Our goal: understanding the theory (1)
of physical fields (such as, e.g., scalar, electromagnetic / Yang-Mills, spinor &c. ones) over spacetime in the lagrangean formulation & - subsequently - its derived canonical description, with emphasis on the underlying geometry.

The path starts at classical mechanics which we reformulate conveniently so that it guides our intuition...

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I The lagrangean formulation:

We define the theory of interest through Principle of Least Action for a given lagrangean (density)

$$L(t, \vec{q}, \dot{\vec{q}}),$$

taking numerical values on

$t \in P \subset \mathbb{R}$, $P \supset [t_0, t_1]$ (TIME), $\vec{q} \in [P, M]$, with $M \subseteq \mathbb{R}^N$ (here) (GENERALISED COÖRDS / POSITIONS)

Above, the $\dot{\vec{q}}$ are GENERALISED VELOCITY

The aforementioned ACTION is the functional

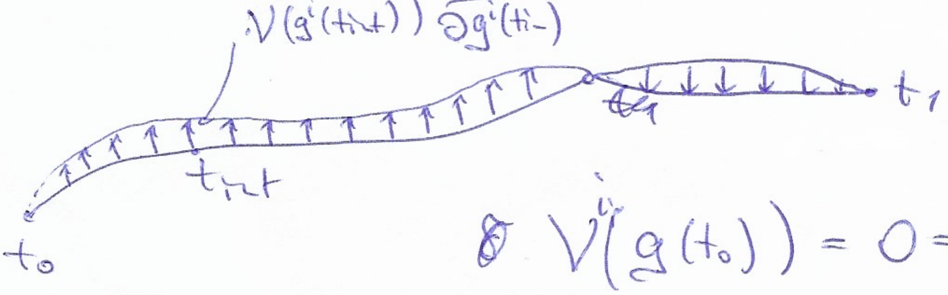
$$S_{[t_0,t_1]}[\cdot] : [\mathcal{P}, M] \longrightarrow \mathbb{R}$$

$$: q \longmapsto \int_{t_0}^{t_1} dt\, L(t, \vec{q}, \dot{\vec{q}})$$

that singles out CLASSICAL PATHS

given by its critical points q_*, i.e.,
(with $\delta^2 S > 0$)

and that for ~~any~~ $V(q(\cdot)) = V^i_{(q(\cdot))} \frac{\delta}{\delta q^i(\cdot)}$



& $V^i(q(t_0)) = 0 = V^i(q(t_1))$

We obtain

$$V \lrcorner\, \delta S_{[t_0,t_1]}[q_*] = 0 \qquad \forall V$$

Here, $q \longmapsto q + V$

$\Downarrow$

$\dot{q} \longmapsto \dot{q} + \dot{V}$, so that

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$$0 = \int_{t_0}^{t_1} dt \left[V^i(q(t)) \frac{\partial L}{\partial q^i}(t, q(t), \dot q(t)) + \dot V^i(q(t)) \frac{\partial L}{\partial \dot q^i}(t, q(t), \dot q(t)) \right]$$

$$= V^i(q(t)) \frac{\partial L}{\partial \dot q^i}(t, q(t), \dot q(t)) \Big|_{t=t_0}^{t=t_1}$$

$$+ \int_{t_0}^{t_1} dt\, V^i(q(t)) \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot q^i} \right)(t, q(t), \dot q(t)),$$

whence – ~~for~~ $V^i(q(t_0)) = 0 = V^i(q(t_1))$ – ($\&$ V^i arbitrary in $]t_0, t_1[$)

we obtain –

The Euler – Lagrange Equations :

$$\boxed{\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot q^i} = 0, \quad i \in \overline{1,N}} \quad \text{(EL)}$$

NB. $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^i}\right) \equiv \frac{\partial^2 L}{\partial t \partial \dot q^i} + \frac{\partial^2 L}{\partial q^j \partial \dot q^i} \dot q^j + \frac{\partial^2 L}{\partial \dot q^j \partial \dot q^i} \ddot q^j$

Thus, as long as

REGULARITY COND$^{\underline{n}}$ $\quad \boxed{\det\left(\frac{\partial^2 L}{\partial \dot q^i \partial \dot q^j}\right) \neq 0},$

(EL) are Picard-Lindelöf ~~ODE~~ ODE's of 2$^{\underline{nd}}$ order, with sol$^{\underline{n}}$s determined by ($\&$ depending smoothly on) INITIAL COND$^{\underline{n}}$s $(q(t_0), \dot q(t_0))$

Example: A point-like particle (4)
in Euclidean ${}^{r}\mathcal{M}$, $\dim \mathcal{M} = N$, moving in a potential V,

$$L(t, q, \dot{q}) = \frac{1}{2} m \|\dot{q}\|^2 - V(q),$$

with $\frac{\partial L}{\partial \dot{q}^i} = m \dot{q}^i \delta_{ij}$, $\frac{\partial L}{\partial q^i} \equiv \partial_i V(q)$,

so that

$$(EL): \quad m \ddot{q}^i \delta_{ij} = - \partial_j V(q), \quad j \in \overline{1,N}$$

The Newton Eq^{n}s

we shall, next, change the perspective...

X

II The hamiltonian approach:

Define covectors with components

$$p \equiv p_i \delta q^i \qquad p_i := \frac{\partial L}{\partial \dot{q}^i}$$

Treating $F_{t,q}(\dot{q}, p) \equiv \left(\frac{\partial L}{\partial \dot{q}^i}(t, q, \dot{q}) - p_i \right)_{i \in \overline{1,N}}$

as an implicit rel^{n}

between $\dot{q}$ & p,

we conclude that the ~~app~~ assignment

$$\dot{q} \longmapsto p$$

is invertible iff $0 \neq \det \left(\frac{\partial F_i}{\partial \dot{q}^j} \right) = \det \left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right)$

i.e., under the assumption of regularity of L, we may equivalently describe the physical system by (5)

$$\begin{cases} q^i & - \text{ GENERALISED POSITIONS (as earlier)} \\ p_i & - \quad \text{———}\text{"}\text{———} \text{ MOMENTA} \end{cases}$$

Write

$$\underbrace{H(t,q,p) := \dot{q}^{(t,q,p)} \lrcorner\, p}_{\text{HAMILTONIAN}} - L(t,q,\dot{q}(t,q,p))$$

$$\dot{q} \lrcorner\, p \equiv \dot{q}^i p_i$$

to obtain

$$\delta H(t,q,p) \equiv \delta t \frac{\partial H}{\partial t} + \delta q^i \frac{\partial H}{\partial q^i} + \delta p_i \frac{\partial H}{\partial p_i}$$

$$\equiv \delta t \left(\cancel{\ddot{q}^i p_i} - \frac{\partial L}{\partial t} - \cancel{\frac{\partial L}{\partial \dot{q}^i} \ddot{q}^i} \right)$$

$$+ \delta q^i \left(\cancel{\frac{\partial \dot{q}^j}{\partial q^i} p_j} - \frac{\partial L}{\partial q^i} - \cancel{\frac{\partial L}{\partial \dot{q}^j} \frac{\partial \dot{q}^j}{\partial q^i}} \right)$$

$$+ \delta p_i \left(\cancel{\frac{\partial \dot{q}^j}{\partial p_i} p_j} + \dot{q}^i - \cancel{\frac{\partial L}{\partial \dot{q}^j} \frac{\partial \dot{q}^j}{\partial p_i}} \right)$$

$$\equiv +\delta t \left(-\frac{\partial L}{\partial t} \right) + \delta q^i \left(-\frac{\partial L}{\partial q^i} \right) + \delta p_i \dot{q}^i$$

Thus,

$$(H)\quad \begin{cases} \dfrac{\partial H}{\partial q^i} = -\dfrac{\partial L}{\partial q^i} = -\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}^i}\right) = -\dot{p}_i \\[2ex] \dfrac{\partial H}{\partial p_i} = \dot{q}^i \end{cases}$$

HAMILTON Eqⁿs $\overset{\text{regularity}}{\Longleftrightarrow}$ E-L Eqⁿs

Example — ctd.

