ASSOCIATIONS IN ACTION, & THE TRIDENT
(DDD '24/25 VII, VIII & IX [RRS])

Figure 1. Plato and Aristotle, the Ur-fathers of the idea of association through a (convergent) sequence of actions—here, depicted in a *fresco* by Raffaello Sanzio da Urbino from 1511, with the title "*Scuola di Atene*" (*Stanze di Raffaello*, Palazzo Apostolico, Vaticano). Accompanied by a silver coin of the Wise from the X$^{\text{th}}$ or XI$^{\text{th}}$ century.

Contents

In the last lecture, a geometric object was introduced, which we heralded as a model of a 'space of local (observation/description) frames' P over the spacetime B of a field theory with a global symmetry (so far, modelled solely on a Lie-group action $\lambda : \mathrm{G} \longrightarrow \mathrm{Diff}(M)$) rendered local, or gauged. Below, we employ it, as a fundamental substrate, in a methodical construction of two more physically (more evidently) relevant geometric entities, to wit: a 'space of symmetry transformations between the local frames' and a 'space of fields amenable to observation/description in the local frames', the latter admitting an intuitively understandable action of the former—to be locally modelled on λ. In so doing, we shall justify the Homeric description of P and the extensive attention devoted to it in our hitherto considerations. Both constructions shall be sought after, as was P, in the category $\mathbf{Bun}(B)$ of fibre bundles over spacetime B. Moreover, physically motivated operations on them, such as, *e.g.*, a 'transformation between frames', and composition of such operations shall be internalised in the corresponding slice category $\mathbf{Bun}(B)/B$ as long as they represent actions to be performed by each observer (as represented by a point in B) separately *within* his 'space of observations/descriptions' (as represented by the correponding fibre of the relevant bundle). We shall also, whenever possible, formulate our findings in a manner which emphasises the groupoidal underpinning of our constructions and admits a natural -oïdification, or—more generally—categorification.

Secretely, our construction—essentially based on The Quotient Manifold Theorem (or, more generally, on Godement's Criterion)—can be seen as a natural field-theoretic avatar (or a Yonedian sample over a given spacetime) of a universal construction of the so-called **homotopy quotient**, *i.e.*, a homotopy model of the symmetry-orbispace $M//\mathrm{G}$ of the configuration space of the field theory under consideration. This construction plays a prime rôle in the modelling of differential geometry (or, at any rate, of the associated homotopically stable structures—such as cohomology or its higher-geometric realisations) of the said orbispace—this is the context in which it was first conceived by Henri Cartan in [Car50], and subsequently elaborated and popularised by Émile Borel, whence its name: the Cartan–Borel model (or mixing construction), *cp.* [Tu20].

1. Technical motivation & warm-up

The reconstruction of a smooth distribution, over a given smooth base B, of objects with a fixed common isotype M from the set of (B-)local distributions of frames of description (or coordinatisation) of M, related by an action of a symmetry structure (*e.g.*, a group), and that in a way which encodes the freedom of the local choice of a representative of a symmetry class (of frames), has an elementary prototype, with a simple and helpful linear-algebraic intuition behind it: the reconstruction of a vector bundle $\mathbb{V}$ over B from its frame bundle $\mathsf{F}_{\mathrm{GL}}\mathbb{V}$, which we consider below as a conceptual and technical base for subsequent abstraction.

We begin our reconstruction by introducing the manifestly well-defined and smooth evaluation map

$$\widehat{\mathrm{ev}} \;:\; \mathsf{F}_{\mathrm{GL}}\mathbb{V}\times\mathbb{K}^{\times r}\longrightarrow\mathbb{V} \;:\; \big((\beta_x,x),v\big)\longmapsto\beta_x(v)\in\mathbb{V}_x\,.$$

This map is constant on orbits of the action

$$\widetilde{\mathrm{ev}} \;:\; \mathrm{GL}(r;\mathbb{K})\times\big(\mathsf{F}_{\mathrm{GL}}\mathbb{V}\times\mathbb{K}^{\times r}\big)\longrightarrow\mathsf{F}_{\mathrm{GL}}\mathbb{V}\times\mathbb{K}^{\times r} \;:\; \big(\chi,\big((\beta_x,x),v\big)\big)\longmapsto\big(\big(\beta_x\circ\chi^{-1},x\big),\chi(v)\big)\,,$$

expressed in terms of the natural (defining) action of the group $\mathrm{GL}(r;\mathbb{K})$ on $\mathbb{K}^{\times r}$,

$$\mathrm{ev} \;:\; \mathrm{GL}(r;\mathbb{K})\times\mathbb{K}^{\times r}\longrightarrow\mathbb{K}^{\times r} \;:\; (\chi,v)\longmapsto\chi(v)\,.$$

This implies that $\widehat{\mathrm{ev}}$ descends to the quotient manifold $(\mathsf{F}_{\mathrm{GL}}\mathbb{V}\times\mathbb{K}^{\times r})/\mathrm{GL}(r;\mathbb{K})$, the latter being defined relative to the action $\widetilde{\mathrm{ev}}$. Smoothness of the quotient manifold is ensured by the reasoning from Rem. VI.2., based on Thm. I.21. In other words, $\widehat{\mathrm{ev}}$ determines a map

$$\begin{aligned}[\widehat{\mathrm{ev}}] \quad &: \quad (\mathsf{F}_{\mathrm{GL}}\mathbb{V}\times\mathbb{K}^{\times r})/\mathrm{GL}(r;\mathbb{K})\longrightarrow\mathbb{V}\\ &: \quad \big[\big((\beta_x,x),v\big)\big]\longmapsto\widehat{\mathrm{ev}}\big((\beta_x,x),v\big)\equiv\beta_x(v)\end{aligned}\tag{2}$$

which closes the following commutative diagram:

$$\begin{array}{ccc} & & \mathbb{V}\\ & \overset{\widehat{\mathrm{ev}}}{\nearrow} & \uparrow{\scriptstyle[\widehat{\mathrm{ev}}]}\\ \mathsf{F}_{\mathrm{GL}}\mathbb{V}\times\mathbb{K}^{\times r} & \xrightarrow[\pi_\sim]{} & (\mathsf{F}_{\mathrm{GL}}\mathbb{V}\times\mathbb{K}^{\times r})/\mathrm{GL}(r;\mathbb{K}) \end{array}\quad.$$

The canonical (orbit) projection $\pi_\sim$ in this diagram is—in virtue of Thm. I.21.—a submersion, and so, by Thm. V.2., smoothness of the induced map $[\widehat{\mathrm{ev}}]$ is implied by the same property of $\widehat{\mathrm{ev}}$. It is also strightforward to see that a restriction of the map to an arbitrary fibre $(\mathrm{Iso}_{\mathbb{K}}(\mathbb{K}^{\times r},\mathbb{V}_x)\times\mathbb{K}^{\times r})/\mathrm{GL}(r;\mathbb{K}),\ x\in B$ is a bijecion. Indeed, let us fix a basis $\beta^\star_x\in\mathrm{Iso}_{\mathbb{K}}(\mathbb{K}^{\times r},\mathbb{V}_x)$ and consider the set $S \coloneqq \{ ((\beta_x^*,x),v) \quad | \quad v \in \mathbb{K}^{\times r} \}$. Orbits of any two of its elements, $\mathrm{GL}(r;\mathbb{K}) \triangleright ((\beta_x^*,x),v_1)$ and $\mathrm{GL}(r;\mathbb{K}) \triangleright ((\beta_x^*,x),v_2)$, either coincide with one another, or are disjoint (as classes of an equivalence relation). The former happens iff

$$\big((\beta_x^*,x),v_2\big) \in \mathrm{GL}(r;\mathbb{K}) \triangleright \big((\beta_x^*,x),v_1\big)$$

$$\Longleftrightarrow \qquad \exists_{\chi \in \mathrm{GL}(r;\mathbb{K})} \; : \; \big((\beta_x^*,x),v_2\big) = \big((\beta_x^* \circ \chi^{-1},x),\chi(v_1)\big) \Longleftrightarrow \qquad (\; \chi = \mathrm{id}_{\mathbb{K}^{\times r}} \quad \wedge \quad v_2 = v_1 \;) ,$$

and so different elements of S belong to disjoint orbits. Hence, the fibre $(\mathrm{Iso}_{\mathbb{K}}(\mathbb{K}^{\times r},\mathbb{V}_x)\times\mathbb{K}^{\times r})/\mathrm{GL}(r;\mathbb{K})$ covers the corresponding fibre $\mathbb{V}_x$. It now remains to check injectivity of $[\widehat{\mathrm{ev}}]$. To this end, consider implications of the equality

$$\beta_x^1(v_1) \equiv [\widehat{\mathrm{ev}}]\big(\big[\big((\beta_x^1,x),v_1\big)\big]\big) = [\widehat{\mathrm{ev}}]\big(\big[\big((\beta_x^2,x),v_2\big)\big]\big) \equiv \beta_x^2(v_2) \, .$$

The latter is equivalent to

$$v_2 = \beta_x^{2\,-1} \circ \beta_x^1(v_1) \, ,$$

which, in turn, infers

$$v_2 \in \mathrm{GL}(r;\mathbb{K}) \triangleright v_1 \, ,$$

and so also

$$\big((\beta_x^2,x),v_2\big) = \big((\beta_x^1 \circ \big(\beta_x^{2\,-1} \circ \beta_x^1\big)^{-1},x),\beta_x^{2\,-1} \circ \beta_x^1(v_1)\big) \in \mathrm{GL}(r;\mathbb{K}) \triangleright \big((\beta_x^1,x),v_1\big) \, .$$

Thus, we establish the identity

$$\big[\big((\beta_x^1,x),v_1\big)\big] = \big[\big((\beta_x^2,x),v_2\big)\big] \, ,$$

which shows injectivity of $[\widehat{\mathrm{ev}}]$. Altogether, then, we have a smooth bijection. We shall next construct its smooth inverse. For this purpose, we employ local trivialisations $\tau_i \; : \; \pi_{\mathsf{F}_{\mathrm{GL}}\mathbb{V}}^{-1}(O_i) \xrightarrow{\cong} O_i \times \mathrm{GL}(r;\mathbb{K}), \; i \in I$ of the frame bundle, associated with an open cover $\{O_i\}_{i\in I}$. In virtue of Prop. VI.3., these induce local sections

$$\sigma_i \; : \; O_i \longrightarrow \mathsf{F}_{\mathrm{GL}}\mathbb{V} \; : \; x \longmapsto \tau_i^{-1}(x,e) \equiv \big(\beta_i(x),x\big) \in \mathrm{Iso}_{\mathbb{K}}(\mathbb{K}^{\times r},\mathbb{V}_x) \times \{x\} \, ,$$

where the field of bases β_i depends (locally) smoothly on the point in $O_i \subset B$. We readily convince ourselves that a map locally (over $O_i \ni x$) given by

$$\Sigma_i \restriction_{\mathbb{V}_x} \; : \; \mathbb{V}_x \longrightarrow (\mathrm{Iso}_{\mathbb{K}}(\mathbb{K}^{\times r},\mathbb{V}_x) \times \mathbb{K}^{\times r})/\mathrm{GL}(r;\mathbb{K}) \; : \; \nu \longmapsto \big[\big((\beta_i(x),x),\beta_i(x)^{-1}(\nu)\big)\big]$$

is an inverse of $[\widehat{\mathrm{ev}}]$ (a local one), since

$$\begin{aligned} \Sigma_i \circ [\widehat{\mathrm{ev}}]\big(\big[\big((\beta_x,x),v\big)\big]\big) &= \Sigma_i \circ \beta_x(v) = \big[\big((\beta_i(x),x),\beta_i(x)^{-1} \circ \beta_x(v)\big)\big] \\ \equiv \quad & \big[\big(\big(\beta_x \circ \big(\beta_i(x)^{-1} \circ \beta_x\big)^{-1},x\big),\beta_i(x)^{-1} \circ \beta_x(v)\big)\big] = \big[\big((\beta_x,x),v\big)\big] \end{aligned}$$

and, for $\nu \in \mathbb{V}_x$ arbitrary,

$$[\widehat{\mathrm{ev}}] \circ \Sigma_i(\nu) = [\widehat{\mathrm{ev}}]\big(\big[\big((\beta_i(x),x),\beta_i(x)^{-1}(\nu)\big)\big]\big) = \beta_i(x)\big(\beta_i(x)^{-1}(\nu)\big) = \nu \, .$$

Last, we check that the local maps Σ_i are restrictions (to the respective elements $\pi_{\mathbb{V}}^{-1}(O_i)$ of an open cover of the total spce $\mathbb{V}$) of a globally smooth map. For that, we first need to derive the transformation rule that relates the local choices of basis β_i. Let $g_{ij} \; : \; O_{ij} \longrightarrow \mathrm{GL}(r;\mathbb{K})$ be the transition maps for the formerly fixed local trivialisations $\mathsf{F}_{\mathrm{GL}}\mathbb{V}$, *i.e.*, for every $x \in O_{ij}$ and $\chi \in \mathrm{GL}(r;\mathbb{K})$,

$$\tau_i \circ \tau_j^{-1}(x,\chi) = \big(x,g_{ij}(x) \circ \chi\big) \, .$$

We then calculate

$$\begin{aligned} \big(\beta_j(x),x\big) \quad &\equiv \quad \tau_j^{-1}(x,\mathrm{id}_{\mathbb{K}^{\times r}}) = \tau_i^{-1}\big(x,g_{ij}(x)\big) = \tau_i^{-1}(x,\mathrm{id}_{\mathbb{K}^{\times r}}) \triangleleft g_{ij}(x) = \big(\beta_i(x),x\big) \triangleleft g_{ij}(x) \\ &\equiv \quad \big(\beta_i(x) \circ g_{ij}(x),x\big) \, , \end{aligned}$$

so that

$$\beta_j(x) = \beta_i(x) \circ g_{ij}(x) \, ,$$

whence we readily obtain, for $\nu \in \mathbb{V}_x,\ x \in O_{ij}$ arbitrary, the desired identity

$$\begin{aligned}\Sigma_j(\nu) &= \big[\big((\beta_j(x),x),\beta_j(x)^{-1}(\nu)\big)\big] = \big[\big((\beta_i(x)\circ g_{ij}(x),x),g_{ij}(x)^{-1}\circ\beta_i(x)^{-1}(\nu)\big)\big] \\ &= \big[\big((\beta_i(x),x),\beta_i(x)^{-1}(\nu)\big)\big] \equiv \Sigma_i(\nu)\,.\end{aligned}$$

Our hitherto considerations enable us to write out directly local trivialisations

$$[\tau_i]\ :\ \big(\pi^{-1}_{\mathsf{F}_{\rm GL}\mathbb{V}}(O_i)\times\mathbb{K}^{\times r}\big)/{\rm GL}(r;\mathbb{K}) \xrightarrow{\cong} O_i\times\mathbb{K}^{\times r}\ :\ \big[\big((\beta_x,x),v\big)\big]\longmapsto\big(x,\beta_i(x)^{-1}\circ\beta_x(v)\big)\,,$$

alongside with their inverses

$$[\tau_i]^{-1}\ :\ O_i\times\mathbb{K}^{\times r}\xrightarrow{\cong}\big(\pi^{-1}_{\mathsf{F}_{\rm GL}\mathbb{V}}(O_i)\times\mathbb{K}^{\times r}\big)/{\rm GL}(r;\mathbb{K})\ :\ (x,v)\longmapsto\big[\big((\beta_i(x),x),v\big)\big]\,,$$

and thus identify the structure of a fibre bundle on the quotient manifold $(\mathsf{F}_{\rm GL}\mathbb{V}\times\mathbb{K}^{\times r})/{\rm GL}(r;\mathbb{K})$. It is clearly a vector bundle with base B and base field $\mathbb{K}$. Note, parenthetically, that the transition maps for the above trivialisations take, for $(x,v)\in O_{ij}\times\mathbb{K}^{\times r}$ arbitrary, the form

$$[\tau_i]\circ[\tau_j]^{-1}(x,v) = [\tau_i]\big(\big[\big((\beta_j(x),x),v\big)\big]\big) = \big(x,\beta_i(x)^{-1}\circ\beta_j(x)(v)\big) = \big(x,g_{ij}(x)(v)\big)\,,$$

i.e., they are identical to those of $\mathbb{V}$. Due to its manifest $\mathbb{K}$-linearity, the map $[\widehat{\rm ev}]$ now becomes a vector-bundle isomorphism

$$[\widehat{\rm ev}]\ :\ (\mathsf{F}_{\rm GL}\mathbb{V}\times\mathbb{K}^{\times r})/{\rm GL}(r;\mathbb{K})\xrightarrow{\cong}\mathbb{V}\,.$$

Based on the above and earlier considerations, we may now articulate simple yet structural

Proposition 1. There is a one-to-one correspondence between (local) sections (and so also (local) trivialisations) of the frame bundle of a vector bundle and (local) trivialisations of the latter.

Proof: An arbitrary local section $\sigma\ :\ O\longrightarrow\pi^{-1}_{\mathsf{F}_{\rm GL}\mathbb{V}}(O)\subset\mathsf{F}_{\rm GL}\mathbb{V},\ O\in\mathscr{T}(B)$ gives rise to a map

$$\tau_\sigma\ :\ \pi_{\mathbb{V}}^{-1}(O)\longrightarrow O\times\mathbb{K}^{\times r}\ :\ v\longmapsto\big(\pi_{\mathbb{V}}(v),(\sigma\circ\pi_{\mathbb{V}})(v)^{-1}(v)\big)\,,$$

manifestly $\mathbb{K}$-linear and smooth, with an inverse

$$\tau_\sigma^{-1}\ :\ O\times\mathbb{K}^{\times r}\longrightarrow\pi_{\mathbb{V}}^{-1}(O)\ :\ (x,V)\longmapsto\sigma(x)(V)\,,$$

which is also smooth (and $\mathbb{K}$-linear). The stated properties permit us to identify τ_σ as a local trivialisation of $\mathbb{V}$ associated with the local section σ of the frame bundle.

