstructed bundle P_{G}. Under this trivialisation, we obtain – for an arbitrary vector $X\in\mathfrak{g}$ – the identity

$$
\mathcal{A}\circ\widetilde{\mathrm{Vert}}_{(x,g)}(X)\equiv\mathcal{A}\big(0_{\mathsf{T}_xB},L_X(g)\big)=\big(x,g,L_X\lrcorner\,\theta_L(g)\big)\equiv(x,g,X)\,.
$$

Finally, we verify G-equivariance of the postulated principal connection form. Thus, we take into account, once more, Prop. 4.9 in the equality

$$
\begin{aligned}
&\mathcal{A}\circ\mathsf{T}_{(x,g)}(\mathrm{id}_B\times\wp_h)(v,V)=\mathcal{A}\circ\big(\mathrm{id}_{\mathsf{T}B}\oplus\mathsf{T}_g\wp_h\big)(v,V)\\
=\ &\big(x,g\cdot h,v\lrcorner\big(\mathrm{id}_{\mathsf{T}^*B}\otimes\mathsf{T}_e\mathrm{Ad}_{(g\cdot h)^{-1}}\big)\circ\mathsf{A}_i(x)+\mathsf{T}_g\wp_h(V)\lrcorner\,\theta_L(g\cdot h)\big)\,,
\end{aligned}
$$

whereupon we arrive at

$$
\mathsf{T}_g\wp_h(V)\lrcorner\,\theta_L(g\cdot h)=V\lrcorner\,(\wp_h^*\otimes\mathrm{id}_{\mathfrak{g}})\theta_{\mathrm{L}}(g)=V\lrcorner\big(\mathrm{id}_{\mathsf{T}^*\mathrm{G}}\otimes\mathsf{T}_e\mathrm{Ad}_{h^{-1}}\big)\circ\theta_{\mathrm{L}}(g)\,,
$$

and so also

$$
\begin{aligned}
\mathcal{A}\circ\mathsf{T}_{(x,g)}(\mathrm{id}_B\times\wp_h)(v,V)\ &=\ \big((\mathrm{id}_B\times\wp_h)\times\big(\mathrm{id}_{\mathsf{T}^*B}\otimes\mathsf{T}_e\mathrm{Ad}_{h^{-1}}\big)\big)\big(x,g,\\
&\qquad v\lrcorner\big(\mathrm{id}_{\mathsf{T}^*B}\otimes\mathsf{T}_e\mathrm{Ad}_{g^{-1}}\big)\circ\mathsf{A}_i(x)+V\lrcorner\,\theta_L(g)\big)\\
&\equiv\ \big((\mathrm{id}_B\times\wp_h)\times\big(\mathrm{id}_{\mathsf{T}^*B}\otimes\mathsf{T}_e\mathrm{Ad}_{h^{-1}}\big)\big)\circ\mathcal{A}(v,V)\,.
\end{aligned}
$$

□

The same scheme may subsequently be applied to morphisms.

Theorem 5. Adopt the hitherto notation and let $(\mathsf{P}_{\mathrm{G}}^\alpha,B,\mathrm{G},\pi_{\mathsf{P}_{\mathrm{G}}^\alpha})$, $\alpha\in\{1,2\}$ be two principal bundles (with a common structural group G) with principal connections over a common base B, with the respective local trivialisations $\tau_i^\alpha\ :\ \pi_{\mathsf{P}_{\mathrm{G}}^\alpha}^{-1}(\mathcal{O}_i)\xrightarrow{\cong}\mathcal{O}_i\times\mathrm{G}$ associated with a common trivialising cover $\mathscr{O}=\{\mathcal{O}_i\}_{i\in I}$. Introduce two families of local sections:

$$
s_{(i)}^\alpha:=\tau_i^{\alpha\,-1}(\cdot,e)\ :\ \mathcal{O}_i\longrightarrow\mathsf{P}_{\mathrm{G}}^\alpha\,,\quad\alpha\in\{1,2\}\,,
$$

with the corresponding transition maps $g_{ij}^\alpha\ :\ \mathcal{O}_{ij}\longrightarrow\mathrm{G}$, $\alpha\in\{1,2\}$ and 1-forms $\mathsf{A}_i^\alpha\in\Omega^1(\mathcal{O}_i)\otimes_{\mathbb{R}}\mathfrak{g}$, $\alpha\in\{1,2\}$ with values in the Lie algebra $\mathfrak{g}$ of the Lie group G. An arbitrary morphism of principal bundles with connection