Here, $p = m\dot{q}$, &c so

$$H(t,q,p) = \dot{q}^i p_i - \frac{1}{2} m \|\dot{q}\|^2 + V(q)$$

$$= \underbrace{\frac{\|p\|^2}{2m}}_{\text{kinetic energy}} + \underbrace{V(q)}_{\text{potential energy}} \equiv \text{total energy}$$

Hamilton eqⁿs:

$$\begin{cases} \dfrac{dq}{dt} = \dfrac{p}{m} \\[2ex] \dfrac{dp}{dt} = -\nabla V \end{cases}$$

This is where the standard course on classical mechanics leaves off...

III The poissonian structure on the phase space

Def. The PHASE SPACE $\mathcal{P}_L$ of the physical model is the space of solutions to (EL) $\underset{\text{regularity (assumed)}}{\Longleftrightarrow}$ (H). ~~This space~~

Clearly, $\mathcal{P}_L$ is coordinatised by $(q_i(t), \dot{q}_i(t))$ for any fixed $t \in \mathcal{T}$, or – in the regular case – by $(q(t), p(t)) \in T^*M$

$\Rightarrow$ functions on $\mathcal{P}_L$ are

$$C^\infty(T^*M, \mathbb{R}).$$

On this $\mathbb{R}$-algebra, we define

$$\{\cdot,\cdot\} : C^\infty(T^*M, \mathbb{R}) \times C^\infty(T^*M, \mathbb{R}) \to C^\infty(T^*M, \mathbb{R})$$

$$(f, g) \longmapsto \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial \dot{q}_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial \dot{q}_i}$$

POISSON BRACKET,

ˏmk properties:

(PB1) $\forall f, g \in C^\infty(T^*M, \mathbb{R})$: $\{g, f\} = -\{f, g\}$
(antisymmetry)
⇓ bi-
(linearity)

(PB2) $\forall \lambda_1, \lambda_2 \in \mathbb{R}\ \forall f_1, f_2, g \in C^\infty(T^*M, \mathbb{R})$:

$$\{\lambda_1 f_1 + \lambda_2 f_2, g\} = \lambda_1 \{f_1, g\} + \lambda_2 \{f_2, g\}$$

pointwise bilinear structure on $C^\infty(T^*M, \mathbb{R})$

(PB3) $\forall f, g, h \in C^\infty(T^*M, \mathbb{R})$: (the Leibniz rule)

$$\{f, g \cdot h\} = \{f, g\} h + g \{f, h\}$$

pointwise multiplication ($\Leftrightarrow$ $\{f, \cdot\}$ is a derivation on $C^\infty(T^*M, \mathbb{R})$)

(PB4) $\forall f, g, h \in C^\infty(T^*M, \mathbb{R})$:

JACOBIATOR

$$\mathrm{Jac}(f, g, h) := \{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\}$$

$$\boxed{\mathrm{Jac}(f, g, h) \equiv 0}$$ the Jacobi identity

~~Naturality~~ of $\{\cdot,\cdot\}$ can be seen (9)
Structurality

as follows : (H) rewrite as

$$(H\text{-}P)\begin{cases}\dfrac{dq^i}{dt}=\{q^i,H\}\\[2ex]\dfrac{dp_i}{dt}=\{p_i,H\}\end{cases}$$

More generally, ~~as~~ for any

$$f\in C^\infty(\mathbb{R}_t\times T^*M,\mathbb{R}),$$

we find

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial q^i}\frac{dq^i}{dt}+\frac{\partial f}{\partial p_i}\frac{dp_i}{dt}$$

$$=\frac{\partial f}{\partial t}+\{f,H\}$$

We shall, next, geometrize the canonical analysis, with view to field-theoretic applications or ... quantization.

guiding intuition : $\{f, \cdot\} \in \mathrm{Der}\, C^\infty(\mathcal{P}_L, \mathbb{R})$

But for any $X \in \mathrm{Ob}\,\mathbf{Man}$:

$$\Gamma(TX) \equiv \mathrm{Der}\, C^\infty(X, \mathbb{R}) .$$

Hence $\{f, \cdot\}$ should be related to a vector field on $\mathcal{P}_L$ ...

Recapitulation of rudiments of differential geometry :

X - manifold

$\Updownarrow$

topological space locally modelled on $(\mathbb{R}^n, \mathcal{T}(\mathbb{R}^n))$ e.g., generated by o-balls

$\exists\ \{\mathcal{O}_i\}_{i\in I} \equiv \mathcal{O}_X$ - open cover of X , $\bigcup_{i\in I} \mathcal{O}_i \equiv X$
$\cap$
$\mathcal{T}(X)$

LOCAL CHARTS

s.th. $\forall\, i \in I\ \exists\ \kappa_i$ ~~$\in$~~ : $\mathcal{O}_i \xrightarrow{\cong} U_i \subset \mathbb{R}^n$ (homeomorphisms)

s.th. $\forall\, i,j \in I : \big(\mathcal{O}_i \cap \mathcal{O}_j \equiv \mathcal{O}_{ij} \neq \emptyset \implies \kappa_i \circ \kappa_j^{-1}$ is a diffeomorphism

~~On~~ X thus structured, we have (11)

the following entities ...

$\cancel{TM} \equiv$ (smooth) derivations of $C^\infty(X,R)$

or ~~res~~ Leibniz rule

smooth assignments co-tangency

or local coords — of a tangent vector (class of paths)

to a point in $\cancel{M}$

~~$T^*X \cong$~~

<u>NB.</u> TX is a (locally free,) ~~algebra~~

$C^\infty(X,R)$ - module!

$T^*\cancel{M} \equiv C^\infty(X,R)$ - linear maps

$TX \longrightarrow C^\infty(X,R)$.

or

smooth assignments

as above — of a covector (dual to a tangent vector)

to a point in X

On the above, we have

~~some~~ natural 'operations' ...

1. Lie bracket of vector fields: ℝ

$$[\cdot,\cdot] : \Gamma(TX)\times\Gamma(TX) \longrightarrow \Gamma(TX)$$

$$: (\cancel{X}^i\partial_i, \cancel{W}^j\partial_j) \longmapsto (\cancel{X}^j\partial_j \cancel{W}^i - \cancel{W}^j\partial_j\cancel{X}^i)\partial_i$$

loc. coord presentation

2. Pushforward of a vector field along a diffeomorphism:

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle \pi_{TX}}\downarrow & & \downarrow{\scriptstyle \pi_{TY}} \\ X & \longrightarrow & Y \end{array}$$

(induced by)

$(f_* \cancel{V})(f(x)) \equiv T_x f(\cancel{V}(x))$, where local coord presentation

Let $\underset{\Gamma(TX)}{X} \overset{\text{loc.}}{=} X^i(x)\partial_i$ f

~~Then~~, $T_x f(\cancel{X}^i(x)\partial_i(x)) = \cancel{X}^i(x)\,\frac{\partial f^A}{\partial x^i}(x)\,\partial_A(f(x))$

~~3. Exterior product of forms~~

3. Pull back of a 1-form:

$$f^* : \Gamma(T^*\cancel{X}) \longrightarrow \Gamma(T^*X)$$

~~f^*~~ Let $\underset{\Gamma(T^*Y)}{\omega} \xrightarrow{\text{loc.}} \omega_A(y)\,dy^A$