Reversing the reasoning, we associate to any trivialisation $\tau\ :\ \pi_{\mathbb{V}}^{-1}(O)\xrightarrow{\cong}O\times\mathbb{K}^{\times r}$ a (local) section

$$\sigma_\tau\ :\ O\longrightarrow\pi^{-1}_{\mathsf{F}_{\rm GL}\mathbb{V}}(O)\ :\ x\longmapsto\tau^{-1}(x,\cdot)\,.$$

Next, we verify that the two maps written out above are each other's inverses. Indeed, we establish the equality

$$\forall_{(x,V)\in O\times\mathbb{K}^{\times r}}\ :\ \sigma_{\tau_\sigma}(x)(V)=\tau_\sigma^{-1}(x,V)=\sigma(x)(V)\,,$$

from which we derive the identity

$$\sigma_{\tau_\sigma}=\sigma\,.$$

Furthermore,

$$\begin{aligned}\forall_{v\in\pi_{\mathbb{V}}^{-1}(O)}\ :\ \tau_{\sigma_\tau}(v) &= \big(\pi_{\mathbb{V}}(v),(\sigma_\tau\circ\pi_{\mathbb{V}})(v)^{-1}(v)\big)=\big(\pi_{\mathbb{V}}(v),\tau^{-1}\big(\pi_{\mathbb{V}}(v),\cdot\big)^{-1}(v)\big) \\ &\equiv \tau\circ\tau^{-1}\big(\pi_{\mathbb{V}}(v),\tau^{-1}\big(\pi_{\mathbb{V}}(v),\cdot\big)^{-1}(v)\big)=\tau(v)\,,\end{aligned}$$

whence

$$\tau_{\sigma_\tau}=\tau\,.$$

□

and

Proposition 2. Every family of local trivialisations of the frame bundle of a vector bundle induces a family of local trivialisations of the vector bundle (associated with the same family of open sets in their common base) with the same transition maps.

Proof: Let $(\mathbb{V}, B, \mathbb{K}^{\times r}, \pi_{\mathbb{V}})$ be a vector bundle, and let $(\mathsf{F}_{\rm GL}\mathbb{V},\ B, {\rm GL}(r;\mathbb{K}),\ \pi_{\mathsf{F}_{\rm GL}\mathbb{V}})$ be its frame bundle. Fix two local trivialisations $\tau_i\ :\ \pi_{\mathsf{F}_{\rm GL}\mathbb{V}}^{-1}(O_i) \xrightarrow{\cong} O_i \times {\rm GL}(r;\mathbb{K}),\ O_i \in \mathscr{T}(B),\ i \in \{1,2\}$ of the latter bundle, with a nonempty intersection of domains, $O_1 \cap O_2$, over which we find transition maps $g_{12}\ :\ O_1 \cap O_2 \longrightarrow {\rm GL}(r;\mathbb{K})$. To each of the two trivialisations, associate a local section, as in Prop. V.3.,

$$\sigma_i\ :\ O_i \longrightarrow \pi_{\mathsf{F}_{\rm GL}\mathbb{V}}^{-1}(O_i) \subset \mathsf{F}_{\rm GL}\mathbb{V}\ :\ y \longmapsto \tau_i^{-1}(y, \mathbf{1}_n)\,,$$

and subsequently use them to construct the respective local trivialisations of $\mathbb{V}$ according to the prescription given in the proof of Prop. 1,

$$\tau_{\sigma_i}\ :\ \pi_{\mathbb{V}}^{-1}(O_i) \xrightarrow{\cong} O_i \times \mathbb{K}^{\times r}\ :\ v \longmapsto \big(\pi_{\mathbb{V}}(v), (\sigma_i \circ \pi_{\mathbb{V}})(v)^{-1}(v)\big)\,, \quad i \in \{1,2\}\,.$$

That the the transition map between these is the desired one is readily confirmed in a direct calculation, carried out for arbitrary $(y, V) \in O_{12} \times \mathbb{K}^{\times r}$,

$$\tau_{\sigma_1} \circ \tau_{\sigma_2}^{-1}(y, V) = \tau_{\sigma_1}\big(\sigma_2(y)(V)\big) = \Big(\pi_{\mathbb{V}}\big(\sigma_2(y)(V)\big), (\sigma_1 \circ \pi_{\mathbb{V}})\big(\sigma_2(y)(V)\big)^{-1}\big(\sigma_2(y)(V)\big)\Big)$$

$$= \quad \big(y, \sigma_1(y)^{-1} \circ \sigma_2(y)(V)\big)\,.$$

Taking into account the following chain of equalities:

$$\sigma_2(y) \quad \equiv \quad \tau_2^{-1}(y, \mathbf{1}_n) = \tau_1^{-1}\big(y, g_{12}(y)\big) = \tau_1^{-1}(y, \mathbf{1}_n) \triangleleft g_{12}(y) \equiv \tau_1^{-1}(y, \mathbf{1}_n) \circ g_{12}(y) \equiv \sigma_1(y) \circ g_{12}(y)\,,$$

we reproduce the anticipated result

$$\tau_{\sigma_1} \circ \tau_{\sigma_2}^{-1}(y, V) = \big(y, \sigma_1(y)^{-1} \circ \sigma_1(y) \circ g_{12}(y)(V)\big) \equiv \big(y, g_{12}(y)(V)\big)\,.$$

□

2. The symmetry-module object in the category of spacetime bundles

We are now fully prepared—both conceptually and technically—to carry out a systematic construction of the two spaces of immediate physical interest: a 'space of fields amenable to observation/description in the local frames' and a 'space of symmetry transformations between the local frames'. As it happens, both arise from a procedure of association of a G-manifold to the underlying 'space of local observation/description frames' P, which we now abstract from the above motivating considerations.

Definition 1. Let $(\mathsf{P}, B, {\rm G}, \pi_{\mathsf{P}})$ be a principal bundle, and M – a manifold with a smooth (left) action $\lambda\ :\ {\rm G} \times M \longrightarrow M$ of the Lie group G. A **bundle associated with P by λ** is a fibre bundle

$$\big(\mathsf{P} \times_\lambda M, B, M, \pi_{\mathsf{P} \times_\lambda M}\big)$$

composed of

- the total space $\mathsf{P} \times_\lambda M \equiv (\mathsf{P} \times M)/{\rm G}$ given by the quotient manifold determined by the action of Eq. (VI.5);
- the base projection

$$\pi_{\mathsf{P} \times_\lambda M}\ :\ \mathsf{P} \times_\lambda M \longrightarrow B\ :\ [(p, m)] \longmapsto \pi_{\mathsf{P}}(p)\,.$$

Here, local trivialisations $\tau_i\ :\ \pi_{\mathsf{P}}^{-1}(O_i) \xrightarrow{\cong} O_i \times {\rm G},\ i \in I$ of the principal bundle P associated with an open cover $\{O_i\}_{i \in I}$ of the base B induce local trivialisations

$$[\tau_i]\ :\ \pi_{\mathsf{P} \times_\lambda M}^{-1}(O_i) \xrightarrow{\cong} O_i \times M\ :\ [(p, m)] \longmapsto \big(\pi_{\mathsf{P}}(p), \lambda_{{\rm pr}_2 \circ \tau_i(p)}(m)\big)\,,$$

with the ensuing transition maps

$$[\tau_i] \circ [\tau_j]^{-1} \;:\; O_{ij} \times M \circlearrowleft \;:\; (x,m) \longmapsto \big(x, \lambda_{g_{ij}(x)}(m)\big)\,.$$

Upon fixing (arbitrarily) a point $x \in B$, we choose (also arbitrarily) $p_* \in (\mathsf{P})_x$. Diffeomorphisms

$$[p_*]_\lambda \;:\; M \xrightarrow{\cong} (\mathsf{P} \times_\lambda M)_x \;:\; m \longmapsto [(p_*, m)]\,,$$

with inverses

$$[p_*]_\lambda^{-1} \;:\; (\mathsf{P} \times_\lambda M)_x \xrightarrow{\cong} M \;:\; [(p,m)] \longmapsto \lambda_{\phi_{\mathsf{P}}(p_*,p)}(m)$$

and the obvious property

$$\forall_{g \in \mathrm{G}} \;:\; [p_* \triangleleft g]_\lambda = [p_*]_\lambda \circ \lambda_g\,, \tag{3}$$

are called **fibre-modelling isomorphisms**. These induce **fibre-transport isomorphisms**

$$[p_2, p_1]_\lambda \equiv [p_2]_\lambda \circ [p_1]_\lambda^{-1} \;:\; (\mathsf{P} \times_\lambda M)_{\pi_{\mathsf{P}}(p_1)} \xrightarrow{\cong} (\mathsf{P} \times_\lambda M)_{\pi_{\mathsf{P}}(p_2)} \;:\; [(p,m)] \longmapsto \big[\big(p_2, \lambda_{\phi_{\mathsf{P}}(p_1,p)}(m)\big)\big]\,,$$

defined for all pairs $(p_1, p_2) \in \mathsf{P}^{\times 2}$.

For any pair $(\mathsf{P} \times_{\lambda_\alpha} M_\alpha, B, M_\alpha, \pi_{\mathsf{P} \times_{\lambda_\alpha} M_\alpha})$, $\alpha \in \{1,2\}$ of bundles associated with the same principal bundle $(\mathsf{P}, B, \mathrm{G}, \pi_{\mathsf{P}})$, we also define the **associated-bundle invariant** given by the bundle morphism

$$(\Phi, \mathrm{id}_B) \;:\; \mathsf{P} \times_{\lambda_1} M_1 \longrightarrow \mathsf{P} \times_{\lambda_2} M_2$$

with the fundamental property expressed by the commutative diagram (written for any $p_1, p_2 \in \mathsf{P}$)

$$\begin{array}{ccc} \big(\mathsf{P} \times_{\lambda_1} M_1\big)_{\pi_{\mathsf{P}}(p_1)} & \xrightarrow{[p_2,p_1]_{\lambda_1}} & \big(\mathsf{P} \times_{\lambda_1} M_1\big)_{\pi_{\mathsf{P}}(p_2)} \\ \Big\downarrow {\scriptstyle \Phi\restriction_{(\mathsf{P}\times_{\lambda_1} M_1)_{\pi_{\mathsf{P}}(p_1)}}} & & \Big\downarrow {\scriptstyle \Phi\restriction_{(\mathsf{P}\times_{\lambda_1} M_1)_{\pi_{\mathsf{P}}(p_2)}}} \\ \big(\mathsf{P} \times_{\lambda_2} M_2\big)_{\pi_{\mathsf{P}}(p_1)} & \xrightarrow[{[p_2,p_1]_{\lambda_2}}]{} & \big(\mathsf{P} \times_{\lambda_2} M_2\big)_{\pi_{\mathsf{P}}(p_2)} \end{array}\,.$$

Remark 1. The existence of the structure of a smooth manifold on the space of orbits $\mathsf{P} \times_\lambda M$ of the action $\widetilde{\lambda}$ is a direct consequence of Thm. I.21., which can be invoked in the present context in virtue of Cor. VI.1. The smoothness of the base projection $\pi_{\mathsf{P} \times_\lambda M}$ is readily inferred from Thm. V.2. upon noting that the projection closes the commutative diagram

$$\begin{array}{ccc} & & B \\ & {\scriptstyle \pi_{\mathsf{P}} \circ \mathrm{pr}_1} \nearrow & \Big\uparrow {\scriptstyle \pi_{\mathsf{P} \times_\lambda M}} \\ \mathsf{P} \times M & \xrightarrow[\pi_\sim]{} & \mathsf{P} \times_\lambda M \end{array}\,,$$

in which $\pi_\sim$ is a surjective submersion (by the very same Thm. I.21.), and $\pi_{\mathsf{P}} \circ \mathrm{pr}_1$ is manifestly smooth. As the latter map is also submersive, this property is inherited by $\pi_{\mathsf{P} \times_\lambda M}$, a fact that can also be demonstrated directly by applying the tangent functor T to the above diagram.

We shall, next, examine the local trivialisations, beginning with a check of their well-definedness. For that, we must verify that the value taken by the map $[\tau_i]$ on the class $[(p,m)]$ does not depend on the choice of the representative thereof. Thus, we compute

$$\begin{aligned} \big(\pi_{\mathsf{P}}(p \triangleleft g), \lambda\big(\mathrm{pr}_2 \circ \tau_i(p \triangleleft g), \lambda\big(g^{-1}, m\big)\big)\big) &= \big(\pi_{\mathsf{P}}(p), \lambda\big(\mathrm{pr}_2 \circ \tau_i(p) \cdot g, \lambda\big(g^{-1}, m\big)\big)\big) \\ &= \big(\pi_{\mathsf{P}}(p), \lambda\big(\mathrm{pr}_2 \circ \tau_i(p) \cdot g \cdot g^{-1}, m\big)\big) = \big(\pi_{\mathsf{P}}(p), \lambda\big(\mathrm{pr}_2 \circ \tau_i(p), m\big)\big)\,. \end{aligned}$$

Furthermore, since maps

$$\underline{\tau}_i \;:\; \pi_{\mathsf{P}}^{-1}(O_i) \times M \longrightarrow O_i \times M \;:\; (p,m) \longmapsto \big(\pi_{\mathsf{P}}(p), \lambda_{\mathrm{pr}_2 \circ \tau_i(p)}(m)\big)\,, \quad i \in \{1,2\}$$

are manifestly smooth, and related to $[\tau_i]$ through the commutative diagram

$$\begin{array}{ccc} & & O_i\times M \\ & \nearrow^{\underline{\tau}_i} & \uparrow{\scriptstyle [\tau_i]} \\ \pi_{\mathsf{P}}^{-1}(O_i)\times M & \xrightarrow[\pi_\sim]{} & \pi^{-1}_{\mathsf{P}\times_\lambda M}(O_i) \end{array}\ ,$$

in which the canonical projection $\pi_\sim$ is smooth by Thm. I.21. and Cor. VI.1, we conclude that also the maps $[\tau_i]$ are smooth in virtue of Thm. V.2. There is no doubt about smoothness (also local) of their inverses

$$[\tau_i]^{-1}\ :\ O_i\times M\longrightarrow\pi^{-1}_{\mathsf{P}\times_\lambda M}(O_i)\ :\ (x,m)\longmapsto\big[\big(\tau_i^{-1}(x,e),m\big)\big].$$

In all the hitherto considerations, we have *implicitly* assumed well-definedness of the definition of the maps $[\tau_i]$ and $[\tau_i]^{-1}$, and that calls for a separate verification—the latter justifies *a posteriori* our identification of the typical fibre

$$\pi^{-1}_{\mathsf{P}\times_\lambda M}\big(\{\pi_{\mathsf{P}\times_\lambda M}([(p,m)])\}\big)\cong M\,,\quad [(p,m)]\in\mathsf{P}\times_\lambda M$$

of the fibre bundle under reconstruction. We readily demonstrate the desired properties: Thus, for any $(x,m)\in O_i\times M$, we have

$$[\tau_i]\circ[\tau_i]^{-1}(x,m)=[\tau_i]\big(\big[\big(\tau_i^{-1}(x,e),m\big)\big]\big)=\big(\pi_{\mathsf{P}}\circ\tau_i^{-1}(x,e),\lambda\big(\mathrm{pr}_2\circ\tau_i\circ\tau_i^{-1}(x,e),m\big)\big)$$

$$=\quad\big(x,\lambda(e,m)\big)=(x,m)\,,$$

and we obtain, for $[(p,m)]\in\mathsf{P}\times_\lambda M$, $p=\tau_i^{-1}(x,g)$,

$$[\tau_i]^{-1}\circ[\tau_i]\big([(p,m)]\big)=[\tau_i]^{-1}\big(\pi_{\mathsf{P}}(p),\lambda\big(\mathrm{pr}_2\circ\tau_i(p),m\big)\big)=\big[\big(\tau_i^{-1}\big(\pi_{\mathsf{P}}(p),e\big),\lambda\big(\mathrm{pr}_2\circ\tau_i(p),m\big)\big)\big]$$

$$=\quad\big[\big(\tau_i^{-1}(x,e),\lambda(g,m)\big)\big]=\big[\big(\tau_i^{-1}(x,e)\triangleleft g,m\big)\big]=\big[\big(\tau_i^{-1}(x,g),m\big)\big]\equiv[(p,m)]\,.$$

Finally, we calculate

$$[\tau_i]\circ[\tau_j]^{-1}(x,m)\equiv[\tau_i]\circ[\tau_j]^{-1}\big(\pi_{\mathsf{P}}\circ\tau_j^{-1}(x,e),\lambda\big(\mathrm{pr}_2\circ\tau_j(\tau_j^{-1}(x,e)),m\big)\big)=[\tau_i]\big(\big[\tau_j^{-1}(x,e),m\big]\big)$$

$$=\quad\big(x,\lambda\big(\mathrm{pr}_2\circ\tau_i\circ\tau_j^{-1}(x,e),m\big)\big)=\big(x,\lambda\big(\mathrm{pr}_2(x,g_{ij}(x)),m\big)\big)\equiv\big(x,\lambda\big(g_{ij}(x),m\big)\big)\,.$$

The construction of the associated bundle is, therefore, well-defined.

Let us, next, consider the map

$$[p_*]_\lambda^{-1}\ :\ (\mathsf{P}\times_\lambda M)_x\longrightarrow M\ :\ [(p,m)]\longmapsto\lambda_{\phi_{\mathsf{P}}(p_*,p)}(m)\,,\quad p_*\in(\mathsf{P})_x\,.$$

It is well-defined as for any representative $(\widetilde{p},\widetilde{m})\in[(p,m)]$ we get

$$\lambda_{\phi_{\mathsf{P}}(p_*,\widetilde{p})}(\widetilde{m})=\lambda_{\phi_{\mathsf{P}}(p_*,p)}\circ\lambda_{\phi_{\mathsf{P}}(p,\widetilde{p})}(\widetilde{m})=\lambda_{\phi_{\mathsf{P}}(p_*,p)}(m)\,.$$

Moreover, it is bijective because of the implication

$$[p_*]_\lambda^{-1}\big([(p_2,m_2)]\big)=[p_*]_\lambda^{-1}\big([(p_1,m_1)]\big)\qquad\Longleftrightarrow\qquad m_2=\lambda_{\phi_{\mathsf{P}}(p_2,p_1)}(m_1)$$

$$\Longrightarrow\qquad[(p_2,m_2)]=\big[\big(p_2,\lambda_{\phi_{\mathsf{P}}(p_2,p_1)}(m_1)\big)\big]=\big[\big(p_2\triangleleft\phi_{\mathsf{P}}(p_2,p_1),m_1\big)\big]=[(p_1,m_1)]\,,$$

showing injectivity of $[p_*]_\lambda^{-1}$, and because any point $m\in M$ may be written as

$$m=[p_*]_\lambda^{-1}\big([(p_*,m)]\big)\,,$$

which testifies to the map's surjectivity, simultaneously indicating its inverse

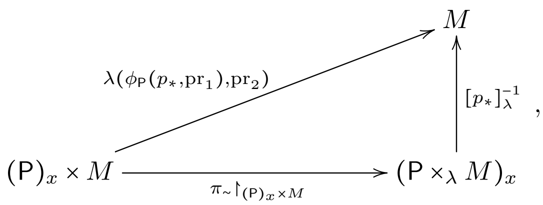
$$[p_*]_\lambda\ :\ M\longrightarrow(\mathsf{P}\times_\lambda M)_x\ :\ m\longmapsto[(p_*,m)]\,.$$

Indeed, the map $[p_*]_\lambda$ satisfies the identities

$$[p_*]_\lambda^{-1}\circ[p_*]_\lambda(m)\quad=\quad\lambda_{\phi_{\mathsf{P}}(p_*,p_*)}(m)=\lambda_e(m)=m\,,$$

$$[p_*]_\lambda\circ[p_*]_\lambda^{-1}\big([(p,m)]\big)\quad=\quad\big[\big(p_*,\lambda_{\phi_{\mathsf{P}}(p_*,p)}(m)\big)\big]\equiv\big[\big(p_*\triangleleft\phi_{\mathsf{P}}(p_*,p),m\big)\big]=[(p,m)]\,.$$

It is manifestly smooth as a superposition of the immersion $(p_*, \mathrm{id}_M) : M \longrightarrow \{p_*\} \times M \subset (\mathsf{P})_{\pi_{\mathsf{P}}(p_*)} \times M$ and the surjective submersion $\pi_{(\mathsf{P}\times M)/\mathrm{G}} : \mathsf{P} \times M \longrightarrow (\mathsf{P} \times M)/\mathrm{G}$. Smoothness of $[p_*]_\lambda^{-1}$, on the other hand, follows from Thm. V.2. referred to the commutative diagram



with a surjective submersion on the horizontal edge. The construction of the diffeomorphism $[p_*]_\lambda^{-1}$ thus provides us with an independent proof of the identification of the typical fibre of the associated bundle advanced above.