$$\begin{array}{ccc} \mathsf{P}^1_{\mathrm{G}} & \xrightarrow{\ \Phi\ } & \mathsf{P}^2_{\mathrm{G}} \\ {\scriptstyle \pi_{\mathsf{P}^1_{\mathrm{G}}}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathsf{P}^2_{\mathrm{G}}}} \\ B & \underset{\mathrm{id}_B}{=\!=\!=} & B \end{array}$$

determines a family $\{h_i\}_{i\in I}$ of maps (locally) of class C^∞

$$h_i\ :\ \mathcal{O}_i \longrightarrow \mathrm{G}\,,\quad i\in I$$

satisfying conditions: (5.3) and

$$\forall_{x\in\mathcal{O}_i}\ :\ \mathsf{A}^2_i(x) = \big(\mathrm{id}_{\mathsf{T}^*B}\otimes \mathsf{T}_e\mathrm{Ad}_{h_i(x)}\big)\circ \mathsf{A}^1_i(x) + \big((\mathrm{Inv}\circ h_i)^*\otimes \mathrm{id}_{\mathfrak{g}}\big)\theta_{\mathrm{L}}(x)\,. \tag{16}$$

Conversely, every such family determines a unique morphism of the type described.

Proof: On taking into account Thm. 6.1, it remains to verify the postulated transformation formula for the potential of the connection form. To this end, we invoke condition (PFCM4) from Thm. 3 and substitute it in definition (13) of the potential, in which we also use the simple relation

$$\tau_i^{2\,-1}\big(x,h_i(x)\big) = \Phi\big(\tau_i^{1\,-1}(x,e)\big)\,,$$

which we rewrite in the present notation as

$$\Phi\circ s^1_{(i)}(x) = r_{h_i(x)}\big(s^2_{(i)}(x)\big)\,.$$

In this manner, we obtain, on the one hand, the equality

$$\begin{aligned} \big(\mathrm{id}_{\mathsf{T}^*B}\otimes \underline{\mathcal{A}}_2\circ \mathsf{T}_{s^1_{(i)}(x)}\Phi\big)\circ \mathsf{T}_x s^1_{(i)} &= \big(\mathrm{id}_{\mathsf{T}^*B}\otimes \mathrm{pr}_2\circ(\Phi\times\mathrm{id}_{\mathfrak{g}})\circ\mathcal{A}_1\big)\circ \mathsf{T}_x s^1_{(i)} \\ &= \big(\mathrm{id}_{\mathsf{T}^*B}\otimes \underline{\mathcal{A}}_1\big)\circ \mathsf{T}_x s^1_{(i)} \equiv \mathsf{A}^1_i(x)\,, \end{aligned}$$

and, on the other hand, in the light of the detailed calculations carried out in the body of Remark 2,

$$\begin{aligned} \big(\mathrm{id}_{\mathsf{T}^*B}\otimes \underline{\mathcal{A}}_2\circ \mathsf{T}_{s^1_{(i)}(x)}\Phi\big)\circ \mathsf{T}_x s^1_{(i)} &= (\mathrm{id}_{\mathsf{T}^*B}\otimes \underline{\mathcal{A}}_2)\circ \mathsf{T}_x\big(r_{h_i(\cdot)}\big(s^2_{(i)}(\cdot)\big)\big) \\ &= \big(\mathrm{id}_{\mathsf{T}^*B}\otimes \mathsf{T}_e\mathrm{Ad}_{h_i(x)^{-1}}\big)\circ \mathsf{A}^2_i(x) + (h_i^*\otimes\mathrm{id}_{\mathfrak{g}})\theta_{\mathrm{L}}(x)\,. \end{aligned}$$

Putting the above together, we obtain – through reference to Prop. 4.9 – the desired identity

$$\begin{aligned} \mathsf{A}^2_i(x) &= \big(\mathrm{id}_{\mathsf{T}^*B}\otimes \mathsf{T}_e\mathrm{Ad}_{h_i(x)}\big)\circ \mathsf{A}^1_i(x) - \big(\mathrm{id}_{\mathsf{T}^*B}\otimes \mathsf{T}_e\mathrm{Ad}_{h_i(x)}\big)\circ(h_i^*\otimes\mathrm{id}_{\mathfrak{g}})\theta_{\mathrm{L}}(x) \\ &= \big(\mathrm{id}_{\mathsf{T}^*B}\otimes \mathsf{T}_e\mathrm{Ad}_{h_i(x)}\big)\circ \mathsf{A}^1_i(x) - (h_i^*\otimes\mathrm{id}_{\mathfrak{g}})\theta_{\mathrm{R}}(x) \\ &= \big(\mathrm{id}_{\mathsf{T}^*B}\otimes \mathsf{T}_e\mathrm{Ad}_{h_i(x)}\big)\circ \mathsf{A}^1_i(x) + \big((\mathrm{Inv}\circ h_i)^*\otimes\mathrm{id}_{\mathfrak{g}}\big)\theta_{\mathrm{R}}(x)\,. \end{aligned}$$