Then, $(f^*\omega)(x) := \omega_A(f(x))\frac{\partial f^A}{\partial x^i}(x)\,dx^i$

The pullback lifts to

4. EXTERIOR ALGEBRA of DIFFERENTIAL FORMS on X : (13)

$$\left(\Gamma(\Lambda^\bullet T^*M) \equiv\right) \Lambda^\bullet \Gamma(T^*X) \leq \bigotimes_{C^\infty(X,\mathbb{R})} \Gamma(T^*X) \Big/ \langle dx^i \otimes dx^j - dx^j \otimes dx^i \rangle_{i,j \in \overline{1,N}}^{C^\infty(X,\mathbb{R})}$$

$$\rotatebox{90}{$\subseteq$}\ \Omega^\bullet(X)$$

with EXTERIOR/WEDGE product descended from $\otimes_{C^\infty(X,\mathbb{R})}$

$$\wedge : \Omega^m(X) \times \Omega^n(X) \longrightarrow \Omega^{m+n}(X)$$

with $\dim \Omega^m = \begin{cases} \binom{N}{m} & \text{for } 0 \leq m \leq N \\ 0 & \text{for } m > N \end{cases}$

We have

$$f^*(\underset{(p)}{\omega} \wedge \underset{(q)}{\eta}) = f^* \underset{(p)}{\omega} \wedge f^* \underset{(q)}{\eta}$$

(extended $C^\infty(X,\mathbb{R})$-lin. to $\Omega^\bullet(X)$)

On $\Omega^\bullet(X)$, we have

5. EXTERIOR DERIVATIVE :

$$d : \Omega^\bullet(X) \circlearrowleft \quad , \quad \boxed{d^2 = 0}$$

$$: \Omega^m(X) \longrightarrow \Omega^{m+1}(X)$$

$$d \xrightarrow[\text{loc.}]{} dx^\mu \wedge \partial_\mu \quad , \text{ i.e.,}$$

$$d\omega(x) = dx^\mu \wedge \partial_\mu \omega_{\mu_1 \mu_2 \ldots \mu_m} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_m}$$

A structural rel^{n} between the two entities: $\Gamma(TX)$ & $\Omega^\circ(X)$ is established as follows:

Let $\Phi_V \xi :]-\varepsilon,\varepsilon[\times M \longrightarrow M$, $\varepsilon > 0$

("time" at t; initial cond^{n})

be the (local) ~~FLOW~~ FLOW of V,

$$\frac{d}{dt}\Big|_{t=0} \Phi_V(t,x) = V(x) ,$$

so that $\Phi_V(t,\cdot) \in \mathrm{Diff}_{loc}(M)$. path with velocity

We then have – for

$$\frac{d}{dt}\Big|_{t=0} \Phi_V(t,\cdot)^* \omega =: \mathcal{L}_V \omega$$

LIE DERIVATIVE of ω along V

easy to check by writing – locally – $\Phi_V(t,x)^i = x^i + tV^i(x) + O(t^2)$ & performing Taylor's expansion

$$\mathcal{L}_V \omega \equiv V \lrcorner\, d\omega + d(V \lrcorner\, \omega),$$

or ~~sim~~ simply

$$\mathcal{L}_V = (V \lrcorner) \circ d + d \circ (V \lrcorner)$$

CARTAN'S 'MAGICAL' FORMULA

Remember also

$$f^*\big(T_x f(V(x)) \lrcorner\, \omega(f(x))\big) \equiv V(x) \lrcorner (f^*\omega)(x).$$

Back to our physical considerations... 15

~~Define~~ Def^{n}:

A SYMPLECTIC MANIFOLD is a pair $(\cancel{X}, \omega)$

s.th. $\cancel{X} \in$ ob Man, $\omega \in \Omega^2(\cancel{X})$;

(SM1) $d\omega = 0$ (ω is <u>closed</u>) — PRE-SYMPLECTIC FORM

(SM2) $\omega : T\cancel{X} \xrightarrow{\cong} T^*\cancel{X}$ is non-degenerate, i.e., — SYMPLECTIC FORM

locally $\omega \underset{loc.}{\longrightarrow} \omega_{\mu\nu}(x)\, dx^\mu \wedge dx^\nu$

$\forall x \in \cancel{X} : \det(\omega_{\mu\nu}(x)) \neq 0$

Necessary cond^{n} : $\boxed{\dim X \in 2\mathbb{N}}$! $\Leftarrow$ $\det(\omega_{\mu\nu}(x)) = \det(\omega^{T}_{\mu\nu}(x)) = \det(\omega_{\nu\mu}(x)) = \det(-\omega_{\mu\nu}(x)) = (-1)^{\dim X} \det(\omega_{\mu\nu}(x))$

A SYMPLECTOMORPHISM (or a CANONICAL TRASFORMATION) ↳ sometimes reserved for ENDO-symplectomorphisms, is a map

$$(X_1, \omega_1) \xrightarrow{f} (X_2, \omega_2) \quad \text{s.th.} \quad f^*\omega_2 = \omega_1$$

A symplectic form distinguishes (or CANONICAL) LOCALLY HAMILTONIAN VECTOR FIELDS :

$V \in \mathrm{Ham}_{loc}(X,\omega) \Leftrightarrow \mathcal{L}_V \omega = 0$ (16)

Their (local) flows preserve ω,

$$\Phi_V(t,\cdot)^* \omega = \omega \quad , t \in]-\epsilon, \epsilon[$$

Let us take a closer look at these...

$$0 = \mathcal{L}_V \omega \equiv V \lrcorner \overset{=0}{d\omega} + d(V \lrcorner \omega)$$

$$= d(V \lrcorner \omega)$$

In virtue of the Poincaré Lemma, locally, ...

convention!

$$\forall x \in X \ \exists \mathcal{O}_x \in \mathcal{T}(x) \ : \ V \lrcorner \omega|_{\mathcal{O}_x} = -dQ_V ,$$

$$\underset{x}{\overset{\Psi}{}} \quad Q_V \in \Omega^0(\mathcal{O}_x) \equiv C^\infty(\mathcal{O}_x, \mathbb{R})$$

need not extend beyond $\mathcal{O}_x$

Q_V is determined up to a constant

Conversely, non-degeneracy of ω implies

$$\forall f \in C^\infty(X,\mathbb{R}) \ \exists V_f \in \Gamma(TX) \ : \ df = -V_f \lrcorner \omega$$

We may use a symplectic ~~structure~~ / form ω on X to define a Poisson bracket:

$$\{\cdot,\cdot\}_\omega : C^\infty(X,\mathbb{R}) \times C^\infty(X,\mathbb{R}) \longrightarrow C^\infty(X,\mathbb{R})$$

$$(f,g) \longmapsto V_f \lrcorner V_g \lrcorner \omega = \omega(V_g, V_f)$$

We have: - antisymmetry & $\mathbb{R}$-linearity is obvious

- the Leibniz property:

$$-V_{fg} \lrcorner \omega \equiv d(fg) = f\,dg + g\,df \equiv -(f \cdot V_g + g \cdot V_f) \lrcorner \omega$$

& so we ~~may / take~~ have the unique answer (ω is non-deg!)