Example 1.

(1) A vector bundle $\mathbb{V}$ (of rank n) can be viewed/reconstructed as a bundle associated with the principal bundle of frames $\mathsf{F}_{\mathrm{GL}}\mathbb{V}$ by evaluation,

$$\mathbb{V} \cong \mathsf{F}_{\mathrm{GL}}\mathbb{V} \times_{\mathrm{ev}} \mathbb{K}^{\times n}\,.$$

(2) The **adjoint bundle**

$$(\mathrm{Ad}\,\mathsf{P} \equiv \mathsf{P} \times_{\mathrm{Ad}} \mathrm{G}, B, \mathrm{G}, \pi_{\mathrm{Ad}\,\mathsf{P}})\,.$$

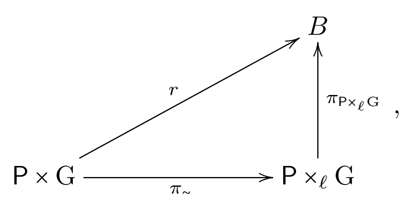
(3) The principal bundle P can be realised as an associated bundle

$$(\mathsf{P} \times_\ell \mathrm{G}, B, \mathrm{G}, \pi_{\mathsf{P}\times_\ell \mathrm{G}})\,,$$

where $\ell : \mathrm{G} \times \mathrm{G} \longrightarrow \mathrm{G}$ is the left regular action of G on itself. The relevant bundle isomorphism is given by

$$\widetilde{\imath} : \mathsf{P} \times_\ell \mathrm{G} \longrightarrow \mathsf{P} : [(p,g)] \longmapsto p \triangleleft g\,,$$

its smoothness following from the fact that it closes the commutative diagram



in which $\pi_\sim$ is a surjective submersion, and r is a smooth map. The inverse $\widetilde{\imath}$ is given, in a manifestly smooth form, by

$$\widetilde{\imath}^{-1} : \mathsf{P} \longrightarrow \mathsf{P} \times_\ell \mathrm{G} : p \longmapsto [(p,e)]\,.$$

On the associated bundle $\mathsf{P} \times_\ell \mathrm{G}$, we find the following right action:

$$\widetilde{r} : (\mathsf{P} \times_\ell \mathrm{G}) \times \mathrm{G} \longrightarrow \mathsf{P} \times_\ell \mathrm{G} : \big([(p,g)], h\big) \longmapsto [(p, g \cdot h)]\,.$$

Relative to it, each fibre is a torsor. The isomorphism $\widetilde{\imath}$ is G-equivariant,

$$\widetilde{\imath} \circ \widetilde{r}\big([(p,g)], h\big) = \widetilde{\imath}\big([(p, g\cdot h)]\big) = p \triangleleft (g \cdot h) = (p \triangleleft g) \triangleleft h = r \circ \widetilde{\imath}\big([(p,g)], h\big)\,,$$

and so we do, indeed, have a principal-bundle isomorphism.

In a search for automorphisms of the associated bundle $\mathsf{P} \times_\ell \mathrm{G}$, we note that due to mutual commutativity of the left $\ell.$ and right $\wp. : \mathrm{G} \times \mathrm{G} \longrightarrow \mathrm{G} : (g,h) \longmapsto g \cdot h$ regular actions, the latter induces—in virtue of Prop. 3, and for any $g \in \mathrm{G}$—an associated-bundle invariant

$$\Phi[r_g] : \mathsf{P} \times_\ell \mathrm{G} \circlearrowleft : [(p,h)] \longmapsto \Phi[r_g]^{\pi_{\mathsf{P}}(p)}\big([(p,h)]\big)\,,$$

with

$$\Phi[r_g]^{\pi_{\mathsf{P}}(p)}\big([(p,h)]\big) \;=\; [p]_{\mathsf{P}\times_\ell \mathrm{G}} \circ r_g \circ [p]_{\mathsf{P}\times_\ell \mathrm{G}}^{-1}\big([(p,h)]\big) = [p]_{\mathsf{P}\times_\ell \mathrm{G}} \circ r_g \circ \ell_{\phi_{\mathsf{P}}(p,p)}(h) = [p]_{\mathsf{P}\times_\ell \mathrm{G}} \circ r_g(h)$$

$$= \quad [p]_{\mathsf{P}\times_\ell \mathrm{G}}(h\cdot g) = [(p,h\cdot g)] \equiv \widetilde{r}_g\big([(p,h)]\big)\,,$$

whence

$$\Phi[r_g] \equiv \widetilde{r}_g\,,$$

and since

$$[(p,h)] = [(p \triangleleft h, e)] \equiv \widetilde{\iota}^{-1}(p \triangleleft h)$$

and

$$[(p,h\cdot g)] = [(p \triangleleft h\cdot g, e)] = \big[\big((p\triangleleft h)\triangleleft g, e\big)\big] = \big[\big(r_g(p\triangleleft h), e\big)\big] \equiv \widetilde{\iota}^{-1}\circ r_g(p\triangleleft h)\,,$$

we obtain

$$\widetilde{\iota}\circ\Phi[r_g]\circ\widetilde{\iota}^{-1} = r_g\,.$$

It is in this sense that automorphisms $\Phi[r_g]$ are induced by $r_\cdot$, and the latter can be regarded as a model associated-bundle invariant.

As announced before, the practical (*e.g.*, physical) purpose behind the construction of the associated bundle is to obtain a smooth distribution of manifolds of a predetermined (iso)type M over a give base B (*e.g.*, a spacetime), endowed with a distinguished action of a fixed Lie group G (*e.g.*, of symmetries of a physical theory), the latter being local over the base. In other words, it is to obtain a manifold locally modelled on $O\times M$, $O\subset B$ with an action of G locally modelled on λ. That the goal thus defined has been attained is demonstrated convincingly in the following two propositions.

Proposition 3. Fix a principal bundle $((\mathsf{P},B,\mathrm{G},\pi_{\mathsf{P}}),r)$. The **category of bundles associated with** P, with associated-bundle invariants as morphisms, to be denoted as

$$\mathbf{AssBun}(\mathsf{P})\,.$$

is canonically equivalent to the category $\mathbf{Man}_{\mathrm{G}}$ of manifolds with a left action of G, with G-equivariant maps as morphisms.

Proof: The first part of the statement is merely an indication of the class of morphisms to be considered, and as such it does not require a separate proof (associated-bundle invariants can be composed, and the identity map is—of course—an associated-bundle invariant). Also the one-to-one correspondence between objects of the category $\mathbf{AssBun}(\mathsf{P})$ and G-manifolds is obvious. Thus, the only thing that needs to be checked is the relevant bijective correspondence between associated-bundle invariants and G-equivariant maps.

Let $(\Phi,\mathrm{id}_B)\;:\;\mathsf{P}\times_{\lambda_1}M_1\longrightarrow\mathsf{P}\times_{\lambda_2}M_2$ be an associated-bundle invariant. We may define, for some (arbitrary) point $p\in\mathsf{P}$, a map—manifestly smooth—

$$\chi[\Phi] := [p]_{\lambda_2}^{-1}\circ\Phi\circ[p]_{\lambda_1}\;:\;M_1\xrightarrow{\cong}(\mathsf{P}\times_{\lambda_1}M_1)_{\pi_{\mathsf{P}}(p)}\longrightarrow(\mathsf{P}\times_{\lambda_2}M_2)_{\pi_{\mathsf{P}}(p)}\xrightarrow{\cong}M_2\,,$$

which, owing to the defining property of Φ,

$$\Phi\circ[p_2]_{\lambda_1}\circ[p_1]_{\lambda_1}^{-1} = [p_2]_{\lambda_2}\circ[p_1]_{\lambda_2}^{-1}\circ\Phi\,,$$

does not depend on the choice of the point p used in its definition. The G-equivariance of the thus determined map follows from a direct computation, invoking Eq. (3) and carried out for arbitrary $(p,g)\in\mathsf{P}\times\mathrm{G}$,

$$\begin{aligned}\chi[\Phi]\circ\lambda_{1\,g} \;&\equiv\; [p]_{\lambda_2}^{-1}\circ\Phi\circ\big([p]_{\lambda_1}\circ\lambda_{1\,g}\big) = [p]_{\lambda_2}^{-1}\circ\Phi\circ[p\triangleleft g]_{\lambda_1} \equiv \big([p\triangleleft g]_{\lambda_2}\circ\lambda_{2\,g^{-1}}\big)^{-1}\circ\Phi\circ[p\triangleleft g]_{\lambda_1}\\ &=\; \lambda_{2\,g}\circ[p\triangleleft g]_{\lambda_2}^{-1}\circ\Phi\circ[p\triangleleft g]_{\lambda_1} = \lambda_{2\,g}\circ[p]_{\lambda_2}^{-1}\circ\Phi\circ[p]_{\lambda_1} \equiv \lambda_{2\,g}\circ\chi[\Phi]\,.\end{aligned}$$

Thus,

$$\chi[\Phi]\in\mathrm{Hom}_{\mathrm{G}}(M_1,M_2)\,.$$

Conversely, to every map $\chi\in\mathrm{Hom}_{\mathrm{G}}(M_1,M_2)$, we may associate a (smooth) map

$$\Phi[\chi]^{\pi_{\mathsf{P}}(p)} := [p]_{\lambda_2}\circ\chi\circ[p]_{\lambda_1}^{-1}\;:\;(\mathsf{P}\times_{\lambda_1}M_1)_{\pi_{\mathsf{P}}(p)}\longrightarrow(\mathsf{P}\times_{\lambda_1}M_2)_{\pi_{\mathsf{P}}(p)}\;:\;[(p,m)]\longmapsto\big[\big(p,\chi(m)\big)\big]\,,$$

depending on $p \in \mathsf{P}$ exclusively through its projection to B,

$$\Phi[\chi]^{\pi_{\mathsf{P}}(p \triangleleft g)} = [p \triangleleft g]_{\lambda_2} \circ \chi \circ [p \triangleleft g]_{\lambda_1}^{-1} = [p]_{\lambda_2} \circ \big(\lambda_{2\,g} \circ \chi \circ \lambda_{1\,g^{-1}}\big) \circ [p]_{\lambda_1}^{-1}$$

$$= [p]_{\lambda_2} \circ \chi \circ \big(\lambda_{1\,g} \circ \lambda_{1\,g^{-1}}\big) \circ [p]_{\lambda_1}^{-1} = [p]_{\lambda_2} \circ \chi \circ [p]_{\lambda_1}^{-1} \equiv \Phi[\chi]^{\pi_{\mathsf{P}}(p)}\,,$$

and hence defining an associated-bundle invariant by the formula

$$\Phi[\chi] \;:\; \mathsf{P} \times_{\lambda_1} M_1 \longrightarrow \mathsf{P} \times_{\lambda_2} M_2 \;:\; [(p,m)] \longmapsto \Phi[\chi]^{\pi_{\mathsf{P}}(p)}\big([(p,m)]\big)\,.$$

Indeed, we calculate

$$\Phi[\chi] \circ [p_2, p_1]_{\lambda_1} \equiv \big([p_2]_{\lambda_2} \circ \chi \circ [p_2]_{\lambda_1}^{-1}\big) \circ \big([p_2]_{\lambda_1} \circ [p_1]_{\lambda_1}^{-1}\big) = [p_2]_{\lambda_2} \circ \chi \circ [p_1]_{\lambda_1}^{-1}$$

$$= \big([p_2]_{\lambda_2} \circ [p_1]_{\lambda_2}^{-1}\big) \circ \big([p_1]_{\lambda_2} \circ \chi \circ [p_1]_{\lambda_1}^{-1}\big) \equiv [p_2, p_1]_{\lambda_2} \circ \Phi[\chi]\,.$$

The two assignments given above:

$$\mathrm{Hom}_{\mathbf{AssBun}(\mathsf{P})}(\mathsf{P} \times_{\lambda_1} M_1, \mathsf{P} \times_{\lambda_2} M_2) \longrightarrow \mathrm{Hom}_{\mathrm{G}}(M_1, M_2) \;:\; (\Phi, \mathrm{id}_B) \longmapsto \chi[\Phi]$$

and

$$\mathrm{Hom}_{\mathrm{G}}(M_1, M_2) \longrightarrow \mathrm{Hom}_{\mathbf{AssBun}(\mathsf{P})}(\mathsf{P} \times_{\lambda_1} M_1, \mathsf{P} \times_{\lambda_2} M_2) \;:\; \chi \longmapsto \big(\Phi[\chi], \mathrm{id}_B\big)$$

are mutually inverse, and each of them is functorial. Indeed, given a manifold M with an action $\lambda : \mathrm{G} \times M \longrightarrow M$, we obtain, over an arbitrary point $p \in \mathsf{P}$, the equality

$$\Phi[\mathrm{id}_M]^{\pi_{\mathsf{P}}(p)} = [p]_{\lambda_2} \circ \mathrm{id}_B \circ [p]_{\lambda_1}^{-1} = [p]_{\lambda_2} \circ [p]_{\lambda_1}^{-1} = \mathrm{id}_{(\mathsf{P} \times_\lambda M)_{\pi_{\mathsf{P}}(p)}}\,,$$

and so also

$$\Phi[\mathrm{id}_M] = \mathrm{id}_{\mathsf{P} \times_\lambda M}\,.$$

Furthermore, for any pair of G-equivariant maps $\chi_\alpha : M_\alpha \longrightarrow M_{\alpha+1}$, $\alpha \in \{1,2\}$ between G-manifolds M_β, $\beta \in \{1,2,3\}$ with the respective actions $\lambda_\beta : \mathrm{G} \times M_\beta \longrightarrow M_\beta$, we obtain the expected identity (written out for an arbitrary point $p \in \mathsf{P}$) we arrive at the commutative diagram (for an arbitrary point $p \in \mathsf{P}$)

$$\Phi[\chi_2 \circ \chi_1] \equiv [p]_{\lambda_3} \circ (\chi_2 \circ \chi_1) \circ [p]_{\lambda_1}^{-1} = [p]_{\lambda_3} \circ \chi_2 \circ [p]_{\lambda_2}^{-1} \circ [p]_{\lambda_2} \circ \chi_1 \circ [p]_{\lambda_1}^{-1} \equiv \Phi[\chi_2] \circ \Phi[\chi_1]\,.$$

An analogous argument convinces us of the functoriality of the inverse map. □

Remark 2. The term "adjoint bundle" is sometimes used in the literature with regard to another associated bundle, to wit,

$$(\mathrm{ad}\,\mathsf{P} \equiv \mathsf{P} \times_{\mathrm{T}_e\mathrm{Ad}} \mathfrak{g}, B, \mathfrak{g}, \pi_{\mathsf{P} \times_{\mathrm{T}_e\mathrm{Ad}} \mathfrak{g}})\,,$$

with the typical fibre given by the Lie algebra $\mathfrak{g}$ of the structure Lie group G.

Remark 3. Prior to advancing in our discussion of the procedure of association, we pause to investigate the important question about the anatomy of (global) sections of an associated bundle. Let $\phi \in \Gamma(\mathsf{P} \times_\lambda M)$ be such a section. Upon restriction to the domain O_i $(i \in I)$ of a local section $\sigma_i : O_i \longrightarrow \mathsf{P}$ of the mother principal bundle, *i.e.*, to the latter's trivialisation chart $\widetilde{\tau}_i \equiv \widetilde{\tau}_{\sigma_i} : \pi_{\mathsf{P} \times_\lambda M}^{-1}(O_i) \xrightarrow{\cong} O_i \times M$, the section ϕ becomes a section of the bundle $\mathsf{P} \times_\lambda M\restriction_{O_i}$, and so under $\widetilde{\tau}_i$ takes the form

$$\widetilde{\tau}_i \circ \phi\restriction_{\mathcal{O}_i} = \big(\mathrm{id}_{\mathcal{O}_i}, \mu_i\big)$$

for some (locally) smooth map $\mu_i : \mathcal{O}_i \longrightarrow M$. Invoking the explicit form of the iverse of the induced trivialisation $[\tau_i]$, we recover the local presentation of the global section:

$$\phi\restriction_{\mathcal{O}_i} = \widetilde{\tau}_i^{-1} \circ \big(\mathrm{id}_{\mathcal{O}_i}, \mu_i\big) = \big[\big(\tau_i^{-1}(\cdot, e), \mu_i(\cdot)\big)\big]\,,$$

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$$\phi\restriction_{\mathcal{O}_i} = \big[\big(\sigma_i(\cdot), \mu_i(\cdot)\big)\big]\,,$$

where $\sigma_i \equiv \sigma_{\tau_i}$ is the flat unital section of P (associated with τ_i). Accordingly, we may (and shall, henceforth) *locally* represent an arbitrary global section as

$$\phi =_{\text{loc.}} \left[(\sigma,\mu)\right],$$

i.e., as a G-orbit of a pair composed of a local section $\sigma : O \longrightarrow \mathsf{P}$ of the principal bundle P (over some open set $O \in \mathscr{T}(B)$) and a locally smooth map $\mu : O \longrightarrow M$. Thus equipped, we return to the study of the structure of associated bundles.

We have the fundamental

Theorem 1. There exists on $\mathrm{Ad}\,\mathsf{P}$ a canonical structure of a group object in the (slice) category $\mathbf{Bun}(B)/B$, locally modelled on G. This structure canonically induces that of a (Fréchet–Lie) group on the space $\Gamma(\mathrm{Ad}\,\mathsf{P})$ of sections of the bundle. The latter admits a realisation on the space $\Gamma(\mathsf{P}\times_\lambda M)$ of sections of the associated bundle $\mathsf{P}\times_\lambda M$, which, in turn, is induced from a canonical structure of a left $\widehat{\mathrm{Ad}\,\mathsf{P}}$-module-object in $\mathbf{Bun}(\Sigma)/\Sigma$ that exists on $\mathsf{P}\times_\lambda M$, itself being locally modelled on λ.