And conversely, having a family $h_i\ :\ \mathcal{O}_i \longrightarrow \mathrm{G},\ i\in I$ of maps as in the statement of the proposition under consideration, we define local maps

$$\Phi_i\quad:\quad \pi^{-1}_{\mathsf{P}^1_{\mathrm{G}}}(\mathcal{O}_i)\longrightarrow \pi^{-1}_{\mathsf{P}^2_{\mathrm{G}}}(\mathcal{O}_i)\ :\ \tau_i^{1\,-1}(x,g)\longmapsto \tau_i^{2\,-1}\big(x,h_i(x)\cdot g\big)\,,\quad i\in I\,.$$

We readily convince ourselves that these are, in fact, restrictions of a globally smooth map $\Phi\ :\ \mathsf{P}^1_{\mathrm{G}}\longrightarrow \mathsf{P}^2_{\mathrm{G}}$ to the respective elements of the trivialising cover, $\Phi\restriction_{\pi^{-1}_{\mathsf{P}^1_{\mathrm{G}}}(\mathcal{O}_i)} = \Phi_i$, as, indeed, at an arbitrary point $x\in\mathcal{O}_{ij}$, we obtain the equality

$$\begin{aligned} \Phi_j\circ s^1_{(i)}(x) &= \Phi_j\circ\tau_j^{1\,-1}\big(x,g^1_{ji}(x)\big) = \tau_j^{2\,-1}\big(x,h_j(x)\cdot g^1_{ji}(x)\big) \\ &= \tau_j^{2\,-1}\big(x,g^2_{ji}(x)\cdot h_i(x)\big) = \tau_i^{1\,-1}\big(x,h_i(x)\big)\equiv \Phi_i\circ s^1_{(i)}(x)\,. \end{aligned}$$

Moreover, the maps Φ_i are G-equivariant, which we verify for arbitrary $x \in \mathcal{O}_i$ and $g,h \in \mathrm{G}$,

$$\begin{aligned}\Phi_i \circ r^1_h \circ \tau_i^{1\,-1}(x,g) &= \Phi_i \circ \tau_i^{1\,-1}(x,g\cdot h) = \tau_i^{2\,-1}\big(x,h_i(x)\cdot g\cdot h\big)\\ &= r^2_h\big(\tau_i^{2\,-1}\big(x,h_i(x)\cdot g\big)\big) \equiv r^2_h \circ \Phi_i \circ \tau_i^{1\,-1}(x,g)\,.\end{aligned}$$

We conclude the proof by checking condition (PFCM4) of Thm. 3. We do that in the picture of a local trivialisation τ_i, with the help of Prop. 4. Thus, we establish the equality

$$\begin{aligned}&\underline{\mathcal{A}}_2 \circ \mathsf{T}\Phi \circ \mathsf{T}_{\tau_i^{1\,-1}(x,g)}\tau_i^{1\,-1}\\ \equiv\ &\big(\underline{\mathcal{A}}_2 \circ \mathsf{T}_{\tau_i^{2\,-1}(x,g)}\tau_i^{2\,-1}\big) \circ \mathsf{T}_{\tau_i^{1\,-1}(x,g)}\big(\tau_i^2 \circ \Phi \circ \tau_i^{1\,-1}\big)\\ \equiv\ &\big(\tau_i^2 \circ \Phi \circ \tau_i^{1\,-1}\big)^*\big(\big(\mathrm{id}_{\mathsf{T}^*B} \otimes \mathsf{T}_e\mathrm{Ad}_{\mathrm{Inv}\circ\mathrm{pr}_2(\cdot)}\big) \circ \mathrm{pr}_1^*\mathsf{A}_i^2 + \mathrm{pr}_2^*\theta_{\mathrm{L}}\big)(x,g)\\ =\ &\big(\big(\mathrm{id}_{\mathsf{T}^*B} \otimes \mathsf{T}_e\mathrm{Ad}_{\mathrm{Inv}\circ\mathrm{pr}_2(\cdot)}\big) \circ \mathrm{pr}_1^*\mathsf{A}_i^2 + \mathrm{pr}_2^*\theta_{\mathrm{L}}\big)\big(x,h_i(x)\cdot g\big)\\ =\ &\big(\mathrm{id}_{\mathsf{T}^*B} \otimes \mathsf{T}_e\mathrm{Ad}_{(h_i(x)\cdot g)^{-1}}\big) \circ \mathsf{A}_i^2(x) + \theta_{\mathrm{L}}\big(h_i(x)\cdot g\big)\\ =\ &\big(\mathrm{id}_{\mathsf{T}^*B} \otimes \mathsf{T}_e\mathrm{Ad}_{g^{-1}}\big) \circ \big(\mathrm{id}_{\mathsf{T}^*B} \otimes \mathsf{T}_e\mathrm{Ad}_{h_i(x)^{-1}}\big) \circ \mathsf{A}_i^2(x) + \theta_{\mathrm{L}}\big(h_i(x)\cdot g\big)\,,\end{aligned}$$