$$\boxed{V_{fg} = f \cdot V_g + g \cdot V_f}\ ,\ \text{whence}$$

$$\{f, gh\} \equiv V_f \lrcorner (g \cdot V_h + h \cdot V_g) \lrcorner \omega$$

$$= g\,(V_f \lrcorner V_h \lrcorner \omega) + h\,(V_f \lrcorner V_g \lrcorner \omega)$$

$$\equiv g\,\{f,h\} + h\,\{f,g\} \quad \checkmark$$

- the Jacobi identity follows from the identity (18)

$$V_3 \lrcorner V_2 \lrcorner V_1 \lrcorner d\omega = \mathcal{L}_{V_1}(V_3 \lrcorner V_2 \lrcorner \omega) + \mathcal{L}_{V_2}(V_1 \lrcorner V_3 \lrcorner \omega)$$
$$+ \mathcal{L}_{V_3}(V_2 \lrcorner V_1 \lrcorner \omega)$$
$$- V_3 \lrcorner [V_1, V_2] \lrcorner \omega - V_2 \lrcorner [V_3, V_1] \lrcorner \omega$$
$$- V_1 \lrcorner [V_2, V_3] \lrcorner \omega$$
$$(\text{valid } \forall \omega \in \Omega^2(X))$$

$$\Uparrow$$

$$[\mathcal{L}_{V_1}, \mathcal{L}_{V_2}] = \mathcal{L}_{[V_1, V_2]}$$

&

$$V_{\{f,g\}} = -[V_f, V_g] \Leftarrow \mathcal{L}_V(W \lrcorner \eta) = [V, W] \lrcorner \eta + W \lrcorner \mathcal{L}_V \eta$$

that latter following from

$$\cancel{V_{\{f,g\}} \lrcorner \omega = -d\{f,g\} = -d(V_f \lrcorner V_g \lrcorner \omega)}$$
$$\cancel{= -\mathcal{L}_{V_f}(V_g \lrcorner \omega) + V_f \lrcorner d(\underbrace{V_g \lrcorner \omega}_{=-dg})} \; {=0}$$
$$\cancel{= -[V_f, V_g] \lrcorner \omega - V_g \lrcorner \mathcal{L}_{V_f} \omega}$$
$$\cancel{+ V_f \lrcorner d(V_g \lrcorner \omega) + V_f \lrcorner V_g \lrcorner d\omega}$$

$$V_{\{f,g\}} \lrcorner \omega \equiv -d\{f,g\} \equiv -d\left(V_f \lrcorner V_g \lrcorner \omega\right) \quad (19)$$

$$= -\mathcal{L}_{V_f}\left(V_g \lrcorner \omega\right) + V_f \lrcorner d\left(V_g \lrcorner \omega\right)$$

$$\equiv -\mathcal{L}_{V_f}\left(V_g \lrcorner \omega\right) - V_f \lrcorner d^2 g^{=0}$$

$$= -\mathcal{L}_{V_f}\left(V_g \lrcorner \omega\right) = -[V_f, V_g] \lrcorner \omega - V_g \lrcorner \mathcal{L}_{V_f}\omega$$

$$= -[V_f, V_g] \lrcorner \omega \Leftarrow \mathcal{L}_{V_f}\omega \equiv V_f \lrcorner d\overset{\text{“}0}{\omega} + d(\underbrace{V_f \lrcorner \omega}_{=-df})$$

$\Leftarrow$ de Rham

Thus, the non-degenerate de Rham 2-cocycle

$$\omega \underset{\text{loc.}}{=} \omega_{\mu\nu}(x)\, dx^\mu \wedge dx^\nu$$

induces the Poisson bi-vector

$$\Pi = \frac{1}{4}\,\omega^{-1\,\mu\nu}(x)\, \partial_\mu \wedge \partial_\nu$$

s. th.

$$\{f,g\}_\Pi = df \lrcorner dg \lrcorner \Pi$$

$$\underset{\text{loc.}}{\longrightarrow} \frac{1}{4}\partial_\mu f\, \partial_\nu g\, dx^\mu \wedge dx^\nu \lrcorner \omega^{-1\alpha\beta}(x)\, \partial_\alpha \wedge \partial_\beta$$

$df = +2V_f^\beta \omega_{\alpha\beta}\, dx^\alpha$

$\shortparallel$

$\partial_\alpha f\, dx^\alpha$

$$\longrightarrow \frac{1}{2}\partial_\alpha f\, \partial_\beta g\, \omega^{-1\alpha\beta}(x)$$

$$= 2\,\omega_{\alpha\gamma}\,\omega_{\beta\delta}\, V_f^\gamma V_g^\delta\, \omega^{-1\alpha\beta}(x)$$

$$= 2\,\omega_{\delta\gamma}\, V_f^\gamma V_g^\delta(x) \equiv V_f \lrcorner V_g \lrcorner \omega(x) \equiv \{f,g\}(x)$$

We shall, next, derive a canonical (20)
(pre-) symplectic form on $\mathcal{P}_L$ ...

Recall

$$\Theta = V \lrcorner \delta S_{[t_0,t_1]}[q_*]$$

variable around classical trajectory

but now drop the assumption

$$V\restriction_{\partial q_*([t_0,t_1])} = 0$$

to obtain

assume regularity

ACTION 1-form

1-form on $\mathcal{P}_L$ $\rightarrow$

$$\delta S_{[t_0,t_1]}[q_*] = \theta_{T^*M}(t_1, q_*(t_1), p_*(t_1)) - \theta_{T^*M}(t_0, q_*(t_0), p_*(t_0))$$

$$\equiv \Theta_{t_1}[q_*] - \Theta_{t_0}[q_*]$$

$\Downarrow$

$$0 = \delta^2 S_{[t_0,t_1]}[q_*] = \delta\theta(t_1, q_*(t_1), p_*(t_1)) - \delta\theta(t_0, q_*(t_0), p_*(t_0))$$

$\Downarrow$

We have a definition of a closed 2-form on $\mathcal{P}_L$:

$$\Omega_L[q_*, p_*] := \delta p_{*i}(t) \wedge \delta q^i_*(t)$$

evaluated at an <u>arbitrary</u> time = equitemporal slice carrying the Cauchy data $(q_*, p_*)(t)$

What about the non-degeneracy (21)
of Ω_L ???

It is not granted but actually carries important structural information on local symmetries of the theory... The notion shall be defined 'rigorously' later on. Meanwhile, consider an example...

Example: Relativistic point-like particle

$$L(t,q,\dot{q}) := m\sqrt{-\dot{q}^\mu \dot{q}^\nu \eta_{\mu\nu}}$$

> $(\eta_{\mu\nu}) = \mathrm{diag}(-1,1,1,1)$

- Minkowski metric on $\mathbb{R}^4$

→ q taken s.th. $\dot{q}$ is ~~time~~ time-like,

$\eta_{\mu\nu}\dot{q}^\mu\dot{q}^\nu < 0$ (material particle)

$\Downarrow$

$$\boxed{\dot{q}^0 \neq 0}$$

We find

$$(EL) \qquad \frac{dp_\mu}{dt} = 0 \qquad \text{for } p_\mu \equiv -\frac{m \dot{q}^\nu \eta_{\mu\nu}}{\sqrt{-\eta_{\alpha\beta}\dot{q}^\alpha \dot{q}^\beta}}$$

Note the presence of CONSTRAINTS;

$$\eta^{-1\,\mu\nu} p_\mu p_\nu = -m^2 \quad \text{, implying}$$

$$\eta^{-1\,\mu\nu} p_\mu \, \delta p_\nu = 0$$

Consider a reparametrisation of the world line,

$$t \longmapsto \tilde{t}(t)$$

$$\tilde{t}'(t) > 0$$

We then have $-$ for $\tilde{t}(t) = t + \epsilon T(t) + O(\epsilon^2)$.