Proof: Before we proceed, let us remark that a variant of the cartesian product adapted to the structure of the slice category $\mathbf{Bun}(B)/B$ is the B-fibred cartesian product (which exists owing to submersivity of base projections of fibre bundles). Moreover, the terminal object in that category has a natural model

$$\begin{array}{ccc} \bullet & \hookrightarrow & B\times\bullet\equiv B \\ & & \Big\downarrow \mathrm{id}_B \\ & & B \end{array} .$$

With these preparatory observations in mind, we may now dive headlong into details of the constructive proof below.

Consider, first, a binary operation

$$[M] \quad : \quad \begin{array}{ccc} \mathrm{Ad}\,\mathsf{P}\times_B \mathrm{Ad}\,\mathsf{P} & \longrightarrow & \mathrm{Ad}\,\mathsf{P} \\ & \searrow \quad \swarrow & \\ & B & \end{array} \quad : \ \left([(p_1,g_1)],[(p_2,g_2)]\right) \longmapsto \left[\left(p_1, g_1\cdot \mathrm{Ad}_{\phi_{\mathsf{P}}(p_1,p_2)}(g_2)\right)\right],$$

alongside the slice-category constant

$$[\varepsilon] \ : \quad \begin{array}{ccc} B\times\bullet\equiv B & \longrightarrow & \mathrm{Ad}\,\mathsf{P} \\ & \searrow \quad \swarrow & \\ & B & \end{array} \quad : \ x \longmapsto [(p,e)], \quad p\in\mathsf{P}_x ,$$

and the unary operation

$$[\mathrm{Inv}] \ : \quad \begin{array}{ccc} \mathrm{Ad}\,\mathsf{P} & \longrightarrow & \mathrm{Ad}\,\mathsf{P} \\ & \searrow \quad \swarrow & \\ & B & \end{array} \quad : \ [(p,g)] \longmapsto \left[\left(p,g^{-1}\right)\right].$$

We begin by verifying that all three maps are well-defined. Thus, let $(p_3,g_3)\in[(p_1,g_1)]$, so that $(p_3,g_3) = (p_1 \triangleleft g_{13}, \mathrm{Ad}_{g_{13}^{-1}}(g_1))$ and $(p_4,g_4)\in[(p_2,g_2)]$, *i.e.*, $(p_4,g_4) = (p_2 \triangleleft g_{24}, \mathrm{Ad}_{g_{24}^{-1}}(g_2))$, where we are using the notation $g_{ij}\equiv\phi_{\mathsf{P}}(p_i,p_j)$, $(i,j)\in\{(1,3),(2,4)\}$ for the sake of brevity. In virtue of Prop. VI.1., we obtain

$$\left[\left(p_3, g_3\cdot\mathrm{Ad}_{g_{34}}(g_4)\right)\right] = \left[\left(p_1, \mathrm{Ad}_{g_{13}}\left(g_3\cdot\mathrm{Ad}_{g_{34}}(g_4)\right)\right)\right] = \left[\left(p_1, \mathrm{Ad}_{g_{13}}\left(\mathrm{Ad}_{g_{13}^{-1}}(g_1)\cdot\mathrm{Ad}_{g_{34}\cdot g_{24}^{-1}}(g_2)\right)\right)\right]$$

$$= \quad \left[\left(p_1, g_1\cdot\mathrm{Ad}_{g_{13}\cdot g_{34}\cdot g_{42}}(g_2)\right)\right] = \left[\left(p_1, g_1\cdot\mathrm{Ad}_{g_{12}}(g_2)\right)\right]$$

and

$$\big[\big(p_3, g_3^{-1}\big)\big] = \big[\big(p_1, \mathrm{Ad}_{g_{13}}\big(g_3^{-1}\big)\big)\big] = \big[\big(p_1, \mathrm{Ad}_{g_{13}}(g_3)^{-1}\big)\big] = \big[\big(p_1, g_1^{-1}\big)\big] .$$

Besides, we readily establish that the value taken by the constant $[\varepsilon]$ does not depend on the choice of the point in the fibre over x as for any $\widetilde{p} = p \triangleleft \phi_{\mathsf{P}}(p, \widetilde{p})$, we get

$$[(\widetilde{p}, e)] = \big[\big(p \triangleleft \phi_{\mathsf{P}}(p, \widetilde{p}), e\big)\big] = \big[\big(p, \mathrm{Ad}_{\phi_{\mathsf{P}}(p,\widetilde{p})}(e)\big)\big] = [(p, e)] .$$

Our proof of the claim that the above structure is locally modelled on G boils down to demonstrating that the fibre-modelling isomorphisms

$$[p_*]_{\mathrm{Ad}} \ :\ \big(\mathrm{Ad}\,\mathsf{P}\big)_x \longrightarrow \mathrm{G} \ :\ [(p, g)] \longmapsto \mathrm{Ad}_{\phi_{\mathsf{P}}(p_*,p)}(g) , \quad x \in B ,$$

are group homomorphisms, which we show below (for an arbitrary pair of points $(p_1, g_1), (p_2, g_2) \in \mathsf{P} \times \mathrm{G}$ such that $p_1, p_2 \in (\mathsf{P})_x$), invoking Prop. VI.1. again along the way,

$$\begin{aligned} & [p_*]_{\mathrm{Ad}} \circ [M]\big([(p_1, g_1)], [(p_2, g_2)]\big) = [p_*]_{\mathrm{Ad}}\big(\big[\big(p_1, g_1 \cdot \mathrm{Ad}_{\phi_{\mathsf{P}}(p_1,p_2)}(g_2)\big)\big]\big) \\ = \ & \mathrm{Ad}_{\phi_{\mathsf{P}}(p_*,p_1)}\big(g_1 \cdot \mathrm{Ad}_{\phi_{\mathsf{P}}(p_1,p_2)}(g_2)\big) = \mathrm{Ad}_{\phi_{\mathsf{P}}(p_*,p_1)}(g_1) \cdot \mathrm{Ad}_{\phi_{\mathsf{P}}(p_*,p_1)\cdot\phi_{\mathsf{P}}(p_1,p_2)}(g_2) \\ = \ & \mathrm{Ad}_{\phi_{\mathsf{P}}(p_*,p_1)}(g_1) \cdot \mathrm{Ad}_{\phi_{\mathsf{P}}(p_*,p_2)}(g_2) \equiv M\big([p_*]_{\mathrm{Ad}}\big([p_1, g_1]\big), [p_*]_{\mathrm{Ad}}\big([p_2, g_2]\big)\big) . \end{aligned}$$

The first step towards a reconstruction of the fibrewise action of the group $\Gamma(\mathrm{Ad}\,\mathsf{P})$ on the space $\Gamma(\mathsf{P} \times_\lambda M)$ consists in identifying the following left action of the adjoint bundle on P:

$$(4) \qquad [r]. \ :\ \begin{array}{ccc} \mathrm{Ad}\,\mathsf{P} \times_B \mathsf{P} & \longrightarrow & \mathsf{P} \\ & \searrow \ \swarrow & \\ & B & \end{array} \ :\ \big([(p, g)], \widetilde{p}\big) \longmapsto r_{\mathrm{Ad}_{\phi_{\mathsf{P}}(\widetilde{p},p)}(g)}(\widetilde{p}) .$$

The latter is defined unequivocally since for any representative $(p_2, g_2) \in [(p_1, g_1)]$, we obtain

$$r_{\mathrm{Ad}_{\phi_{\mathsf{P}}(\widetilde{p},p_2)}(g_2)}(\widetilde{p}) = r_{\mathrm{Ad}_{\phi_{\mathsf{P}}(\widetilde{p},p_1)\cdot\phi_{\mathsf{P}}(p_1,p_2)}(g_2)}(\widetilde{p}) = r_{\mathrm{Ad}_{\phi_{\mathsf{P}}(\widetilde{p},p_1)}(\mathrm{Ad}_{\phi_{\mathsf{P}}(p_1,p_2)}(g_2))}(\widetilde{p}) = r_{\mathrm{Ad}_{\phi_{\mathsf{P}}(\widetilde{p},p_1)}(g_1)}(\widetilde{p}) .$$

Its smoothness is ensured by Thm. V.2.—indeed, $[r].$ is the (only) smooth map induced by the (manifestly smooth) map

$$\widetilde{r}. \ :\ \begin{array}{ccc} (\mathsf{P} \times \mathrm{G}) \times_B \mathsf{P} & \longrightarrow & \mathsf{P} \\ & \searrow \ \swarrow & \\ & B & \end{array} \ :\ \big((p, g), \widetilde{p}\big) \longmapsto r_{\mathrm{Ad}_{\phi_{\mathsf{P}}(\widetilde{p},p)}(g)}(\widetilde{p}) ,$$

constant on level sets of the canonical projection $\pi_\sim \ :\ \mathsf{P} \times \mathrm{G} \longrightarrow (\mathsf{P} \times \mathrm{G})/\mathrm{G}$. We readily convince ourselves that $[r].$ has properties analogous to the defining ones of a (left) group action: The neutral element acts trivially,

$$[r]_{[(p,e)]}(\widetilde{p}) = r_{\mathrm{Ad}_{\phi_{\mathsf{P}}(\widetilde{p},p)}(e)}(\widetilde{p}) = r_e(\widetilde{p}) = \widetilde{p} ,$$

and $[r].$ is multiplicative in the first argument, *i.e.*, for any pair $[(p_1, g_1)] \equiv [(\widetilde{p}, \widetilde{g}_1)], [(p_2, g_2)] \equiv [(\widetilde{p}, \widetilde{g}_2)] \in (\mathsf{P}_{\mathrm{G}})_{\pi_{\mathsf{P}_{\mathrm{G}}}(\widetilde{p})}$, we find the identity

$$\begin{aligned} & [r]_{[M]([(p_1,g_1)],[(p_2,g_2)])}(\widetilde{p}) \equiv [r]_{[M]([(\widetilde{p},\widetilde{g}_1)],[(\widetilde{p},\widetilde{g}_2)])}(\widetilde{p}) = [r]_{[(\widetilde{p},\widetilde{g}_1\cdot\widetilde{g}_2)]}(\widetilde{p}) = r_{\widetilde{g}_1\cdot\widetilde{g}_2}(\widetilde{p}) \\ = \ & r_{\widetilde{g}_2\cdot\mathrm{Ad}_{\widetilde{g}_2^{-1}}(\widetilde{g}_1)}(\widetilde{p}) = r_{\mathrm{Ad}_{\widetilde{g}_2^{-1}}(\widetilde{g}_1)} \circ r_{\widetilde{g}_2}(\widetilde{p}) \equiv [r]_{[(r_{\widetilde{g}_2}(\widetilde{p}),\mathrm{Ad}_{\widetilde{g}_2^{-1}}(\widetilde{g}_1))]}\big(r_{\widetilde{g}_2}(\widetilde{p})\big) = [r]_{[(\widetilde{p},\widetilde{g}_1)]}\big(r_{\widetilde{g}_2}(\widetilde{p})\big) \\ \equiv \ & [r]_{[(p_1,g_1)]}\big([r]_{[(p_2,g_2)]}(\widetilde{p})\big) . \end{aligned}$$

It ought to be underlined that the action of the adjoint bundle defined above commutes with the defining (right) action $r.$—indeed, for any $\widetilde{p} \in (\mathsf{P}_{\mathrm{G}})_{\pi_{\mathsf{P}_{\mathrm{G}}}(p)}, [(p, g)] \equiv [(\widetilde{p}, \widetilde{g})] \in \mathrm{Ad}\,\mathsf{P}_{\mathrm{G}}$ and $h \in \mathrm{G}$, we conclude that

$$[r]_{[(p,g)]} \circ r_h(\widetilde{p}) = r_{\mathrm{Ad}_{\phi_{\mathsf{P}_{\mathrm{G}}}(r_h(\widetilde{p}),\widetilde{p})}(\widetilde{g})}\big(r_h(\widetilde{p})\big) = r_{\mathrm{Ad}_{h^{-1}}(\widetilde{g})}\big(r_h(\widetilde{p})\big) = r_{\widetilde{g}\cdot h}(\widetilde{p}) = r_h \circ r_{\widetilde{g}}(\widetilde{p}) \equiv r_h \circ [r]_{[(p,g)]}(\widetilde{p}) .$$

The significance of the above fact follows from the fact that it implies commutativity of the action induced by $[r].$ on $\mathsf{P}\times M$ as

$$\widetilde{[r]}. \ : \quad \begin{array}{ccc} \mathrm{Ad}\,\mathsf{P}\times_B(\mathsf{P}\times M) & \longrightarrow & \mathsf{P}\times M \\ & \searrow \quad \swarrow & \\ & B & \end{array} \quad : \ \big([(p_1,g_1)],(p_2,m_2)\big)\longmapsto\big(r_{\mathrm{Ad}_{\phi_{\mathsf{P}}(p_2,p_1)}(g_1)}(p_2),m_2\big) \tag{5}$$

with the action $\widetilde{\lambda}.$ defined in Eq. (VI.5) that serves as the basis of the construction of the associated bundle $\mathsf{P}\times_\lambda M$. Consequently, the induced action descends to the smooth quotient $\mathsf{P}\times_\lambda M\equiv(\mathsf{P}\times M)/\mathrm{G}$ in the form

$$[\lambda]. \ : \quad \begin{array}{ccc} \mathrm{Ad}\,\mathsf{P}\times_B(\mathsf{P}\times_\lambda M) & \longrightarrow & \mathsf{P}\times_\lambda M \\ & \searrow \quad \swarrow & \\ & B & \end{array} \quad : \ \big([(p_1,g_1)],[(p_2,m_2)]\big)\longmapsto\big[\big(p_2,\lambda_{\mathrm{Ad}_{\phi_{\mathsf{P}}(p_2,p_1)}(g_1)}(m_2)\big)\big], \tag{6}$$

where $[(p_2,\lambda_{\mathrm{Ad}_{\phi_{\mathsf{P}}(p_2,p_1)}(g_1)}(m_2))]\equiv[\widetilde{[r]}_{[](p_1,g_1)]}(p_2,m_2)]$.

We may, next, transpose the above actions, without losing any of their desired properties verified above, from the total space of the adjoint bundle to the space of its (global) sections, according to the prescription

$$\Gamma[r]. \ : \ \Gamma(\mathrm{Ad}\,\mathsf{P})\times\mathsf{P}\longrightarrow\mathsf{P} \ : \ (\sigma,p)\longmapsto[r]_{\sigma\circ\pi_{\mathsf{P}}(p)}(p)\,.$$

The space $\Gamma(\mathrm{Ad}\,\mathsf{P})$ (equipped with the natural structure of a Fréchet manifold) thus assumes the rôle of the support of the structure of a (Fréchet–Lie) group with group operations

$$\begin{aligned} \Gamma[M] \quad &: \quad \Gamma(\mathrm{Ad}\,\mathsf{P})\times\Gamma(\mathrm{Ad}\,\mathsf{P})\longrightarrow\Gamma(\mathrm{Ad}\,\mathsf{P}) \ : \ (\sigma_1\sigma_2)\longmapsto[M]\circ(\sigma_1,\sigma_2)\,, \\ \Gamma[\mathrm{Inv}] \quad &: \quad \Gamma(\mathrm{Ad}\,\mathsf{P}) \circlearrowleft \ : \ \sigma\longmapsto[\mathrm{Inv}]\circ\sigma\,, \\ \Gamma[\varepsilon] \quad &: \quad \bullet\longrightarrow\Gamma(\mathrm{Ad}\,\mathsf{P}) \ : \ \bullet\longmapsto\big[\big(\sigma(\cdot),e\big)\big]\,, \end{aligned}$$

induced, in an obvious (pointwise) manner, from the respective operations on $\mathrm{Ad}\,\mathsf{P}$, and, at the same time, that of a subgroup of the group of automorphisms of the principal bundle P (covering the identity on the base, *i.e.*, as expected in the slice category). Here, the map $\Gamma[r]_\sigma$ is identified with the automorphism $(\Gamma[r]_\sigma,\mathrm{id}_{\mathrm{G}},\mathrm{id}_B)$ in the notation of Def. VI.1. We may subsequently extend, again in an obvious way, the thus understood action of the group of sections of the adjoint bundle on P to the bundle $\mathsf{P}\times M$ over the same base B by setting

$$\Gamma[\widetilde{r}].:=\Gamma[r].\times\mathrm{id}_M \ : \ \Gamma(\mathrm{Ad}\,\mathsf{P})\times(\mathsf{P}\times M)\longrightarrow\mathsf{P}\times M \ : \ \big(\sigma,(p,m)\big)\longmapsto\big([r]_{\sigma\circ\pi_{\mathsf{P}}(p)}(p),m\big)\,.$$

As before, we find the key property of the latter action: its commutativity with the action $\widetilde{\lambda}.$. Indeed, for any $\sigma\equiv[(\pi,\gamma)]\in\Gamma(\mathrm{Ad}\,\mathsf{P})$, $g\in\mathrm{G}$ and $(p,m)\in\mathsf{P}\times M$, we find—upon invoking relative commutativity of the actions: $[r].$ i $r.$, checked formerly—the identity

$$\begin{aligned} \Gamma[\widetilde{r}]_\sigma\circ\widetilde{\lambda}_g(p,m) \quad &= \quad \big([r]_{\sigma\circ\pi_{\mathsf{P}}(r_g(p))}\big(r_g(p)\big),\lambda_{g^{-1}}(m)\big)\equiv\big([r]_{\sigma\circ\pi_{\mathsf{P}}(p)}\circ r_g(p),\lambda_{g^{-1}}(m)\big) \\ &= \quad \big(r_g\circ[r]_{\sigma\circ\pi_{\mathsf{P}}(p)}(p),\ell_{g^{-1}}(m)\big)=\widetilde{\lambda}_g\circ\Gamma[\widetilde{r}]_\sigma(p,m)\,. \end{aligned}$$

As a result of the above, the induced action $\Gamma[\widetilde{r}].$ descends to the quotient manifold $(\mathsf{P}\times M)/\mathrm{G}\equiv\mathsf{P}\times_\lambda M$ just like its bundle precursor did, *i.e.*, it canonically induces a left action of the group $\Gamma(P_{\mathrm{Ad.}}\mathrm{G})$ on the manifold $\mathsf{P}\times_\lambda M$, given by

$$[\Gamma[\widetilde{r}]]^\lambda_. \ : \ \Gamma(\mathrm{Ad}\,\mathsf{P})\times\mathsf{P}\times_\lambda M\longrightarrow\mathsf{P}\times_\lambda M \ : \ \big(\sigma,[(p,m)]\big)\longmapsto\big[\big([r]_{\sigma\circ\pi_{\mathsf{P}}(p)}(p),m\big)\big]\,.$$