which we may, in the light of the assumptions made and of Prop. 4.13, rewrite in the desired form

$$\begin{aligned}&\underline{\mathcal{A}}_2 \circ \mathsf{T}\Phi \circ \mathsf{T}_{\tau_i^{1\,-1}(x,g)}\tau_i^{1\,-1}\\ =\ &\big(\mathrm{id}_{\mathsf{T}^*B} \otimes \mathsf{T}_e\mathrm{Ad}_{g^{-1}}\big) \circ \big(\mathsf{A}_i^1(x) - (h_i^* \otimes \mathrm{id}_{\mathfrak{g}})\theta_{\mathrm{L}}(x)\big)\\ &+\theta_{\mathrm{L}}(g) + \big(\mathrm{id}_{\mathsf{T}^*B} \otimes \mathsf{T}_e\mathrm{Ad}_{g^{-1}}\big) \circ (h_i^* \otimes \mathrm{id}_{\mathfrak{g}})\theta_{\mathrm{L}}(x)\\ =\ &\big(\mathrm{id}_{\mathsf{T}^*B} \otimes \mathsf{T}_e\mathrm{Ad}_{g^{-1}}\big) \circ \mathsf{A}_i^1(x) + \theta_{\mathrm{L}}(g) \equiv \underline{\mathcal{A}}_1 \circ \mathsf{T}_{\tau_i^{1\,-1}(x,g)}\tau_i^{1\,-1}\\ =\ &\mathrm{pr}_2 \circ (\Phi \times \mathrm{id}_{\mathfrak{g}}) \circ \mathcal{A}_1 \circ \mathsf{T}_{\tau_i^{1\,-1}(x,g)}\tau_i^{1\,-1}\end{aligned}$$

□

By way of a completion of the main discussion, particularly relevant from the point of view of the various field-theoretic applications of the formalism developed herein, we add

Definition 5. Adopt the notation of Def. 4. Potentials A_i of a principal connection on a principal bundle P_{G} determine a family of locally smooth 2-forms on the base with values in the Lie algebra $\mathfrak{g}$:

$$\mathsf{F}_i := \mathsf{d}\mathsf{A}_i + \mathsf{A}_i \wedge \mathsf{A}_i\,, \quad i \in I$$

that satisfy the conditions

$$\forall_{(i,j)\in\langle I^{\times 2}\rangle_{\mathscr{O}},\ x\in\mathcal{O}_{ij}}\ :\ \mathsf{F}_j(x) = \big(\mathrm{id}_{\mathsf{T}^*B} \otimes \mathrm{Ad}_{g_{ij}(x)^{-1}}\big)\,\mathsf{F}_i(x)\,.$$

These are called **local 2-forms of curvature of the principal connection on the bundle** P_{G}. A **flat principal connection** is one, whose associated local 2-forms of curvature are identically zero.

Homework 1. Prove the above relation.

Our lecture is concluded with an elementary constatation of transformation properties of the curvature with respect to morphisms.

Proposition 5. Adopt the notation of Def. 5 and let $\{\mathsf{F}_i^\alpha\}_{i\in I}$, $\alpha \in \{1,2\}$ be local 2-forms of curvature of a principal connection on principal bundles over a fixed (common) base B, associated with a common trivialising cover $\{\mathcal{O}_i\}_{i\in I}$, assuming existence of an isomorphims between the bundles, covering the identity on the base, with local data $\{h_i\}_{i\in I}$ as defined in the statement of Thm. 5. Then, the following idetities obtain:

$$\mathsf{F}_i^2 = \big(\mathrm{id}_{\mathsf{T}^*B} \otimes \mathrm{Ad}_{h_i}\big)\mathsf{F}_i^1 \,.$$

Proof: Left to the Reader. □