$$q(t) \longmapsto q(t) + \epsilon T(t)\, \dot{q}(t) + O(\epsilon^2)$$

$$\cancel{\dot{q}(t) \longmapsto \dot{q}(t) + \epsilon T(t)\, \ddot{q}(t) + O(\epsilon^2)},$$

~~best (EL) rewrite as~~

$$\tilde{q} \equiv q \circ \tilde{t}$$

Moreover, for $\tilde{q}(t) := q \circ \tilde{t}(t)$ $\quad \tilde{p}_\mu(t) := -\dfrac{m\, \dot{\tilde{q}}^\nu(t)\, \eta_{\mu\nu}}{\sqrt{-\eta_{\alpha\beta}\dot{\tilde{q}}^\alpha \dot{\tilde{q}}^\beta}} =$

$$= -\frac{m\, \dot{q}^{\mu}(\tilde{t}(t))\, \frac{d\tilde{t}}{dt}\, \eta_{\mu\nu}}{\sqrt{-\eta_{\alpha\beta}\, \dot{q}^{\alpha}(\tilde{t}(t))\, \dot{q}^{\beta}(\tilde{t}(t)) \left(\frac{d\tilde{t}}{dt}\right)^2}}$$

$$= p_\mu(\tilde{t}(t))\, \frac{\frac{d\tilde{t}}{dt}}{\left|\frac{d\tilde{t}}{dt}\right|} \equiv p_\mu(\tilde{t}(t))\, \mathrm{sign}\left(\frac{d\tilde{t}}{dt}\right)$$

by assumption

$$\overset{\downarrow}{=} p_\mu(\tilde{t}(t))$$

just like $\tilde{q}(t) = q(\tilde{t}(t))$

Therefore,

$$p_\mu(t) \longmapsto p_\mu(t) + \epsilon T(t)\, \underbrace{\dot{p}_\mu(t)}_{0 \text{ by } (\text{E-L})} + \mathcal{O}(\epsilon^2)$$

$$= p_\mu(t) + \mathcal{O}(\epsilon^2)$$

Consequently, the ~~vector~~ field engendered by $t \mapsto \tilde{t}(t)$ on the phase space
$\doteq t + \epsilon T(t) + \mathcal{O}(\epsilon^2)$

is $\mathcal{J}(q(t), p(t)) = T(t)\, \dot{q}^\mu \frac{\delta}{\delta q^\mu} + 0\, \frac{\delta}{\delta p_\mu}$

We check

$$\mathcal{J}(q_*(t), p_*(t)) \lrcorner\, \Omega_L[q_*(t), p_*(t)] = -T(t)\, \dot{q}^\mu\, \delta p_\mu$$

$$= T(t)\, \frac{\sqrt{-\eta_{\alpha\beta}\, \dot{q}_*^\alpha\, \dot{q}_*^\beta}}{m} \left(\eta^{-1\,\mu\nu}\, p_\mu\, \delta p_\nu\right) \equiv 0$$

Thus, reparametrisations = local gauge transformations are in $\ker \Omega_L$.

Conversely, $\ker \Omega_L$ contains only reparametrisations. Indeed, let

$$V(g,p) = X^\mu \frac{\delta}{\delta g^\mu} + Y_\mu \frac{\delta}{\delta p_\mu}$$

satisfy

$$V \lrcorner \Omega_L(g,p) = 0$$

$$\| $$

$$-X^\mu \delta p_\mu + Y_\mu \delta g^\mu$$

But $-p_0 \delta p_0 + p_i \delta p_j \delta^{ij} = 0$

$$\Downarrow \quad p_0 \sim \dot{g}_0 \neq 0$$

$$\underline{\delta p_0 = \frac{1}{p_0} p_j \delta^{ij} \delta p_i}$$, so that

we obtain

$$0 = -\left(\frac{X^0}{p_0} p_j \delta^{ij} + X^i\right) \delta p_i + Y_\mu \delta g^\mu$$

As there are no further constraints, we find

(25)

$$\begin{cases} X^i = -\dfrac{X^0}{p_0} p_j \delta^{ji} \\ Y_\mu = 0 \end{cases}$$
, or - altogether

$$\begin{cases} \boxed{X^\mu = -\dfrac{X^0}{p_0} p_\nu \eta^{-1\nu\mu}} \\ Y_\mu = 0 \end{cases}$$
as the sol^n

$$\Downarrow$$

$$V(q,p) = -\frac{X^0}{p_0} p_\nu \eta^{-1\nu\mu} \frac{\delta}{\delta q^\mu}$$, or

$$q^\mu \longmapsto q^\mu - \frac{X^0}{p_0} p_\nu \eta^{-1\nu\mu}$$

$$\equiv q^\mu_{(t)} + \left(\frac{m X^0}{p_0 \sqrt{-\eta_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta}} \right) \dot{q}^\mu$$

$$\Downarrow$$

$$q^\mu \xRightarrow[\text{integration}]{} q^\mu \Big(t + \underbrace{\frac{m X^{0t}}{p_0 \sqrt{-\eta_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta}} (t)}_{\text{this has the structure of a reparametrization}} \Big)$$

arbitrary !!!

~~[illegible]~~

We may, next, descend the symplectic structure onto the space of orbits of paths under ~~of~~ reparametrisations, i.e., on the space of leaves of the so-called

START HERE

CHARACTERISTIC FOLIATION.

In the equality

$$\dot{\tilde{q}}(t) = \dot{q}(\tilde{t}(t))\frac{d\tilde{t}}{dt},$$ written for

$$\tilde{q} \equiv q \circ \tilde{t},$$

we solve – for a given

$$\dot{q}^{0} \equiv f$$ –

the equation $$f(\tilde{t}(t))\frac{d\tilde{t}}{dt} = 1,$$

or $$f(\tilde{t})d\tilde{t} = dt,$$ i.e.

$$t = \int_{\tilde{t}(0)}^{\tilde{t}(t)} d\tilde{t}\, f(\tilde{t})$$ (always possible!).

In this manner, we use up some of the gauge freedom (reparametrisation ambiguity. The result,

$$\dot{\tilde{q}}^{0}(t) = 1,$$

integrates as

$$\tilde{q}^0(t) = t + \tilde{q}^0(0),$$

& so a residual reparametrization

$$t \mapsto t - \tilde{q}^0(0) =: \tilde{\tilde{t}}(t)$$

yields

$$\tilde{\tilde{q}}^0(t) := \tilde{q}^0 \circ \tilde{\tilde{t}}(t) = \tilde{q}^0(t - \tilde{q}^0(0))$$

$$= t - \tilde{q}^0(0) + \tilde{q}^0(0) = t,$$

Thus, we may always fix ONE of the coörds, e.g., q^0, to the linear form:

$$q^0(t) = t, \quad \vec{q}(t) \text{ arb.}$$ (the so-called STATIC GAUGE)

Clearly, VERTICAL variations of $q^0(\cdot)$ vanish; it has been fixed (as a function). We are left with the freedom to choose $\vec{q}(t)$

(& $\vec{p}(t)$, ~~but the~~ $p^0(t) = -\frac{m}{\sqrt{1-\dot{\vec{q}}^2}}$ ← fixed in terms of $\vec{q}$)

The descended presymplectic form reads

$$\underline{\Omega}_L[\vec{q}(t), \vec{p}(t)] = \delta p_i(t) \wedge \delta q^i(t)$$

~~so~~ is non-degenerate. The theory descends to $\left(\underline{P}_L = \{(\vec{q}, \vec{p})\}, \underline{\Omega}_L\right)$.

We shall next discuss GLOBAL SYMMETRIES in the geometric picture ...