Our hitherto analysis shows that the latter map is well-defined and has all the requisite properties of a (left) group action. In the last step, we induce with its help the action, postulated in the statement of the proposition, of the group $\Gamma(\mathrm{Ad}\,\mathsf{P})$ on the space of (global) sections of the assciated bundle,

$$\Gamma[\Gamma[\widetilde{r}]]^{\lambda}_{\cdot} \quad : \quad \Gamma(\mathrm{Ad}\,\mathsf{P}) \times \Gamma(\mathsf{P} \times_{\lambda} M) \longrightarrow \Gamma(\mathsf{P} \times_{\lambda} M)$$

$$: \quad \big(\sigma, [(\pi,\mu)]\big) \longmapsto \big[\big([r]_{\sigma\circ\pi_{\mathsf{P}}\circ\pi(\cdot)} \circ \pi(\cdot), \mu(\cdot)\big)\big] \equiv \big[\big([r]_{\sigma(\cdot)} \circ \pi(\cdot), \mu(\cdot)\big)\big]\,. \tag{7}$$

This is, self-evidently, a lift, to the space of sections, of the previously considered map $[\lambda]_{\cdot}$, whose well-definedness and multiplicativity in the first argument is a direct consequence of the respective properties of the action $\Gamma[\Gamma[\widetilde{r}]]_{\cdot}$, checked previously. That the action $[\lambda]_{\cdot}$ is locally modelled on $\lambda_{\cdot}$, as claimed, is most straightforwardly proven with the help of the isomorphisms $[p_*]_{\mathrm{Ad}}$ oraz $[p_*]_{\lambda}$, indicated before. Thus, we carry out the following calculation:

$$\begin{aligned} &\lambda_{[p_*]_{\mathrm{Ad}}([(p_1,g_1)])}\big([p_*]_\lambda\big([(p_2,m_2)]\big)\big) = \lambda_{\mathrm{Ad}_{\phi_{\mathsf{P}}(p_*,p_1)}(g_1)} \circ \lambda_{\phi_{\mathsf{P}}(p_*,p_2)}(m_2) \\ = \;& \lambda_{\phi_{\mathsf{P}}(p_*,p_2)\cdot \mathrm{Ad}_{\phi_{\mathsf{P}}(p_2,p_1)}(g_1)}(m_2) = \lambda_{\phi_{\mathsf{P}}(p_*,p_2)}\big(\lambda_{\mathrm{Ad}_{\phi_{\mathsf{P}}(p_2,p_1)}(g_1)}(m_2)\big) \\ \equiv \;& [p_*]_\lambda\big(\big[\big(p_2, \lambda_{\mathrm{Ad}_{\phi_{\mathsf{P}}(p_2,p_1)}(g_1)}(m_2)\big)\big]\big) \equiv [p_*]_\lambda \circ [\lambda]_{[(p_1,g_1)]}\big([(p_2,m_2)]\big)\,. \end{aligned}$$

□

The above proposition together with its constructive proof demonstrate convincingly that the goal set before has been attained: We have constructed a 'space of fields amenable to observation/description in the local frames' $\mathsf{P} \times_\lambda M$ and 'space of symmetry transformations between the local frames' $\mathrm{Ad}\,\mathsf{P}$ (with the frames in question provided by P). In so doing, the proposition and the proof emphasise the rôle played by the space of smooth sections of the associated bundle, which prompts us to take a closer look at the latter. We do that in

Proposition 4. There exists a bijection

$$\Gamma(\mathsf{P} \times_\lambda M) \cong \mathrm{Hom}_{\mathrm{G}}(\mathsf{P}, M)\,,$$

where $\mathrm{Hom}_{\mathrm{G}}(\mathsf{P}, M)$ is the set of G-equivariant maps $\mathsf{P} \longrightarrow M$ of Def. I.14.

Proof: Invoke Rem. 3 to express a *global* section $\phi \in \Gamma(\mathsf{P} \times_\lambda M)$ locally as $\phi = [(\sigma,\mu)]$ in terms of (local) sections $\sigma \in \Gamma_{\mathrm{loc}}(\mathsf{P})$ and $\mu \in \Gamma_{\mathrm{loc}}(B \times M)$. Using the quotient map and the canonical base projection of P, we define a map

$$\Phi_\lambda[\phi] \;:\; \mathsf{P} \longrightarrow M \;:\; p \longmapsto \lambda_{\phi_{\mathsf{P}}(p,\sigma\circ\pi_{\mathsf{P}}(p))}\big(\mu \circ \pi_{\mathsf{P}}(p)\big)\,.$$

We readily convince ourselves that the above definition makes sense as for any pair $(\sigma',\mu') = (\sigma \triangleleft \mathrm{Inv} \circ \gamma, \gamma \triangleright \mu)$ associated, in an obvious manner, with $\gamma \in \Gamma_{\mathrm{loc}}(B \times \mathrm{G})$, we find—upon invoking the axioms of an action of a group on a set—the desired equality

$$\begin{aligned} &\lambda_{\phi_{\mathsf{P}}(p,\sigma'\circ\pi_{\mathsf{P}}(p))}\big(\mu' \circ \pi_{\mathsf{P}}(p)\big) = \lambda_{\phi_{\mathsf{P}}(p,\sigma\circ\pi_{\mathsf{P}}(p)\triangleleft\gamma\circ\pi_{\mathsf{P}}(p)^{-1})}\big(\lambda_{\gamma\circ\pi_{\mathsf{P}}(p)}\big(\mu\circ\pi_{\mathsf{P}}(p)\big)\big) \\ = \;& \lambda_{\phi_{\mathsf{P}}(p,\sigma\circ\pi_{\mathsf{P}}(p))\cdot\gamma\circ\pi_{\mathsf{P}}(p)^{-1}\cdot\gamma\circ\pi_{\mathsf{P}}(p)}\big(\mu\circ\pi_{\mathsf{P}}(p)\big) = \lambda_{\phi_{\mathsf{P}}(p,\sigma\circ\pi_{\mathsf{P}}(p))}\big(\mu\circ\pi_{\mathsf{P}}(p)\big)\,. \end{aligned}$$

Its G-equivariance follows from the direct calculation:

$$\Phi_\lambda[\phi]\circ r_g(p) = \lambda_{\phi_{\mathsf{P}}(p\triangleleft g,\sigma\circ\pi_{\mathsf{P}}(p\triangleleft g))}\big(\mu\circ\pi_{\mathsf{P}}(p\triangleleft g)\big) = \lambda_{g^{-1}\cdot\phi_{\mathsf{P}}(p,\sigma\circ\pi_{\mathsf{P}}(p))}\big(\mu\circ\pi_{\mathsf{P}}(p)\big) = \lambda_{g^{-1}}\circ\Phi_{\lambda\cdot}[\phi](p)\,,$$

carried out for arbitrary $(p,g) \in \mathsf{P} \times \mathrm{G}$, and using Prop. VI.1. in conjunction with the aforementioned axioms.

In order to construct the inverse of the above assignment, we fix an (arbitrary) open trivialising cover $\{O_i\}_{i\in I}$ for the bundle P, and subsequently assign, to an arbitrary G-equivariant map $f \;:\; \mathsf{P} \longrightarrow M$, the family

$$S_\lambda[f]_i \quad : \quad O_i \longrightarrow \mathsf{P} \times_\lambda M \;:\; x \longmapsto \big[\big(\tau_i^{-1}(x,e), f\circ\tau_i^{-1}(x,e)\big)\big]\,, \quad i \in I$$

of local sections. Each of them is (locally) smooth as a superposition of the respective smooth maps $(\tau_i^{-1}(\cdot,e), f\circ\tau_i^{-1}(\cdot,e)) \;:\; O_i \longrightarrow \mathsf{P} \times M$ and the surjective submersion $\pi_\sim \;:\; \mathsf{P} \times M \longrightarrow \mathsf{P} \times_\lambda M$. We readily establish that these local sections are, in fact, restrictions (to the respective sets O_i) of a global one upon noting that due to G-equivariance of the maps τ_i and f the following equality holds, at an arbitrary point $x \in O_{ij}$,

$$S_\lambda[f]_j(x) = \big[\big(\tau_j^{-1}(x,e), f \circ \tau_j^{-1}(x,e)\big)\big] = \big[\big(\tau_i^{-1}\big(x, g_{ij}(x)\big), f \circ \tau_i^{-1}\big(x, g_{ij}(x)\big)\big)\big]$$

$$= \big[\big(\tau_i^{-1}(x,e) \triangleleft g_{ij}(x), f\big(\tau_i^{-1}(x,e) \triangleleft g_{ij}(x)\big)\big)\big] = \big[\big(\tau_i^{-1}(x,e) \triangleleft g_{ij}(x), g_{ij}(x)^{-1} \triangleright f \circ \tau_i^{-1}(x,e)\big]$$

$$= \big[\big(\tau_i^{-1}(x,e), f \circ \tau_i^{-1}(x,e)\big)\big] \equiv S_\lambda[f]_i(x)\,.$$

A direct calculation of both superpositions:

$$\Phi_\lambda\big[S_\lambda[f]\big] \;:\; \mathsf{P} \longrightarrow M \;:\; p \longmapsto \lambda_{\phi_{\mathsf{P}}(p,p)}\big(f(p)\big) = \lambda_e\big(f(p)\big) = f(p)$$

and

$$S_\lambda\big[\Phi_\lambda\big[[(\sigma,\mu)]\big]\big] \quad : \quad B \longrightarrow \mathsf{P} \times_\lambda M$$

$$: \; x \longmapsto \big[\big(\tau_i^{-1}(x,e), \lambda_{\phi_{\mathsf{P}}(\tau_i^{-1}(x,e), \sigma\circ\pi_{\mathsf{P}}\circ\tau_i^{-1}(x,e))}\big(\mu \circ \pi_{\mathsf{P}} \circ \tau_i^{-1}(x,e)\big)\big)\big]$$

$$\equiv \big[\big(\tau_i^{-1}(x,e), \lambda_{\phi_{\mathsf{P}}(\tau_i^{-1}(x,e),\sigma(x))}\big(\mu(x)\big)\big)\big] = [(\sigma,\mu)](x)$$

reveals the veracity of the desired identities

$$\Phi_\lambda \circ S_\lambda = \mathrm{id}_{\mathrm{Hom}_{\mathrm{G}}(\mathsf{P},M)}\,, \qquad\qquad S_\lambda \circ \Phi_\lambda = \mathrm{id}_{\Gamma(\mathsf{P}\times_\lambda M)}\,.$$

□

A specialisation of the last result to the adjoint bundle turns out to carry further structural information, displayed in

Proposition 5. The bijection

$$\Gamma(\mathrm{Ad}\,\mathsf{P}) \cong \mathrm{Hom}_{\mathrm{G}}(\mathsf{P},\mathrm{G})$$

of Prop. 4 is an isomorphism between the group of sections of the adjoint bundle, with the structure detailed in the proof of Thm. 1, and the group of maps from P to G equivariant relative to the respective (left) actions $r_{\mathrm{Inv}(\cdot)}$ and $\mathrm{Ad}.$, with the natural pointwise group structure.

Proof: Borrowing the notation from the proofs of both statements mentioned above, we check, for any pair of sections $\phi_\alpha = [(\sigma_\alpha,\gamma_\alpha)] \in \Gamma(\mathrm{Ad}\,\mathsf{P})$, $\alpha \in \{1,2\}$ and a point $p \in \mathsf{P}$,

$$\Phi_{\mathrm{Ad}}\big[\Gamma[M](\phi_1,\phi_2)\big](p) = \mathrm{Ad}_{\phi_{\mathsf{P}}(p,\sigma_1\circ\pi_{\mathsf{P}}(p))}\big(\gamma_1 \circ \pi_{\mathsf{P}}(p) \cdot \mathrm{Ad}_{\phi_{\mathsf{P}}(\sigma_1\circ\pi_{\mathsf{P}}(p),\sigma_2\circ\pi_{\mathsf{P}}(p))}\big(\gamma_2 \circ \pi_{\mathsf{P}}(p)\big)\big)$$

$$= \mathrm{Ad}_{\phi_{\mathsf{P}}(p,\sigma_1\circ\pi_{\mathsf{P}}(p))}\big(\gamma_1 \circ \pi_{\mathsf{P}}(p)\big) \cdot \mathrm{Ad}_{\phi_{\mathsf{P}}(p,\sigma_1\circ\pi_{\mathsf{P}}(p))\cdot\phi_{\mathsf{P}}(\sigma_1\circ\pi_{\mathsf{P}}(p),\sigma_2\circ\pi_{\mathsf{P}}(p))}\big(\gamma_2 \circ \pi_{\mathsf{P}}(p)\big)$$

$$= \mathrm{Ad}_{\phi_{\mathsf{P}}(p,\sigma_1\circ\pi_{\mathsf{P}}(p))}\big(\gamma_1 \circ \pi_{\mathsf{P}}(p)\big) \cdot \mathrm{Ad}_{\phi_{\mathsf{P}}(p,\sigma_2\circ\pi_{\mathsf{P}}(p))}\big(\gamma_2 \circ \pi_{\mathsf{P}}(p)\big) = M \circ \big(\Phi_{\mathrm{Ad}}(\phi_1), \Phi_{\mathrm{Ad}}(\phi_2)\big)(p)\,.$$

□

The *relative*-structural character of the bijection of Props. 4 and 5 (*i.e.*, one which, on top of the structure carried by the sets mapped by it, preserves also the structural relation *between* the sets (a realisation of one in terms of permutations of the other)), is best illustrated by

Proposition 6. The pair $(\Phi_{\mathrm{Ad}},\Phi_\lambda)$ defines a (generalised) equivariant bijection between the sets: $\Gamma(\mathrm{Ad}\,\mathsf{P}) \circlearrowright \Gamma(\mathsf{P}\times_\lambda M)$ and $\mathrm{Hom}_{\mathrm{G}}(\mathsf{P},\mathrm{G}) \circlearrowright \mathrm{Hom}_{\mathrm{G}}(\mathsf{P},M)$ endowed with actions of the respective groups, the second of which is induced pointwise from λ, *i.e.*, given by

$$\Phi_{\mathrm{Ad}}\lambda. \;:\; \mathrm{Hom}_{\mathrm{G}}(\mathsf{P},\mathrm{G}) \times \mathrm{Hom}_{\mathrm{G}}(\mathsf{P},M) \longrightarrow \mathrm{Hom}_{\mathrm{G}}(\mathsf{P},M) \;:\; \big(\gamma(\cdot),\mu(\cdot)\big) \longmapsto \lambda_{\gamma(\cdot)}\big(\mu(\cdot)\big)\,,$$

and so we have a commutative diagram

$$\begin{array}{ccc} \Gamma(\mathrm{Ad}\,\mathsf{P})\times\Gamma(\mathsf{P}\times_\lambda M) & \xrightarrow{\ \Gamma[\Gamma[\widetilde{r}]]^\lambda_\cdot\ } & \Gamma(\mathsf{P}\times_\lambda M) \\ \Big\downarrow{\scriptstyle \Phi_{\mathrm{Ad}}\times\Phi_\lambda} & & \Big\downarrow{\scriptstyle \Phi_\lambda} \\ \mathrm{Hom}_{\mathrm{G}}(\mathsf{P},\mathrm{G})\times\mathrm{Hom}_{\mathrm{G}}(\mathsf{P},M) & \xrightarrow[\ \Phi_{\mathrm{Ad}}\lambda_\cdot\ ]{} & \mathrm{Hom}_{\mathrm{G}}(\mathsf{P},M) \end{array} .$$

In other words, bijection Φ_λ is (left) equivariant with respect to the following actions of $\Gamma(\mathrm{Ad}\,\mathsf{P})$: the action $\Gamma[\Gamma[\widetilde{r}]]^\lambda_\cdot$ on the space $\Gamma(\mathsf{P}\times_\lambda M)$, defined in Eq. (7), and the natural action

$$[\Phi_{\mathrm{Ad}}\lambda]_\cdot = \Phi_{\mathrm{Ad}}\lambda_\cdot \circ (\Phi_{\mathrm{Ad}} \times \mathrm{id}_{\mathrm{Hom}_{\mathrm{G}}(\mathsf{P},M)})$$

on the space of G-equivariant maps $\mathrm{Hom}_{\mathrm{G}}(\mathsf{P},M)$.

Proof: Before all else, we convince ourselves that the map $\Phi_{\mathrm{Ad}}\lambda_\cdot$ is well-defined. To this end, we pick up an arbitrary pair $(\gamma,\mu)\in\mathrm{Hom}_{\mathrm{G}}(\mathsf{P},\mathrm{G})\times\mathrm{Hom}_{\mathrm{G}}(\mathsf{P},M)$ and consider the result of the evaluation $\Phi_{\mathrm{Ad}}\lambda_\gamma(\mu)$—we must prove that the latter is G-equivariant, which we do in a direct computation, carried out for arbitrary $(p,g)\in\mathsf{P}\times\mathrm{G}$,

$$\Phi_{\mathrm{Ad}}\lambda_\gamma\circ r_g^*(\mu)(p) = \lambda_{\gamma\circ r_g(p)}\big(\mu\circ r_g(p)\big) = \lambda_{\mathrm{Ad}_{g^{-1}}(\gamma(p))}\circ\lambda_{g^{-1}}\big(\mu(p)\big) = \lambda_{g^{-1}}\big(\lambda_{\gamma(p)}\big(\mu(p)\big)\big)$$

$$\equiv \quad \lambda_{g^{-1}}\circ\Phi_{\mathrm{Ad}}\lambda_\gamma(\mu)(p)\,.$$

It is obvious that the map $\Phi_{\mathrm{Ad}}\lambda_\cdot$ satisfies the axioms of a group action. Therefore, it remains to verify its equivariance. For arbitrary $\gamma=[(\widetilde{\sigma},\widetilde{g})]\equiv[(\sigma,g)]\in\Gamma(\mathrm{Ad}\,\mathsf{P})$ and $\phi=[(\sigma,\mu)]\in\Gamma(\mathsf{P}\times_\lambda M)$ as well as $p\in(\mathsf{P})_x$, we calculate

$$\Phi_\lambda\big[\Gamma[\Gamma[\widetilde{r}]]^\lambda_\gamma(\phi)\big](p) = \lambda_{\phi_{\mathsf{P}}(p,\lambda_{\gamma(x)}(\sigma(x)))}\big(\mu(x)\big) = \lambda_{\phi_{\mathsf{P}}(p,r_{\mathrm{Ad}_{\phi_{\mathsf{P}}(\sigma(x),\sigma(x))}(g(x))}(\sigma(x)))}\big(\mu(x)\big)$$

$$= \quad \lambda_{\phi_{\mathsf{P}}(p,r_{g(x)}(\sigma(x)))}\big(\mu(x)\big) = \lambda_{\phi_{\mathsf{P}}(p,\sigma(x))\cdot g(x)}\big(\mu(x)\big) \equiv \lambda_{\mathrm{Ad}_{\phi_{\mathsf{P}}(p,\sigma(x))}(g(x))\cdot\phi_{\mathsf{P}}(p,\sigma(x))}\big(\mu(x)\big)$$

$$= \quad \lambda_{\mathrm{Ad}_{\phi_{\mathsf{P}}(p,\sigma(x))}(g(x))}\big(\lambda_{\phi_{\mathsf{P}}(p,\sigma(x))}\big(\mu(x)\big)\big) \equiv \lambda_{\mathrm{Ad}_{\phi_{\mathsf{P}}(p,\sigma(x))}(g(x))}\big(\Phi_\lambda[\phi](p)\big)$$

$$= \quad \Phi_{\mathrm{Ad}}\lambda_{\mathrm{Ad}_{\phi_{\mathsf{P}}(\cdot,\sigma\circ\pi_{\mathsf{P}}(\cdot))}(g\circ\pi_{\mathsf{P}}(\cdot))}\big(\Phi_\lambda[\phi]\big)(p) \equiv \Phi_{\mathrm{Ad}}\lambda_{\Phi_{\mathrm{Ad}}[\gamma]}\big(\Phi_\lambda[\phi]\big)(p)\,,$$

which is the anticipated result. □

Our hitherto considerations present $\mathrm{Ad}\,\mathsf{P}$ as a bundle of groups acting on a bundle of manifolds M in a natural manner modelled on λ. The statement that we give below deepens our observations substantially and, simultaneously, paves a way towards a natural physical interpretation of the group $\Gamma(\mathrm{Ad}\,\mathsf{P})$ as the **gauge group** of the field theory.