We ~~start~~ with general ones: consider a ~~transformation~~

$$(t, q) \longmapsto (\tilde{t}(t), \tilde{q}(t, q))$$

covering $t \mapsto \tilde{t}(t)$ in the (space)time of the theory, so define a transformed trajectory over the transformed time:

$$\hat{q}(\tilde{t}(t)) := \tilde{q}(t, q(t))$$

$$\Big\downarrow \frac{d}{dt}$$

We calculate:

$$\boxed{\dot{\hat{q}}^i(\tilde{t}) \frac{d\tilde{t}}{dt} = \frac{\partial \tilde{q}^i}{\partial t} + \frac{\partial \tilde{q}^i}{\partial q^j} \dot{q}^j}$$

(to which we shall return later)

Assume, next, that there exists a function $K(t, q)$ s.th.

$$L\left(\tilde{t}, \hat{q}(\tilde{t}), \hat{q}'(\tilde{t}) \equiv \frac{d\hat{q}}{d\tilde{t}}\right) d\tilde{t} \quad \text{here, implicitly, } \tilde{t} \equiv \tilde{t}(t)$$

$$= L(t, q(t), \dot{q}(t))\, dt + dK(t, q(t))$$

Upon integrating the above between t_0 & t_1 ($\tilde{t}_0 \equiv \tilde{t}(t_0)$ & $\tilde{t}_1 \equiv \tilde{t}(t_1)$) we obtain

$$\boxed{S_{[\tilde{t}_0, \tilde{t}_1]}[\hat{q}] = S_{[t_0, t_1]}[q] + K(t_1, q(t_1)) - K(t_0, q(t_0))} \quad (*)$$

Thus, if q minimizes S, so does $\hat{q}$ (over the transformed segment of time), whence ensues mapping of solutions:

$$\mathcal{P}_L \ni q \longmapsto \hat{q} \in \mathcal{P}_L$$

different parametrization

Differentiation (vertical) of the above equality (*) yields (30)

$$\delta S_{[\tilde t_0,\tilde t_1]}[\hat g] = \delta S_{[t_0,t_1]}[g] + \delta K(t_1,g(t_1)) - \delta K(t_0,g(t_0))$$

$$\Updownarrow$$

$$\Theta_{\tilde t_1}[\hat g] - \Theta_{\tilde t_0}[\hat g] = \Theta_{t_1}[g] - \Theta_{t_0}[g] + \delta K(t_1,g(t_1)) - \delta K(t_0,g(t_0))$$

$$\Updownarrow$$

all at t_1

$$\Theta_{\tilde t(t_1)}[\hat g] - \Theta_{t_1}[g] - \delta K(t_1,g(t_1))$$

all at t_0

$$= \Theta_{\tilde t(t_0)}[\hat g] - \Theta_{t_0}[g] - \delta K(t_0,g(t_0))$$

But $g(t_0)$ & $g(t_1)$ are independent as variables, & so

$$\Downarrow$$

$$\Theta_{\tilde t(t)}[\hat g] - \Theta_t[g] = \delta K(t,g(t))$$

$\cancel{\forall t}$ t_0

$$\Downarrow$$

$$\Omega_L[\hat g(\tilde t)] \equiv \delta\Theta_{\tilde t(t)}[\hat g] = \delta\Theta_t[g] \equiv \Omega_L[g(t)]$$

Thus, we obtain a canonical transformation (for Ω_L symplectic)

We call transformations of this kind

SYMMETRIES.

Those of the type

$$\tilde{T} = \mathrm{Id}_{\mathbb{R}}, \quad \tilde{q}(t,q) = \tilde{q}(q)$$

will be ~~termed~~ GLOBAL SYMMETRIES.

NB. Gauge symmetries describe redundancy that calls for reduction.

Global symmetry describes correspondence.

We are now ready to take the first step towards the quantum regime ...

To this end, consider a situation slightly more general than heretofore, to wit, assume that the action 1-form has a term which (localised in M (depend solely on q)) exists only locally,

$$\Theta_t[q] = \underbrace{\Theta^0_t[q]}_{\simeq\, p^{kin}_\mu \delta q^\mu} + A(q(t))$$

Such a situation is NOT really exotic – it arises in the Lagrangean description of the motion of a charged point-like mass in external gravitational & electromagnetic field,

$$L(\tau, x, \dot{x}) = -\frac{m}{2} g_{\mu\nu}(x(\tau)) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}(\tau) + q\, A_\mu(x(\tau)) \frac{dx^\mu}{d\tau}$$

By the Poincaré Lemma a global (33)
$! \, dF = 0$ ($\wedge \, d{*}F = 0$ in the vacuum)
A_μ for a given $F_{\mu\nu}$ ($=: F_{[\mu\nu]}$) (Maxwell tensor, with $F_{0i} \simeq E_i$, $F_{ij} \sim \frac{1}{2}\epsilon_{ijk}{}^{k\ell} B_{k\ell}$)
may always be found. The problem arises when we compose coboundant trajectories...

Here, we find

$$\boxed{\Omega_L[x,p] = \delta p_\mu \wedge \delta x^\mu + g\, F_{\mu\nu}(x)\, \delta x^\mu \wedge \delta x^\nu}$$

$\Uparrow$ no global primitive...

The only (cohomological) information that we have (& need)

$\triangleright$ $\delta \Omega_L = 0$

$\Downarrow$ locally, by the Poincaré Lemma

$$\Omega_L \big|_{\mathcal{O}_i} = \delta \Theta_i \quad \text{for } \{\mathcal{O}_i\}_{i \in I} \text{ - open cover of } \mathcal{P}_L,$$

for some $\Theta_i \in \Omega^1(\mathcal{O}_i)$

$\Theta_i \mapsto \Theta_i + \delta f_i$ admissible

a good one

The Weil–de Rham Thm says it exists for $\mathcal{P}_L$ of class C^2. $\rightarrow$ i.e. one with all non-empty intersections contractible

Next, we compute

$$(\theta_j - \theta_i)\restriction_{O_{ij}} = \delta X_{ij} \qquad O_{ij} = O_i \cap O_j$$

for some $X_{ij} \in [O_{ij}, \mathbb{R}]$

Whenever
$\theta_i \mapsto \theta_i + \delta f_i$, we get

$$X_{ij} \mapsto X_{ij} + (f_j - f_i)\restriction_{O_{ij}} + 2\pi c_{ij}$$

for some local constants c_{ij}

Moreover,

$$O_{ijk} = O_i \cap O_j \cap O_k$$

$$(X_{jk} - X_{ik} + X_{ij})\restriction_{O_{ijk}} = 2\pi \nu_{ijk}$$

ν_{ijk} – local constants

Freedom:
$$\nu_{ijk} \mapsto \nu_{ijk} + (c_{jk} - c_{ik} + c_{ij})$$

What can we do with these?
Consider locally smooth maps

$$\psi_i : O_i \to \mathbb{C}$$

on which we act with operators

$C^\infty(P_L, \mathbb{R}) \ni h \mapsto \hat{h}_i := -i \mathcal{L}_{\mathcal{X}_h} - \mathcal{X}_h \lrcorner \theta_i + h\restriction_{O_i}$, where $\mathcal{X}_h \lrcorner \Omega_L = -\delta h$

We find $[\hat{h}_{1i}, \hat{h}_{2i}]\psi_i = \ldots = i \, \overbrace{\{h_1, h_2\}_i}^{\mathcal{X}_h}{}^{\text{"}}\nu_h$

$$[\hat{h_{1}}_i, \hat{h_{2}}_i] = i\, \widehat{\{h_1, h_2\}}_i$$

with constraints mapped to

$\hat{c}_i = c\,\mathbb{1}$ $\quad (dc = 0 \Rightarrow X_c = 0)$

LOCALLY

Thus, we have realised Dirac's ~~ec~~ canonical quantisation postulates.

It remains to be seen (38)
how the 'local Hilbert spaces'
relate ~~over~~ over overlaps, & how
they accomodate ~~the freedom~~
to change the local representatives
of $\Omega L|O_i$ ....

1) For $\hat{h}_i \psi_i$ to be locally
well-defined, the ~~redefinition~~
$\Theta_i \longmapsto \Theta_i + \delta f_i$, resulting in
$\hat{h}_i \longmapsto \hat{h}_i - \mathcal{X}_h \lrcorner \delta f_i$,
must be accompanied by
$\psi_i \longmapsto e^{if_i}\psi_i$

Indeed, we then have

$$(\hat{h}_i - \mathcal{X}_h \lrcorner \delta f_i)\, e^{if_i}\psi_i$$
$$= -i\left(\mathcal{L}_{\mathcal{X}_h} e^{if_i}\right)\psi_i - \left(\mathcal{X}_h \lrcorner \delta f_i\right) e^{if_i}\psi_i$$
$$+ e^{if_i}\left(\hat{h}_i\psi_i\right) \equiv e^{if_i}\left(\hat{h}_i\psi_i\right),$$

which is acceptable because we only care about rays.

2) Over O_{ij}, we may choose $\hat{h}_j$ instead of $\hat{h}_i$, but then

$\hat{h}_j = \hat{h}_i - \mathcal{H}_h \lrcorner\, \delta X_{ij}$, so we obtain

$$\hat{h}_j \psi_j = \hat{h}_i \psi_j - (\mathcal{H}_h \lrcorner\, \delta X_{ij}) \psi_j$$

$$= \hat{h}_i \left(\frac{\psi_j}{\psi_i} \cdot \psi_i \right) - (\mathcal{H}_h \lrcorner\, \delta X_{ij}) \psi_j$$

$$= -i \mathcal{H}_h \lrcorner\, d\left(\frac{\psi_j}{\psi_i}\right) \psi_i - (\mathcal{H}_h \lrcorner\, \delta X_{ij}) \psi_j$$

$$+ \frac{\psi_j}{\psi_i} \hat{h}_i \psi_i$$

DO IT WITHOUT DIVISION!!!

Therefore, if we write

$$\forall x \in O_{ij}: \psi_j(x) = e^{i X_{ij}(x)} \psi_i(x),$$

SIMPLY OBSERVE THAT SAVES THE DAY.

we arrive at

$$\hat{h}_j \psi_j = e^{i X_{ij}} \hat{h}_i \psi_i,$$

which – again – is perfectly acceptable (physically).

Thus, consistency seems to require (40)

$$\begin{cases} \theta_i \mapsto \theta_i + \delta f_i \Rightarrow \psi_i \mapsto e^{if_i}\psi_i \\ \psi_j = e^{i\chi_{ij}}\psi_i \end{cases}$$

But then also

$$\psi_k = e^{i\chi_{jk}}\psi_j = e^{i(\chi_{jk}+\chi_{ij})}\psi_i$$

$$\stackrel{?}{=} e^{i\chi_{ik}}\psi_i$$

we are led to impose the ~~constraint~~ :

$$\boxed{\chi_{ijk} \in \mathbb{Z}} \Longrightarrow \boxed{\begin{array}{c} Per(\Omega_L) \subset 2\pi\mathbb{Z} \\ \text{Dirac's Quantization Cond}^n \end{array}}$$

which is more naturally written as $g_{ij} := e^{-i\chi_{ij}}$

$$\begin{cases} \Omega_L\restriction_{\theta_i} = \delta\theta_i \\ (\theta_j - \theta_i)\restriction_{\theta_i} = i\,d\log g_{ij} \\ (g_{jk}\cdot g_{ik}^{-1}\, g_{ij})\restriction_{\theta_{ijk}} = 1 \end{cases}$$

Structure of this kind shall be called LOCAL PRESENTATION of a PRINCIPAL $\mathbb{C}^\times$-BUNDLE (Čech-cohom.)

In this context, the Ψ_i (41)

are LOCAL SECTIONS of the bundle.

We have arrived at WAVE SECTIONS

carrying a realisation

of the (Poisson) algebra of observables.

We might — a bit more

heuristically — go yet another

step further —

Consider the relⁿ from p. (20)

$$\Theta_{t_1}[g_*] - \Theta_{t_0}[g_*] = \delta S_{[t_0,t_1]}[g_*] \text{ for } \underline{\text{local}}$$

Θ_i & S_i

& think of $g_*(t_1)$ as determined

by $g_*(t_0)$ & $p_*(t_0)$

$$\Theta_{i\,t_1}[g_*] = \Theta_i\left(\tilde{g}_{t_1}(g_*(t_0), p_*(t_0)), \tilde{p}_{t_1}(g_*(t_0), p_*(t_0))\right)$$

so that we have $= \Theta_i^{t_1}(g_*(t_0), p_*(t_0))$

$$\boxed{\begin{aligned}&\Theta_i\left(\tilde{g}_{t_1}(g_*(t_0), p_*(t_0)), \tilde{p}_{t_1}(g_*(t_0), p_*(t_0))\right)\\ &- \Theta_i\left(g_*(t_0), p_*(t_0)\right) = \delta S_{i\,[t_0,t_1]}[g_*]\end{aligned}}$$

We may interpret this rel^n as saying that the evolution $(q_*(t_0), p_*(t_0)) \longmapsto \left(\tilde{q}_{t_1}(q_*(t_0), p_*(t_0)), \tilde{p}_{t_1}(q_*(t_0), p_*(t_0))\right)$

is represented by / leads to a 'gauge transformation' / isomorphism of the bundle of wave functions (the PREQUANTUM BUNDLE),

Consequently, for THIS evolution (along the classical picked configuration $q_*([t_0,t_1])$), we obtain

$$\psi\left(t_1, \tilde{q}_{t_1}(q_*(t_0), p_*(t_0)), \tilde{p}_{t_1}(q_*(t_0), p_*(t_0))\right)$$
$$= e^{i S_{[t_0,t_1]}[q_*]}\, \psi\left(t_0, q_*(t_0), p_*(t_0)\right),$$

Thus, the classical action appears in the QM rôle of a transition amplitude between the initial & final 'states'.

We shall elaborate this idea further...

We shall, meanwhile, geometrise the local presentation / trivialisation

$$\Omega_L \xrightarrow[\text{loc.}]{} \left(\mathcal{O}_i \ ,\ X_{ij} \equiv g_{ij}\right)$$

Locally, consider

$$\mathcal{O}_i \times \mathbb{C}^{\times} \ni (\overset{\overset{(g,p)}{\shortparallel}}{x}, z_i)$$

& define

$$\Theta(x, z_i) := \frac{i\,dz_i}{z_i} + \Theta_i(x)$$

to obtain the cond$^{\underline{n}}$ for global smoothness for Θ over $\bigsqcup_{i \in I} (\mathcal{O}_i \times \mathbb{C}^{\times})$:

$$\frac{i\,dz_j}{z_j} + \Theta_j(x) \overset{!}{=} \frac{i\,dz_i}{z_i} + \Theta_i(x)$$

at $x \in \mathcal{O}_{ij}$ where we have both: z_i & z_j ...

The rel$^{\underline{n}}$ between this geometrisation & the ψ_i ~~shall~~ will become clear once we understand the construction of a frame bundle of a given vector bundle.