Proposition 7. There exists a canonical group isomorphism

$$\Gamma(\mathrm{Ad}\,\mathsf{P}) \cong \{\ (\Phi,\mathrm{id}_{\mathrm{G}},f)\in\mathrm{Aut}_{\mathbf{Bun}_{\mathrm{G}}(B)}(\mathsf{P}) \quad | \quad f=\mathrm{id}_B\ \} =: \mathrm{Aut}_{\mathbf{Bun}_{\mathrm{G}}(B)/B}(\mathsf{P})\,.$$

Proof: We begin by establishing a bijection between the sets $\mathrm{Hom}_{\mathrm{G}}(\mathsf{P},\mathrm{G})$ and $\mathrm{Aut}_{\mathbf{Bun}_{\mathrm{G}}(B)/B}(\mathsf{P})$. For that, we pick up (arbitrarily) $\gamma\in\mathrm{Hom}_{\mathrm{G}}(\mathsf{P},\mathrm{G})$ and define a map

$$\Psi[\gamma]\ :\ \mathsf{P}\ \circlearrowleft\ :\ p\longmapsto r_{\gamma(p)}(p)\,.$$

The latter is manifestly G-equivariant,

$$\forall_{(p,g)\in\mathsf{P}\times\mathrm{G}}\ :\ \Psi[\gamma]\circ r_g(p)\equiv r_{\gamma\circ r_g(p)}\big(r_g(p)\big) = r_{\mathrm{Ad}_{g^{-1}}(\gamma(p))}\big(r_g(p)\big) = r_{\gamma(p)\cdot g}(p) = r_g\circ\Psi[\gamma](p)\,,$$

and preserves fibres, and so it defines an automorphism

$$\big(\Psi[\gamma],\mathrm{id}_{\mathrm{G}},\mathrm{id}_B\big)\in\mathrm{Aut}_{\mathbf{Bun}_{\mathrm{G}}(B)/B}(\mathsf{P})\,.$$

Furthermore, it is a group homomorphism—a fact readily inferred from the following direct computation:

$$\Psi\big[\widetilde{M}(\gamma_1,\gamma_2)\big](p) = r_{\gamma_1(p)\cdot\gamma_2(p)}(p) \equiv r_{\gamma_2(p)\cdot\mathrm{Ad}_{\gamma_2(p)^{-1}}(\gamma_1(p))}(p) = r_{\mathrm{Ad}_{\gamma_2(p)^{-1}}(\gamma_1(p))}\circ r_{\gamma_2(p)}(p)$$

$$= \quad r_{\gamma_1(p\triangleleft\gamma_2(p))}\circ r_{\gamma_2(p)}(p) \equiv \Psi[\gamma_1]\circ\Psi[\gamma_2](p)\,,$$

carried out for arbitrary $\gamma_1,\gamma_2\in\mathrm{Hom}_{\mathrm{G}}(\mathsf{P},\mathrm{G})$. At this stage, it suffices to invoke Prop. 4, to obtain the sought-after group homomorphism

$$\alpha.\equiv\big(\Psi[\cdot],\mathrm{id}_{\mathrm{G}},\mathrm{id}_B\big)\circ\Phi_{\mathrm{Ad.}}\ :\ \Gamma(\mathrm{Ad}\,\mathsf{P})\longrightarrow\mathrm{Aut}_{\mathbf{Bun}_{\mathrm{G}}(B)/B}(\mathsf{P})\,.$$

Going in the opposite direction, we associate to an arbitrary automorphism $(\Phi,\mathrm{id}_{\mathrm{G}},\mathrm{id}_B)\in\mathrm{Aut}_{\mathbf{Bun}_{\mathrm{G}}(B)/B}(\mathsf{P})$ a map

$$\chi[(\Phi,\mathrm{id}_{\mathrm{G}},\mathrm{id}_B)]\ :\ \mathsf{P}\longrightarrow\mathrm{G}\ :\ p\longmapsto\phi_{\mathsf{P}}\big(p,\Phi(p)\big)$$

whose G-equivariance is proven on the basis of Prop. VI.1., and for arbitrary $(p,g)\in\mathsf{P}\times\mathrm{G}$, as

$$\chi[(\Phi,\mathrm{id}_{\mathrm{G}},\mathrm{id}_B)]\circ r_g(p)\equiv\phi_{\mathsf{P}}\big(r_g(p),\Phi\circ r_g(p)\big)=\phi_{\mathsf{P}}\big(r_g(p),r_g\circ\Phi(p)\big)=\mathrm{Ad}_{g^{-1}}\big(\phi_{\mathsf{P}}\big(p,\Phi(p)\big)\big)$$

$$\equiv\quad\mathrm{Ad}_{g^{-1}}\circ\chi[(\Phi,\mathrm{id}_{\mathrm{G}},\mathrm{id}_B)](p)\,.$$

It is easy to see that the map

$$\chi\ :\ \mathrm{Aut}_{\mathbf{Bun}_{\mathrm{G}}(B)/B}(\mathsf{P})\longrightarrow\mathrm{Hom}_{\mathrm{G}}(\mathsf{P},\mathrm{G})$$

thus obtained is a group homomorphism—indeed, for any pair of automorphisms $(\Phi_\alpha,\mathrm{id}_{\mathrm{G}},\mathrm{id}_B)\in\mathrm{Aut}_{\mathbf{Bun}_{\mathrm{G}}(B)/B}(\mathsf{P})$, $\alpha\in\{1,2\}$, we calculate

$$\chi\big[(\Phi_1,\mathrm{id}_{\mathrm{G}},\mathrm{id}_B)\circ(\Phi_2,\mathrm{id}_{\mathrm{G}},\mathrm{id}_B)\big](p)=\phi_{\mathsf{P}}\big(p,\Phi_1\circ\Phi_2(p)\big)=\phi_{\mathsf{P}}\big(p,\Phi_1(p)\big)\cdot\phi_{\mathsf{P}}\big(\Phi_1(p),\Phi_1\circ\Phi_2(p)\big)\,,$$

but also

$$\phi_{\mathsf{P}}\big(\Phi_1(p),\Phi_1\circ\Phi_2(p)\big)=\phi_{\mathsf{P}}\big(\Phi_1(p),\Phi_1\big(p\triangleleft\phi_P\big(p,\Phi_2(p)\big)\big)\big)=\phi_{\mathsf{P}}\big(\Phi_1(p),\Phi_1(p)\triangleleft\phi_P\big(p,\Phi_2(p)\big)\big)$$

$$=\quad\phi_P\big(p,\Phi_2(p)\big)\,,$$

and hence

$$\chi\big[(\Phi_1,\mathrm{id}_{\mathrm{G}},\mathrm{id}_B)\circ(\Phi_2,\mathrm{id}_{\mathrm{G}},\mathrm{id}_B)\big](p)=\phi_{\mathsf{P}}\big(p,\Phi_1(p)\big)\cdot\phi_{\mathsf{P}}\big(p,\Phi_2(p)\big)$$

$$\equiv\quad\widetilde{M}\big(\chi\big[(\Phi_1,\mathrm{id}_{\mathrm{G}},\mathrm{id}_B)\big],\chi\big[(\Phi_2,\mathrm{id}_{\mathrm{G}},\mathrm{id}_B)\big]\big)(p)\,,$$

in conformity with our expectations. In the end, we arrive at the group homomorphism

$$S_{\mathrm{Ad.}}\circ\chi\ :\ \mathrm{Aut}_{\mathbf{Bun}_{\mathrm{G}}(B)/B}(\mathsf{P})\longrightarrow\Gamma(\mathrm{Ad}\,\mathsf{P})\,.$$

In order to verify that the latter is the inverse of the previously considered homomorphism $\Psi\circ\Phi_{\mathrm{Ad}}$, it is enough to check that χ is the inverse of the automorphism $\big(\Psi[\cdot],\mathrm{id}_{\mathrm{G}},\mathrm{id}_B\big)$, which we do directly by computing, for arbitrary $(p,g,x)\in\mathsf{P}\times\mathrm{G}\times B$,

$$\big(\Psi[\cdot],\mathrm{id}_{\mathrm{G}},\mathrm{id}_B\big)\circ\chi[(\Phi,\mathrm{id}_{\mathrm{G}},\mathrm{id}_B)](p,g,x)=\big(r_{\phi_{\mathsf{P}}(p,\Phi(p))}(p),g,x\big)=\big(\Phi(p),g,x\big)\equiv(\Phi,\mathrm{id}_{\mathrm{G}},\mathrm{id}_B)(p,g,x)$$

and

$$\chi\circ\big(\Psi[\cdot],\mathrm{id}_{\mathrm{G}},\mathrm{id}_B\big)[\gamma](p)=\phi_{\mathsf{P}}\big(p,r_{\gamma(p)}(p)\big)=\gamma(p)\,.$$

□

3. Groupoidification & the Atiyah sequence

Following the general logic of the course, we shall, next, reformulate and elaborate the findings of the present lecture in the language of Lie groupoids. We base our considerations thus oriented upon, on one hand, the very detailed groupoid-theoretic examination of the principal bundle P carried out in Sec. VI.3., and, on the other hand, the idea of inheritance of the $\mathrm{Ad}\,\mathsf{P}$-module structure by $\mathsf{P}\times M$ from P and of descent of that structure along the symmetry quotient $\mathsf{P}\times M\longrightarrow\mathsf{P}\times_\lambda M$, embodied by the induction of the action $[\lambda]$ from the G-equivariant action $\widetilde{[r]}$. We shall also be concerned with the Lie-groupoidal aspect of the conceptually clear relation between the 'space of symmetry transformations between the local frames' $\mathrm{Ad}\,\mathsf{P}$, introduced in this lecture, and the 'space of (inter-observer) objectivisations' $\mathrm{At}(\mathsf{P})$, encountered in the previous lecture.

By way of a warm-up, note that the action of the adjoint bundle $\mathrm{Ad}\,\mathsf{P}$ on the principal bundle P given in Eq. (4) can be viewed as a groupoid action. For that, we need the fairly obvious

Proposition/Definition 1. The structure of a group object in the slice category $\mathbf{Bun}(B)/B$ carried by the adjoint bundle $\mathrm{Ad}\,\mathsf{P}$ (in virtue of Thm. 1) canonically induces on it the structure of (the arrow manifold of) a Lie-groupoid object in $\mathbf{Bun}(B)/B$ with object manifold given by the terminal object $B\times\bullet\equiv B$ in that category, source and target both equal to $\pi_{\mathrm{Ad}\,\mathsf{P}}$, identity given by $[\varepsilon]$, and inverse equal to $[\mathrm{Inv}]$. The structure is summarised by the following diagram

$$\begin{array}{ccccccc}
\mathrm{Ad}\,\mathsf{P}\times_B\mathrm{Ad}\,\mathsf{P} & \xrightarrow{[\mathsf{M}]} & \mathrm{Ad}\,\mathsf{P} & \xrightarrow{[\mathrm{Inv}]} & \mathrm{Ad}\,\mathsf{P} & \overset{\pi_{\mathrm{Ad}\,\mathsf{P}}}{\underset{\pi_{\mathrm{Ad}\,\mathsf{P}}}{\rightrightarrows}} \;\; (\overset{[\varepsilon]}{\curvearrowleft}) & B \\
\downarrow{\scriptstyle \pi_{\mathrm{Ad}\,\mathsf{P}}\circ\mathrm{pr}_1} & & \downarrow{\scriptstyle \pi_{\mathrm{Ad}\,\mathsf{P}}} & & \downarrow{\scriptstyle \pi_{\mathrm{Ad}\,\mathsf{P}}} & & \| \\
B & = & B & = & B & = & B
\end{array},$$

The above Lie groupoid is called the **adjoint groupoid** of P.

<u>*Proof:*</u> Trivial. □

We may now state

Proposition 8. Every principal bundle P carries a canonical structure of a left module of the corresponding adjoint groupoid $\mathrm{Ad}\,\mathsf{P}\rightrightarrows B$, with momentum id_B and action $[r]$ of Eq. (4). The action preserves fibres of the base projection $\pi_{\mathsf{P}}\ :\ \mathsf{P}\longrightarrow B$, turning P into a (non-principal) $\mathrm{Ad}\,\mathsf{P}$-bundle.

<u>*Proof:*</u> No less trivial. □

A moment's thought reveals that the adjoint groupoid $\mathrm{Ad}\,\mathsf{P}\rightrightarrows B$ can be understood as a restriction of the Atiyah–Ehresmann groupoid $\mathrm{At}(\mathsf{P})\rightrightarrows B$ of Def. VI.5. to the diagonal in the cartesian square $B\times B$ into which $\mathrm{At}(\mathsf{P})$ is mapped by (S,T). The shortest way to this conclusion leads *via* the diffeomorphism

$$(\mathrm{pr}_1,\phi_{\mathsf{P}})\ :\ \mathsf{P}\times_B\mathsf{P}\xrightarrow{\cong}\mathsf{P}\times\mathrm{G}$$

of Prop. VI.1., which we readily verify to intertwine the diagonal action

$$\mathrm{G}\times(\mathsf{P}\times_B\mathsf{P})\longrightarrow\mathsf{P}\times_B\mathsf{P}\ :\ \big(g,(p_2,p_1)\big)\longmapsto\big(r_{g^{-1}}(p_2),r_{g^{-1}}(p_1)\big)$$

with the familiar one

$$\widetilde{\mathrm{Ad}}\ :\ \mathrm{G}\times(\mathsf{P}\times\mathrm{G})\longrightarrow\mathsf{P}\times\mathrm{G}\ :\ \big(g,(p,h)\big)\longmapsto\big(r_{g^{-1}}(p),\mathrm{Ad}_g(h)\big)$$

on the codomain, which defines the quotient

$$(\mathsf{P}\times\mathrm{G})//\widetilde{\mathrm{Ad}}(\mathrm{G})\equiv(\mathsf{P}\times\mathrm{G})//\mathrm{G}\equiv\mathrm{Ad}\,\mathsf{P}\,.$$

Indeed, we obtain, for any $(p_2,p_1)\in\mathsf{P}\times_B\mathsf{P}$ and $g\in\mathrm{G}$,

$$\begin{aligned}(\mathrm{pr}_1,\phi_{\mathsf{P}})\big(r_{g^{-1}}(p_2),r_{g^{-1}}(p_1)\big) &\equiv \big(r_{g^{-1}}(p_2),\phi_{\mathsf{P}}\big(r_{g^{-1}}(p_2),r_{g^{-1}}(p_1)\big)\big)=\big(r_{g^{-1}}(p_2),\mathrm{Ad}_g\big(\phi_{\mathsf{P}}\big(p_2,p_1\big)\big)\big)\\ &\equiv \widetilde{\mathrm{Ad}}_g\circ(\mathrm{pr}_1,\phi_{\mathsf{P}})(p_2,p_1)\,.\end{aligned}$$

The upshot of this chain of thought is the existence of a diffeomorphism

$$\mathrm{Ad}\,\mathsf{P}\equiv(\mathsf{P}\times\mathrm{G})//\mathrm{G}\cong\big(\mathsf{P}\times_B\mathsf{P}\big)//\mathrm{G}\subset(\mathsf{P}\times\mathsf{P})//\mathrm{G}\equiv\mathrm{At}(\mathsf{P})\,.$$

Rather than sticking to the above, though, we shall now follow a more demanding route, which, however, leads through more general results, and as such is of relevance to our later considerations. Its starting point is marked by

Proposition 9. Let $\mathscr{G}_1\rightrightarrows M_1$ and $\mathscr{G}_2\rightrightarrows M_2$ be Lie groupoids, and let

$$\begin{array}{ccc}\mathscr{G}_1 & \xrightarrow{\ \chi\ } & \mathscr{G}_2\\ {\scriptstyle t_1}\downarrow\downarrow{\scriptstyle s_1} & & {\scriptstyle t_2}\downarrow\downarrow{\scriptstyle s_2}\\ M_1 & \xrightarrow[\ \chi_0\ ]{} & M_2\end{array}$$

be a morphism of Lie groupoids, whose component χ is transverse[1] to the identity bisection $\mathrm{Id}(M_2)\subset\mathscr{G}_2$. The preimage $\chi^{-1}(\mathrm{Id}(M_2))$ of $\mathrm{Id}(M_2)$ is a Lie subgroupoid of $\mathscr{G}_1\rightrightarrows M_1$.

Proof: For $\chi^{-1}(\mathrm{Id}(M_2))$ to be a Lie subgroupoid in $\mathscr{G}_1$, the former has to be a submanifold in $\mathscr{G}_1$. In virtue of a classic variant of the Level-Set Theorem for submanifolds, this is ensured by transversality of χ.