We find

$$(\theta_j - \theta_i)(x) = i\,d\log \frac{z_i}{z_j}$$

$$\hookrightarrow i\,d\log g_{ij}(x)$$

$$\Downarrow$$

We may set the gluing cond$^{\underline{n}}$

$$\boxed{z_i = g_{ij}(x)\, z_j}$$

Thus, we pass from the redundant

$$\bigsqcup_i \mathcal{O}_i \times \mathbb{C}^\times$$ (several 'fibres' $\mathbb{C}^\times$ over a single point x in the base M)

with $A_L([(x, z_i)]_{\sim_{g..}}) \equiv \theta(x, z_i) \,(\equiv \theta(x, z_j) \ldots)$

to see well-defined

$$\Big(\bigsqcup_i \mathcal{O}_i \times \mathbb{C}^\times\Big)\Big/\sim_{g..},$$

with $\sim_{g..}$ defined as above

$$[(x, i, z_i)]_{\sim_{g..}} \ni [(x, j, g_{ji}(x) z_i)]$$

As we shall see later, the crucial feature of $g_{..}$ that enables us to equip

$$L := \left(\bigsqcup_i O_i \times \mathbb{C}_i^{\times}\right) / \sim_{g_{..}}$$

with the structure of a diff manifold is the 1-cocycle cond^{n}

$$g_{ij}\, g_{jk}(x) = g_{ik}(x) \quad , x \in O_{ijk}$$

Heuristically, we see it as the consistency cond^{n}:

$$g_{ik}(x)\, z_k = z_i = g_{ij}(x)\, z_j = g_{ij}(x)\, g_{jk}(x)\, z_k$$

$= V$

NB: (1) We have a surjective submersion:

$$\pi_L : \left(\bigsqcup_i O_i \times \mathbb{C}^{\times}\right)/\sim_{g_{..}} \longrightarrow M$$

$$: [(x, i, z_i)]_{\sim_{g_{..}}} \longmapsto x \quad \text{(well-defined)}$$

(2) local trivialisation:

$$L\restriction_{O_i} \cong O_i \times \mathbb{C}^{\times} \quad \text{TYPICAL FIBRE}$$

(3) smooth transition maps

$$g_{ij} : \mathcal{O}_{ij} \longrightarrow \mathbb{C}^\times$$

$$\begin{tikzcd} \mathcal{O}_{ij} \arrow[r] \arrow[dr] & \mathbb{C}^\times \\ & U(1) \arrow[u] \end{tikzcd}$$

(4) connection 1-form:

$$\pi_L^* \cancel{\Omega_L}\, F_L\big([(x,i,z_i)]_\sim\big) = \cancel{\Omega}\, F_L(x) \qquad x \in \mathcal{O}_i$$

$$= d\Theta_i(x) = d\left(\frac{i\,dz_i}{z_i} + \Theta_i(x)\right)$$

$$= dA\big([(x,i,z_i)]_\sim\big) \text{, or:}$$

$$\boxed{\pi_L^* \Omega_L = dA_L}$$

It's not hard to see that ~~the~~ from the above are an abstract

$$\begin{tikzcd} L \times_{\mathcal{O}_L} L \times_{\mathcal{O}_L} L \arrow[r, shift left=2, "pr_{12}"] \arrow[r, "pr_{23}"] \arrow[r, shift right=2, "pr_{13}"'] & L \times_{\mathcal{P}_L} L \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] & L \equiv Y\mathcal{P}_L \,,\, A_L \arrow[d, shift right, "\pi_L"'] \\ & & \mathcal{M}_L \,,\, \Omega_L \arrow[u, shift right, dashed, "s_i"'] \end{tikzcd}$$

$$f_L$$

$$pr_{12}^* f_L \cdot pr_{23}^* f_L = pr_{13}^* f_L$$

$$i\,d\log f_L = (pr_2^* - pr_1^*)\, A_L$$

$$\pi_L^* \Omega_L = dA_L$$

This goes via LOCAL SECTIONS:

$$\begin{array}{ccc}
Y\mathcal{P}_L \times_{\mathcal{O}_L} Y\mathcal{P}_L \times_{\mathcal{O}_L} Y\mathcal{P}_L \overset{pr_i}{\to} Y\mathcal{P}_L \times_{\mathcal{O}_L} Y\mathcal{P}_L & \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} & Y\mathcal{P}_L \\
\uparrow \sigma_{ijk} = (\sigma_i, \sigma_j, \sigma_k) \quad \uparrow \sigma_{ij} := (\sigma_i, \sigma_j) & & \pi_L \downarrow \uparrow \sigma_i \\
& & \mathcal{P}_L \supset \mathcal{O}_i
\end{array}$$

$\pi_L \circ \sigma_i = \mathrm{id}_{\mathcal{O}_i}$

$A_i := \sigma_i^* A_L \Rightarrow dA_i = \sigma_i^* \pi_L^* \Omega_L$

$\equiv (\pi_L \circ \sigma_i)^* \Omega_L$

$\equiv \Omega_L \restriction_{\mathcal{O}_i}$

$g_{ij} := \sigma_{ij}^* f_L$

$\Downarrow$

$i\, d\log g_{ij} = \sigma_{ij}^* (pr_2^* - pr_1^*) A_L$

$\equiv (\sigma_j^* A_L - \sigma_i^* A_L) \restriction_{\mathcal{O}_{ij}}$

$\equiv (A_j - A_i) \restriction_{\mathcal{O}_{ij}}$

$(g_{ij}\, g_{jk}) \restriction_{\mathcal{O}_{ijk}} = \sigma_{ijk}^* (pr_{1,2}^* f_L \cdot pr_{2,3}^* f_L) = \sigma_{ijk}^*\, pr_{1,3}^* f_L$

$= g_{ik} \restriction_{\mathcal{O}_{ijk}}$. ✓

In this semi-formal manner, we have arrived at (a slightly more general, but essentially requested by physics)

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Def^{n} A ~~principal~~ generalised fibration with structural group $\mathbb{C}^\times$ ~~$\mathbb{C}^\times$-bundle~~ over M with ~~principal connection over M~~ potential of curvature $F \in Z^2(M, 2\pi\mathbb{Z})$ is a ~~triple~~ quadruple (π_{YM}, YM, A, f), where

(*) $\pi_{YM} : YM \longrightarrow\!\!\!\!\to M$ is a surjective submersion

(**) $A \in \Omega^1(YM)$:

$$\pi_{YM}^* F = dA$$

(***) $f \in C^\infty(Y^{[2]}M, \mathbb{C}^\times)$:

$$i\, d\log f = (pr_2^* - pr_1^*) A$$

such that

$$pr_{1,2}^* f \cdot pr_{2,3}^* f = pr_{1,3}^* f$$

over $Y^{[3]}M$.

GENERALISED FIBRATION WITH STRUCTURAL GROUP $\mathbb{C}^\times$ over M with POTENTIAL of CURVATURE $F \in Z^2_{dR}(M, 2\pi\mathbb{Z})$

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We shall (sometimes) ~~[illegible]~~ encode the whole structure in the diagram

$$
\begin{tikzcd}
pr_{12}^* f \cdot (pr_{13}^* f)^{-1} \cdot pr_{23}^* f = 1 \arrow[d, dashed] & i\,d\log f := (pr_2^* - pr_1^*)A \arrow[d, dashed] & dA := \pi_{YM}^* F \arrow[d, dashed] \\
Y^{[3]}M \arrow[r, shift left=2, "pr_{A,B}"] \arrow[r] \arrow[r, shift right=2] & Y^{[2]}M, f \arrow[r, shift left, "pr_C"] \arrow[r, shift right] & YM, A \arrow[d] \\
 & & M, F
\end{tikzcd}
$$