At this stage, it remains to check that the subset $\chi^{-1}(\mathrm{Id}(M_2))\subset\mathscr{G}_1$ is a subgroupoid. To this end, consider an arbitrary pair $(g_1,g_1')\in\chi^{-1}(\mathrm{Id}(M_2))\,{}_{s_1}\!\times_{t_1}\chi^{-1}(\mathrm{Id}(M_2))$. There then exist points $m_2,m_2'\in M_2$ such that $\chi(g_1)=\mathrm{Id}_{m_2}$ and $\chi(g_1')=\mathrm{Id}_{m_2'}$. As $\chi_0\circ s_1=s_2\circ\chi$ and $\chi_0\circ t_1=t_2\circ\chi$, we establish the identity

$$m_2'=t_2(\mathrm{Id}_{m_2'})=t_2\big(\chi(g_1')\big)=\chi_0\big(t_1(g_1')\big)=\chi_0\big(s_1(g_1)\big)=s_2\big(\chi(g_1)\big)=s_2(\mathrm{Id}_{m_2})=m_2\,.$$

Since χ is a morphism, this implies

$$\chi(g_1.g_1')=\chi(g_1).\chi(g_1')=\mathrm{Id}_{m_2}.\mathrm{Id}_{m_2'}=\mathrm{Id}_{m_2}\,,$$

which leads to the desired conclusion $g_1.g_1'\in\chi^{-1}(\mathrm{Id}(M_2))$. Similarly, for every $g_1\in\chi^{-1}(\mathrm{Id}(M_2))$, with $\chi(g_1)=\mathrm{Id}_{m_2}$, we have $\chi(g_1^{-1})=\chi(g_1)^{-1}=\mathrm{Id}_{m_2}^{-1}=\mathrm{Id}_{m_2}$, and so also $g_1^{-1}\in\chi^{-1}(\mathrm{Id}(M_2))$. □

In order to jump directly to its end point, we need one more

Definition 2. Let $\mathbf{Gr}_a,\ a\in\{1,2,3\}$ be Lie groupoids and let $\jmath:\mathbf{Gr}_1\longrightarrow\mathbf{Gr}_2$ and $\pi:\mathbf{Gr}_2\longrightarrow\mathbf{Gr}_3$ be Lie-groupoid morphisms. We say that the quintuple $(\mathbf{Gr}_1,\mathbf{Gr}_2,\mathbf{Gr}_3,\jmath,\pi)$ composes a **short exact sequence of Lie groupoids**

$$\mathbf{Gr}_1\xhookrightarrow{\ \jmath\ }\mathbf{Gr}_2\xrightarrow{\ \pi\ }\!\!\!\!\!\rightarrow\mathbf{Gr}_3$$

if the π-preimage of the identity bisection $\mathrm{Id}(\mathrm{Ob}\mathbf{Gr}_3)\subset\mathrm{Mor}\mathbf{Gr}_3$ is canonically isomorphic to the (faithful) $\jmath$-image of $\mathbf{Gr}_1$ in $\mathbf{Gr}_2$. If, moreover, there exists a Lie-groupoid morphism $\sigma:\mathbf{Gr}_3\longrightarrow\mathbf{Gr}_2$ such that the identity $\pi\circ\sigma=\mathrm{id}_{\mathbf{Gr}_3}$ obtains, then we say that the short exact sequence **splits**, and call it a **split short exact sequence (of Lie groupoids)**.

Upon noting that $B\times B$ is the arrow manifold of $\mathrm{Pair}(B)$, we are thus led to the following

[1]Recall that a smooth manifold map $f:M\to N$ is said to be *transverse* to a submanifold $S\subset N$ if at an arbitrary point $m\in f^{-1}(S)$, the following condition is satisfied: $\mathrm{T}_mf(\mathrm{T}_mM)+\mathrm{T}_{f(m)}S=\mathrm{T}_{f(m)}N$.

Theorem 2. For every principal bundle $((\mathsf{P}, B, \mathrm{G}, \pi_{\mathsf{P}}), r)$, the three Lie groupoids: the Atiyah–Ehresmann groupoid $\mathrm{At}(\mathsf{P}) \rightrightarrows B$, the adjoint groupoid $\mathrm{Ad}\,\mathsf{P} \rightrightarrows B$, and the pair groupoid $\mathrm{Pair}(B)$ compose a short exact sequence

(8)
$$\begin{array}{ccccc} \mathrm{Ad}\,\mathsf{P} & \xhookrightarrow{\ \jmath_{\mathrm{Ad}\,\mathsf{P}}\ } & \mathrm{At}(\mathsf{P}) & \xrightarrow{\ (\mathsf{T},\mathsf{S})\ } & B \times B \\ {\scriptstyle \pi_{\mathrm{Ad}\,\mathsf{P}}}\Big\downarrow\Big\downarrow{\scriptstyle \pi_{\mathrm{Ad}\,\mathsf{P}}} & & {\scriptstyle \mathsf{T}}\Big\downarrow\Big\downarrow{\scriptstyle \mathsf{S}} & & {\scriptstyle \mathrm{pr}_1}\Big\downarrow\Big\downarrow{\scriptstyle \mathrm{pr}_2} \\ B & = & B & = & B \end{array} ,$$

in which $\jmath_{\mathrm{Ad}\,\mathsf{P}}$ is an embedding, explicitly given by

$$\jmath_{\mathrm{Ad}\,\mathsf{P}} \ : \ \mathrm{Ad}\,\mathsf{P} \longrightarrow \mathrm{At}(\mathsf{P}) \ : \ [(p,g)] \longmapsto [(p, r_g(p))] \, .$$

Proof: Exactness of sequence (8) at its node $\mathrm{Pair}(B)$ follows from Prop. VI.10. As a set, $\mathrm{Ad}\,\mathsf{P}$ fits into the short exact sequence by definition, in virtue of the argument preceding Prop. 9, and the only thing that remains to be proven is the embedding of $\mathrm{Ad}\,\mathsf{P} \rightrightarrows B$ in $\mathrm{At}(\mathsf{P}) \rightrightarrows B$ as a Lie subgroupoid. Since $((\mathsf{T},\mathsf{S}), \mathrm{id}_B)$ is an epimorphism of Lie groupoids, its arrow component (T,S) is automatically transverse to $\mathrm{Id}(B) \subset B \times B$ (as a submersion), and so we conclude the present proof by invoking Prop. 9. □

Remark 4. While the statement of the above theorem is, by now, clear, we pause to elaborate on the embedding $\jmath_{\mathrm{Ad}\,\mathsf{P}}$. The rationale behind the elaboration is that it leaves us with a handy logical tool, which can be used to further investigate the Atiyah–Ehresmann groupoid. Thus, let us examine the geometric content of the preimage along (T,S) of an arrow (x,x) from the identity submanifold $\mathrm{Id}(B) \subset B \times B$ of the pair groupoid of B. We find

$$(\mathsf{T},\mathsf{S})^{-1}\big(\{\mathrm{Id}(\{x\})\}\big) = \big\{ \ [(p, r_g(p))] \quad | \quad p \in \pi_{\mathsf{P}}^{-1}\big(\{x\}\big) \quad \wedge \quad g \in \mathrm{G} \ \big\} ,$$

and so we may associate with the preimage a pair $(p,g) \in \pi_{\mathsf{P}}^{-1}(\{x\}) \times \mathrm{G}$ up to the following equivalence (determined by the definition of the class of the pair $(p, r_g(p))$ in $\mathrm{At}(\mathsf{P})$)

$$(p,g) \sim \big(r_{h^{-1}}(p), \mathrm{Ad}_h(g)\big) ,$$

or, in other words,

$$[(p,g)] \in (\mathrm{Ad}\,\mathsf{P})_x \, .$$

We are now in a position to sqeeze the Atiyah–Ehresmann for additional information, thereby revealing its absolutely central status in the present discussion.

Proposition 10. Let us fix a point $x_* \in B$ in the base of a principal bundle $((\mathsf{P}, B, \mathrm{G}, \pi_{\mathsf{P}}), r)$, and consider an embedding

$$\mathsf{L}x_* \ : \ B \longrightarrow B \times B \ : \ x \longmapsto (x_*, x) \, .$$

The preimage of the submanifold $\mathsf{L}x_*(B)$ carries a canonical structure of a principal (G-)bundle, and there exists a canonical isomorphism of principal bundles

$$(\mathsf{T},\mathsf{S})^{-1}\big(\mathsf{L}x_*(B)\big) \cong \mathsf{P} \, .$$

Proof: We find, for an arbitrarily *fixed* point $p_* \in \pi_{\mathsf{P}}^{-1}(\{x_*\})$ in the fibre over x_*,

$$(\mathsf{T},\mathsf{S})^{-1}\big(\{(x_*, x)\}\big) = \big\{ \ [(p_* \triangleleft g, p)] \quad | \quad p \in \pi_{\mathsf{P}}^{-1}\big(\{x\}\big) \quad \wedge \quad g \in \mathrm{G} \ \big\} ,$$

from which we extract classes of pairs (p,g) subject to the obvious identification

$$(p,g) \sim \big(r_h(p), g \cdot h\big) ,$$

or, in other words,

$$[(p,g)] \in \big(\mathsf{P} \times_{\wp} \mathrm{G}\big)_x \, .$$

Here, $\mathsf{P}\times_{\wp}\mathrm{G}$ is the bundle associated to P by the right regular action of G on itself. The bundle comes with a natural (right) G-action

$$\big(\mathsf{P}\times_{\wp}\mathrm{G}\big)\times\mathrm{G}\longrightarrow\mathsf{P}\times_{\wp}\mathrm{G}\ :\ \big([(p,g)],h\big)\longmapsto\big[\big(p,h^{-1}\cdot g\big)\big]\,,$$

and we readily verify that the map

$$\mathsf{P}\times_{\wp}\mathrm{G}\longrightarrow\mathsf{P}\ :\ [(p,g)]\longmapsto p\triangleleft g^{-1}$$

is a G-equivariant bundle isomorphism, whence

$$\mathsf{P}\times_{\wp}\mathrm{G}\cong\mathsf{P}$$

as principal G-bundles. We note, in passing, that the G-space structure on $\mathsf{P}\times_{\wp}\mathrm{G}$ neatly accounts for the freedom of choice of the reference point p_*. □

We crown our study of associated bundles with a result in which the induced (gauge) symmetry structure of any such fibre bundle is neatly encapsulated, which we induce from that of the underlying principal bundle, as inscribed in the W-diagram (VI.9) of Thm. VI.2. The result illustrates the deep and universal principle of descent along symmetry quotients (to be extended nontrivially once we endow M with extra cohomological structure, in a physically motivated manner), and—à la fois—reemphasises the rôle of the Atiyah–Ehresmann groupoid in this context. Before the main dish, we serve an appetiser:

Proposition 11. For every principal bundle $((\mathsf{P},B,\mathrm{G},\pi_{\mathsf{P}}),r)$, a bundle $(\mathsf{P}\times_\lambda M,B,M,\pi_{\mathsf{P}\times_\lambda M})$ associated to it by an action $\lambda\ :\ \mathrm{G}\longrightarrow\mathrm{Diff}(M)$ carries a canonical structure of a left $\mathrm{At}(\mathsf{P})$-module with momentum $\pi_{\mathsf{P}\times_\lambda M}$ and action

$$\Lambda\ :\ \mathrm{At}(\mathsf{P})\,{}_{\mathsf{S}}{\times}_{\pi_{\mathsf{P}\times_\lambda M}}(\mathsf{P}\times_\lambda M)\longrightarrow\mathsf{P}\times_\lambda M\ :\ \big([(p_2,p_1)],[(p_1,m)]\big)\longmapsto[(p_2,m)]\,.$$

<u>*Proof:*</u> The only non-obvious aspect of the above statement is the smoothness of the action. This is best seen in the following convenient models of the two geometries involved, reconstructed through (recursive) application of Thm. V.3. for an open cover $\{O_i\}_{i\in I}$ of B trivialising for the principal bundle P:

- the Atiyah–Ehresmann groupoid

$$\mathrm{At}(\mathsf{P})\equiv(\mathsf{P}\times\mathsf{P})/\!/\mathrm{G}\equiv\mathsf{P}\times_{r\circ\mathrm{Inv}}\mathsf{P}\cong\bigsqcup_{j\in I}\big(\mathsf{P}\times O_j\big)/\sim_{r_{\mathrm{Inv}\circ g_{\cdot\cdot}}}\cong\bigsqcup_{i,j\in I}\big(O_i^{(1)}\times\mathrm{G}\times O_j^{(2)}\big)/\sim_{\ell_{g_{\cdot\cdot}^{(1)}}\circ\wp_{\mathrm{Inv}\circ g_{\cdot\cdot}^{(2)}}}\,;$$

- the associated bundle

$$\mathsf{P}\times_\lambda M\cong\bigsqcup_{j\in I}\big(O_j\times M\big)/\sim_{\lambda_{g_{\cdot\cdot}}}\,.$$

As we shall presently consider a far-reaching generalisation of this scenario, we postpone the detailed (and technical) proof until then, and in the meantime, leave the proof as an exercise for an avid Reader. □

Corollary 1. The structure of a left $\mathrm{At}(\mathsf{P})$-module on the associated bundle $(\mathsf{P}\times_\lambda M,B,M,\pi_{\mathsf{P}\times_\lambda M})$ gives rise to an action groupoid $\mathrm{At}(\mathsf{P})\ltimes_\Lambda(\mathsf{P}\times_\lambda M)$ with object manifold $\mathsf{P}\times_\lambda M$, arrow manifold $\mathrm{At}(\mathsf{P})\,{}_{\mathsf{S}}{\times}_{\pi_{\mathsf{P}\times_\lambda M}}(\mathsf{P}\times_\lambda M)$, and the following structure maps:

- the source map

$$\varsigma\ :\ \mathrm{At}(\mathsf{P})\,{}_{\mathsf{S}}{\times}_{\pi_{\mathsf{P}\times_\lambda M}}(\mathsf{P}\times_\lambda M)\longrightarrow\mathsf{P}\times_\lambda M\ :\ \big([(p_2,p_1)],[(p_1,m)]\big)\longmapsto[(p_1,m)]\,;$$

- the target map $\tau:=\Lambda$;
- the identity map

$$\iota\ :\ \mathsf{P}\times_\lambda M\longrightarrow\mathrm{At}(\mathsf{P})\,{}_{\mathsf{S}}{\times}_{\pi_{\mathsf{P}\times_\lambda M}}(\mathsf{P}\times_\lambda M)\ :\ [(p,m)]\longmapsto\big([(p,p)],[(p,m)]\big)\,;$$

- the inverse map

$$\jmath\ :\ \mathrm{At}(\mathsf{P})\,{}_{\mathsf{S}}{\times}_{\pi_{\mathsf{P}\times_\lambda M}}(\mathsf{P}\times_\lambda M)\circlearrowleft\ :\ \big([(p_2,p_1)],[(p_1,m)]\big)\longrightarrow\big([(p_1,p_2)],[(p_2,m)]\big)\,;$$

- the multiplication map

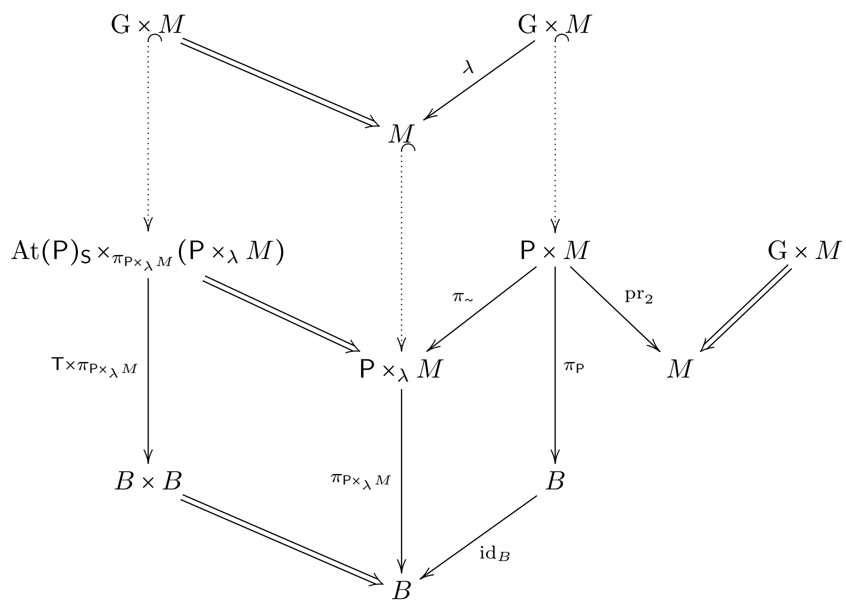
$$m \quad : \quad \big(\mathrm{At}(\mathsf{P})_{\mathsf{S}} \times_{\pi_{\mathsf{P}\times_\lambda M}} (\mathsf{P} \times_\lambda M)\big)_{\varsigma} \times_{\tau} \big(\mathrm{At}(\mathsf{P})_{\mathsf{S}} \times_{\pi_{\mathsf{P}\times_\lambda M}} (\mathsf{P} \times_\lambda M)\big) \longrightarrow \mathrm{At}(\mathsf{P})_{\mathsf{S}} \times_{\pi_{\mathsf{P}\times_\lambda M}} (\mathsf{P} \times_\lambda M)$$

$$: \quad \big(\big([(p_3, p_2)], [(p_2, m)]\big), \big([(p_2, p_1)], [(p_1, m)]\big)\big) \longmapsto \big([(p_3, p_1)], [(p_1, m)]\big)\,.$$

…et maintenant ¡voilà:

Theorem 3. For every principal bundle $((\mathsf{P}, B, \mathrm{G}, \pi_{\mathsf{P}}), r)$ and every smooth manifold M with a smooth action $\lambda \;:\; \mathrm{G} \longrightarrow \mathrm{Diff}(M)$, the manifold $\mathsf{P} \times M$ carries a canonical structure of a principal-$(\mathrm{At}(\mathsf{P}) \ltimes_\Lambda (\mathsf{P} \times_\lambda M), \mathrm{G} \ltimes_\lambda M)$-bibundle object (in the sense of Def. VI.6.) in the category $\mathbf{Bun}(B)/B$, with the left $\mathrm{At}(\mathsf{P})$-action fibring over the canonical left $\mathrm{Pair}(B)$-action on the object manifold B with typical fibre given by the canonical left $\mathrm{G} \ltimes_\lambda M$-action on the object manifold M, as described succinctly by the *extended* W-diagram

(9)



.

<u>*Proof:*</u> Let us start by working out the 'global' right action of the action groupoid. In so doing, we shall be guided by Prop. VI.8., which tells us that the sought-after $\mathrm{G} \ltimes_\lambda M$-action on $\mathsf{P} \times M$ should be such that

$$(\mathsf{P} \times M) // \mathrm{G} \ltimes_\lambda M = \mathsf{P} \times_\lambda M \equiv (\mathsf{P} \times M) // \mathrm{G}\,,$$

where the quotient on the right-hand side is the one with respect to the diagonal action $\widetilde{\lambda}$ of Cor. VI.1. Hence, we simply postulate

$$\varrho \;:\; (\mathsf{P} \times M)_{\mathrm{pr}_2} \times_\lambda (\mathrm{G} \times M) \longrightarrow \mathsf{P} \times M \;:\; \big((p, m), \big(g, \lambda_{g^{-1}}(m)\big)\big) \longmapsto \widetilde{\lambda}_{g^{-1}}(p, m)\,,$$

and subsequently check the axioms of Def. VI.4.:

- axiom (PGr1):

$$(\mathrm{GrM1}) \quad : \quad \mu\big(\widetilde{\lambda}_{g^{-1}}(p, m)\big) \equiv \mathrm{pr}_2\big(\widetilde{\lambda}_{g^{-1}}(p, m)\big) \equiv \mathrm{pr}_2\big(r_g(p), \lambda_{g^{-1}}(m)\big) = \lambda_{g^{-1}}(m) \equiv s\big(g, \lambda_{g^{-1}}(m)\big)\,,$$

$$(\mathrm{GrM2}) \quad : \quad \varrho\big((p, m), (e, m)\big) = \widetilde{\lambda}_{e^{-1}}(p, m) = (p, m)\,,$$

$$(\mathrm{GrM3}) \quad : \quad \varrho\big(\varrho\big((p, m), \big(g, \lambda_{g^{-1}}(m)\big)\big), \big(h, \lambda_{h^{-1}g^{-1}}(m)\big)\big) = \widetilde{\lambda}_{h^{-1}}\big(\widetilde{\lambda}_{g^{-1}}(p, m)\big) = \widetilde{\lambda}_{(gh)^{-1}}(p, m)$$

$$\equiv \varrho\big((p, m), \big(gh, \lambda_{(gh)^{-1}}(m)\big)\big) \equiv \varrho\big((p, m), \big(g, \lambda_h\big(\lambda_{(gh)^{-1}}(m)\big)\big).\big(h, \lambda_{(gh)^{-1}}(m)\big)\big)$$

$$= \varrho\big((p, m), \big(g, \lambda_{g^{-1}}(m)\big).\big(h, \lambda_{h^{-1}g^{-1}}(m)\big)\big)\,;$$

- axiom (PGr2): $\pi_\sim\big(\widetilde{\lambda}_{g^{-1}}(p, m)\big) \equiv [\widetilde{\lambda}_{g^{-1}}(p, m)] = [(p, m)] \equiv \pi_\sim(p, m)$;

- axiom (PGr3): the map

$$\begin{aligned}\big(\mathrm{pr}_1,(\phi_{\mathsf{P}}\circ\mathrm{pr}_{1,3},\mathrm{pr}_4)\big) &: (\mathsf{P}\times M)_{\pi_\sim}\times_{\pi_\sim}(\mathsf{P}\times M)\longrightarrow(\mathsf{P}\times M)_{\mathrm{pr}_2}\times_\lambda(\mathrm{G}\times M)\\ &: \big((p_1,m_1),(p_2,m_2)\big)\longmapsto\big((p_1,m_1),(\phi_{\mathsf{P}}(p_1,p_2),m_2)\big)\,,\end{aligned}$$

which is well-defined (and manifestly smooth) due to the following identities:

$$\pi_{\mathsf{P}}(p_2)\equiv\pi_{\mathsf{P}\times_\lambda}\big([(p_2,m_2)]\big)\equiv\big(\pi_{\mathsf{P}\times_\lambda}\circ\pi_\sim\big)(p_2,m_2)=\big(\pi_{\mathsf{P}\times_\lambda}\circ\pi_\sim\big)(p_1,m_1)\equiv\pi_{\mathsf{P}\times_\lambda}\big([(p_1,m_1)]\big)\equiv\pi_{\mathsf{P}}(p_1)\,,$$

$$[(p_1,\lambda_{\phi_{\mathsf{P}}(p_1,p_2)}(m_2))]=[(p_1\triangleleft\phi_{\mathsf{P}}(p_1,p_2),m_2)]=[(p_2,m_2)]\equiv\pi_\sim(p_2,m_2)=\pi_\sim(p_1,m_1)\equiv[(p_1,m_1)]\,,$$

is an inverse of (pr_1,ϱ), as verified in direct calculation

$$\begin{aligned}&\big((\mathrm{pr}_1,(\phi_{\mathsf{P}}\circ\mathrm{pr}_{1,3},\mathrm{pr}_4))\circ(\mathrm{pr}_1,\varrho)\big)\big((p,m),(g,\lambda_{g^{-1}}(m))\big)\\ =\;&\big(\mathrm{pr}_1,(\phi_{\mathsf{P}}\circ\mathrm{pr}_{1,3},\mathrm{pr}_4)\big)\big((p,m),(r_g(p),\lambda_{g^{-1}}(m))\big)=\big((p,m),(\phi_{\mathsf{P}}(p,r_g(p)),\lambda_{g^{-1}}(m))\big)\\ =\;&\big((p,m),(g,\lambda_{g^{-1}}(m))\big)\,,\end{aligned}$$

$$\begin{aligned}&\big((\mathrm{pr}_1,\varrho)\circ(\mathrm{pr}_1,(\phi_{\mathsf{P}}\circ\mathrm{pr}_{1,3},\mathrm{pr}_4))\big)\big((p_1,m_1),(p_2,m_2)\big)=(\mathrm{pr}_1,\varrho)\big((p_1,m_1),(\phi_{\mathsf{P}}(p_1,p_2),m_2)\big)\\ =\;&\big((p_1,m_1),(p_1\triangleleft\phi_{\mathsf{P}}(p_1,p_2),\lambda_{\phi_{\mathsf{P}}(p_1,p_2)^{-1}}(m_1))\big)=\big((p_1,m_1),(p_2,m_2)\big)\,.\end{aligned}$$

Next, we postulate the left action, along the momentum $\vec{\mu}\equiv\pi_\sim$, in the natural form

$$\begin{aligned}\vec{\lambda} &: \big(\mathrm{At}(\mathsf{P})_{\mathsf{S}}\times_{\pi_{\mathsf{P}\times_\lambda M}}(\mathsf{P}\times_\lambda M)\big)_{\mathrm{pr}_2}\times_{\pi_\sim}(\mathsf{P}\times M)\longrightarrow\mathsf{P}\times M\\ &: \big(([(p_2,p_1)],[(p_1,m)]),(p_1,m)\big)\longmapsto(p_2,m)\,,\end{aligned}$$

and check the axioms of Def. VI.4., adapted to the left action:

- axiom (PGr1):

$$\begin{aligned}\text{(GrM1)} &: \quad \vec{\mu}(p_2,m)=[(p_2,m)]=\Lambda_{[(p_2,p_1)]}\big([(p_1,m)]\big)\,,\\ \text{(GrM2)} &: \quad \vec{\lambda}_{\iota_{[(p,m)]}}(p,m)\equiv\vec{\lambda}_{[(p,p)],[(p,m)])}(p,m)=(p,m)\,,\\ \text{(GrM3)} &: \quad \vec{\lambda}_{([(p_3,p_2)],[(p_2,m)])}\big(\vec{\lambda}_{([(p_2,p_1)],[(p_1,m)])}(p_1,m)\big)=\vec{\lambda}_{([(p_3,p_2)],[(p_2,m)])}\big((p_2,m)\big)\\ &\quad =(p_3,m)\equiv\vec{\lambda}_{([(p_3,p_1)],[(p_1,m)])}(p_1,m)\equiv\vec{\lambda}_{m(([(p_3,p_2)],[(p_2,m)]),([(p_2,p_1)],[(p_1,m)]))}(p_1,m)\,;\end{aligned}$$

- axiom (PGr2): $\mathrm{pr}_2\big(\vec{\lambda}_{([(p_2,p_1)],[(p_1,m)])}(p_1,m)\big)\equiv\mathrm{pr}_2(p_2,m)=m\equiv\mathrm{pr}_2(p_1,m)$;
- axiom (PGr3): the map

$$\begin{aligned}\big(\vec{\phi},\mathrm{pr}_2\big) &: (\mathsf{P}\times M)_{\mathrm{pr}_2}\times_{\mathrm{pr}_2}(\mathsf{P}\times M)\longrightarrow\big(\mathrm{At}(\mathsf{P})_{\mathsf{S}}\times_{\pi_{\mathsf{P}\times_\lambda M}}(\mathsf{P}\times_\lambda M)\big)_{\mathrm{pr}_2}\times_{\pi_\sim}(\mathsf{P}\times M)\\ &: \big((p_2,m),(p_1,m)\big)\longmapsto\big(([(p_2,p_1)],[(p_1,m)]),(p_1,m)\big)\,,\end{aligned}$$

is the inverse of $(\vec{\lambda},\mathrm{pr}_2)$.

Finally, the commutation of the two actions is verified in a direct calculation:

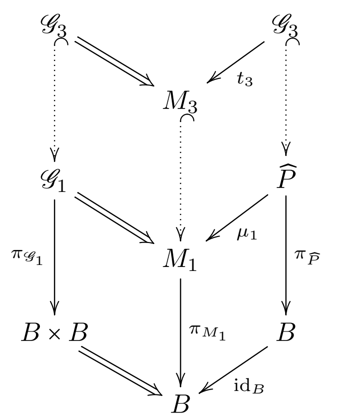
$$\begin{aligned}&\vec{\lambda}_{([(p_2,r_g(p_1))],[(r_g(p_1),\lambda_{g^{-1}}(m))])}\big(\varrho\big((p_1,m),(g,\lambda_{g^{-1}}(m))\big)\big)\\ =\;&\vec{\lambda}_{([(p_2,r_g(p_1))],[(r_g(p_1),\lambda_{g^{-1}}(m))])}\big((r_g(p_1),\lambda_{g^{-1}}(m))\big)=\big(p_2,\lambda_{g^{-1}}(m)\big)\\ \equiv\;&\varrho\big((r_{g^{-1}}(p_2),m),(g,\lambda_{g^{-1}}(m))\big)\equiv\varrho\big(\vec{\lambda}_{([(r_{g^{-1}}(p_2),p_1)],[(p_1,m)])}(p_1,m),(g,\lambda_{g^{-1}}(m))\big)\\ =\;&\varrho\big(\vec{\lambda}_{([(p_2,r_g(p_1))],[(p_1,m)])}(p_1,m),(g,\lambda_{g^{-1}}(m))\big)=\varrho\big(\vec{\lambda}_{([(p_2,r_g(p_1))],[(r_g(p_1),\lambda_{g^{-1}}(m))])}(p_1,m),(g,\lambda_{g^{-1}}(m))\big)\,.\end{aligned}$$

□

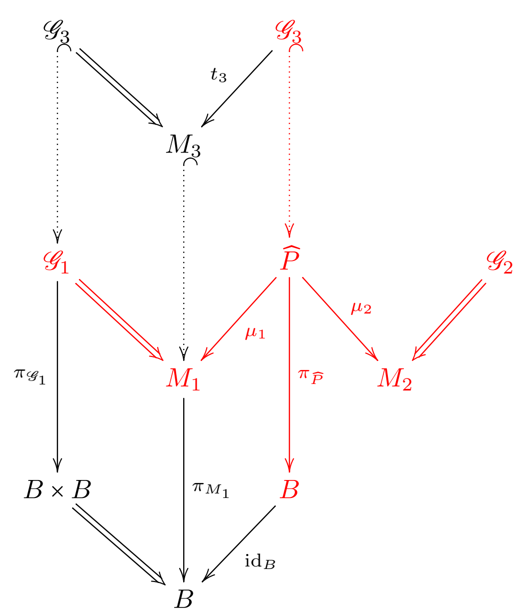
In view of its significance, and amenability to useful generalisations, we formalise the structure encountered above in

Definition 3. Let $\mathscr{G}_A \rightrightarrows M_A,\ A \in \{1,2,3\}$ be Lie groupoids, let $\widehat{P}$ be a principal $(\mathscr{G}_1, \mathscr{G}_2)$-bibundle, as in Def. VI.6., and let B be a manifold. We shall call the quintuple $(\mathscr{G}_1, \widehat{P}, \mathscr{G}_2; B, \mathscr{G}_3)$ a (**left**) **Trident** with base B and (left) fibre $\mathscr{G}_3$ if the following conditions are satisfied:

- $\widehat{P}$ is the total space of a fibre bundle $\pi_{\widehat{P}} : \widehat{P} \to B$ with base B and typical fibre $\mathscr{G}_3$;
- the right $\mathscr{G}_2$-action $\rho_2 \equiv \blacktriangleleft$ preserves $\pi_{\widehat{P}}$-fibres, *i.e.*, $\pi_{\widehat{P}}(p \blacktriangleleft g_2) = \pi_{\widehat{P}}(p)$ for all $(p, g_2) \in \widehat{P}_{\mu_2}\!\times_{t_2} \mathscr{G}_2$;
- the Lie groupoid $\mathscr{G}_1 \rightrightarrows M_1$ is the total space of a fibre-bundle object in the category of Lie groupoids, with base $\mathrm{Pair}(B)$ and typical fibre $\mathscr{G}_3 \rightrightarrows M_3$
- the $\mathscr{G}_1$-module structure on $\widehat{P}$ covers the canonical left $\mathrm{Pair}(B)$-module structure on B, and is modelled on the canonical left $\mathscr{G}_3$-module structure on $\mathscr{G}_3$ (in a local trivialisation), as captured by the diagram:



The Trident shall be represented by the following diagram:

(10)


Upon recalling the original conceptual interpretation of the Atiyah–Ehresmann groupoid in the contex of groupoidal implementation of automorphisms of the underlying principal bundle P, and upon inspection of the above-postulated action of the attendant action groupoid $\mathrm{At}(\mathsf{P}) \ltimes_\Lambda (\mathsf{P} \times_\lambda M)$ on the total space $\mathsf{P} \times M$ of the surjective submersion $\pi_\sim : \mathsf{P} \times M \longrightarrow \mathsf{P} \times_\lambda M$, along which physically relevant automorphisms of its (matter-field) base $\mathsf{P} \times_\lambda M$ are induced from those of $\mathsf{P} \times M$ (realised by that action), it becomes clear that the Trident encodes complete information about the complex mechanism of transmission—*via* association—of gauge symmetries, locally modelled on smooth profiles $O \longrightarrow \mathsf{G} \xrightarrow{\lambda} \mathrm{Diff}(M),\ O \in \mathscr{T}(B)$ (capturing the fibre component of the transformation, which comes on top of an iner-subjectivising diffeomorphism of the base B), from the 'space of local frames' P to the 'space of fields amenable to observation/description in the local frames' $\mathsf{P} \times_\lambda M$. More specifically, we may inscribe symmetry transformations Φ directly into the Trident, and in so doing provide a geometrisation of the induction scheme of Prop. IV.78.—this we acjieve by evaluating/restricting the action of $\mathrm{At}(\mathsf{P}) \ltimes_\Lambda (\mathsf{P} \times_\lambda M)$ on/to images $\beta_\Phi(\mathsf{P} \times_\lambda M)$ of the base $\mathsf{P} \times_\lambda M$ of the groupoid under those of its global bisections, β_Φ, which cover global bisections $(f, \mathrm{id}_B),\ f \in \mathrm{Diff}(B)$ of the base $\mathrm{Pair}(B)$ of the (gauge-)symmetry model

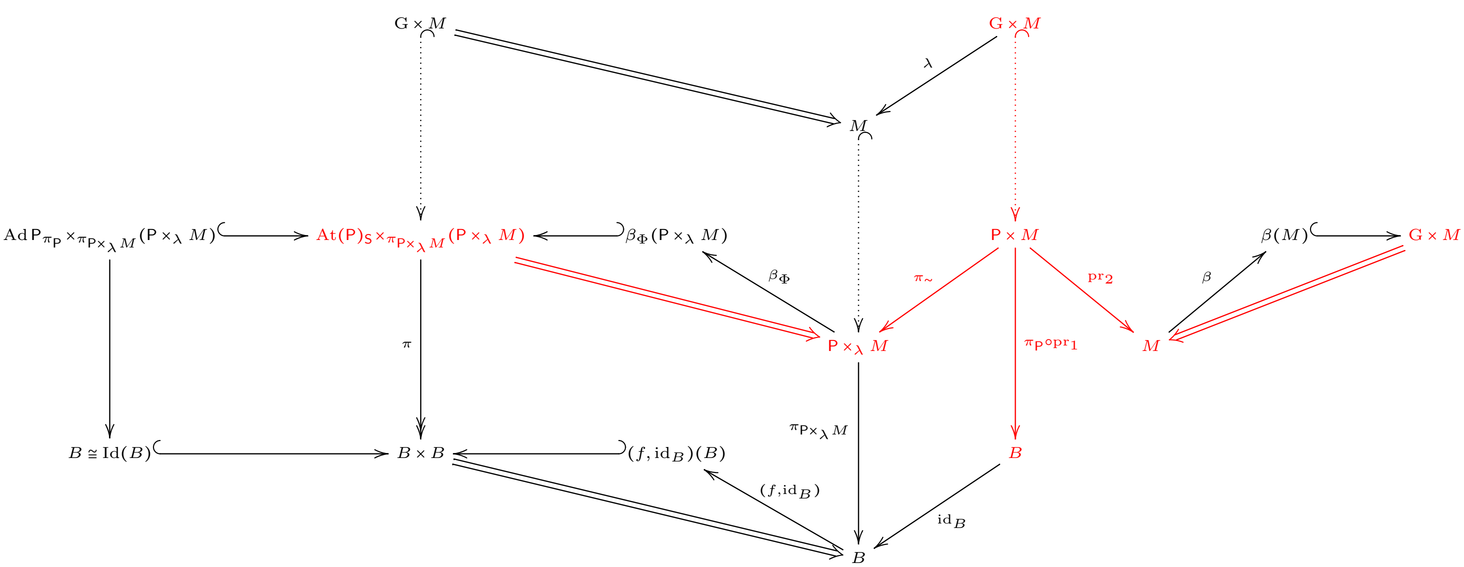
$$\begin{array}{ccc} \mathsf{G} \ltimes_\lambda M & \hookrightarrow & \mathrm{At}(\mathsf{P}) \ltimes_\Lambda (\mathsf{P} \times_\lambda M) \\ & & \Big\downarrow {\scriptstyle (\mathsf{T} \times \pi_{\mathsf{P} \times_\lambda M},\, \pi_{\mathsf{P} \times_\lambda M})} \\ & & \mathrm{Pair}(B) \end{array} .$$

(a bundle object in the category of Lie groupoids, see Prop. VI.10.). Of course, the mechanism just described restricts to the subgroupoid

$$\mathrm{Ad}\,\mathsf{P}_{\pi_\mathsf{P}} \times_{\pi_{\mathsf{P} \times_\lambda M}} (\mathsf{P} \times_\lambda M) \equiv \mathrm{Ad}\,\mathsf{P}_\mathsf{S} \times_{\pi_{\mathsf{P} \times_\lambda M}} (\mathsf{P} \times_\lambda M) \xhookrightarrow{\jmath_{\mathrm{Ad}\,\mathsf{P}}} \mathrm{At}(\mathsf{P})_\mathsf{S} \times_{\pi_{\mathsf{P} \times_\lambda M}} (\mathsf{P} \times_\lambda M)$$

over $\mathrm{Id}(B) \subset B \times B$ and its global bisections, which—accordingly—encodes the 'subjective' component of the gauge symmetry. We may also put global bisections β of the global symmetry model $\mathsf{G} \ltimes_\lambda M$ in the same picture.

Altogether, we arrive at the following beastly Bisection-Extended Trident, or the Diving Falcon:



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(11)

which summarises our findings and sets the stage for generalisations.

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