PRINCIPALITY – A JAZZY AND USEFUL PROPERTY
(DDD '24/25 VI [RRS])

FIGURE 1. *Sua Tremendità* (His Tremendousness) Giorgio I — by education and passion, a flower grower; by office and imagination, the *princeps* of *Principato di Seborga*, a country recognised (according to the 297 Seborgans themselves) by Burkina Faso and flourishing under the pragmatic motto: *Sub umbra sedi!* (Sit in the shade!).

CONTENTS

The concept of (endo-)diffeomorphism—a smooth and smoothly invertible self-mapping of a differentiable manifold M—admits a useful and consequential reinterpretation: Any such map $f \in \mathrm{Diff}(M)$ can be regarded as a coordinate transformation, in which the original (local) coordinates in a chart $O \ni x$ around a given point $x \in M$ are replaced by those in a chart $\widetilde{O} \ni f(x)$ around the image $f(x)$ of that point under the map. This interpretation of diffeomorphisms is sometimes labelled 'passive', in order to distinguish it from the original one (in which points in M are understood to actually map to one another, *i.e.*, to *move*), termed 'active' in this context.

Accordingly, whenever we are given a *global symmetry* of a field theory, realised by distinguished (endo-)diffeomorphisms $\mathrm{G} \subset \mathrm{Diff}(F)$ of the typical fibre F of its configuration bundle (E, B, F, π_E) modelling internal degrees of freedom of the field, we can think of it as a subgroup of coordinate transformations in the latter space preserving the dynamics (as captured by the action functional). This then leads us to contemplate a meaningful redefinition of the theory in which symmetry transformations prescribed *globally* (*i.e.*, determined, each, by a single diffeomorphism for *all* fibres of E) are replaced by *families* of diffeomorphisms $\gamma(x) \in \mathrm{G}$ depending *smoothly* (in conformity with the underlying smooth paradigm of our mathematical modelling of field-theoretic phenomena) on the point $x \in B$ in the spacetime base B of E. This is the conceptual foundation of a field-theoretic procedure known as the **gauging** of the global symmetry G, or rendering the latter *local*. Its deep geometric meaning is encoded in Cartan's mixing construction [Car50], recently reviewed by Tu in his monograph [Tu20] – it boils down to the reduction of the space of internal degrees of freedom from the original one F to its orbispace $F//\mathrm{G}$. The problem here is that the latter is—in general—*not* smooth, and so it cannot be accessed directly within the said smooth paradigm. As shall be argued *simply & intuitively* in the next lecture, the gauging is a method of effectively *modelling* the orbispace ('up to homotopy') by a smooth space, and working with the smooth model instead. Before we get there, though, we need to pass through some mathematical preliminaries, which will provide us, *i.a.*, with a smooth geometric model of a 'space of local (observation/description) frames', also known as **local gauges**, in a field theory with the global symmetry G *gauged* (*i.e.*, rendered local). This we first define for the conventional model of symmetries, that is—a smooth (Lie-)*group* action on F.

1. A motivating canonical construction

Last time, we encountered the construction of a fibre bundle canonically associated with an arbitrary smooth manifold (or, indeed, even of class C^1) given by the tangent bundle of that manifold. The bundle turned out to be endowed with a natural and intuitively anticipated linear structure on the fibre, from which one abstracts the notion of vector bundle, a geometrisation of the algebraic structure of a vector space. The canonical nature of the construction of the tangent bundle (*i.e.*, the lack of any topological obstruction against its realisation) demonstrates the naturality of the construction in the smooth category. We shall now follow the trail of canonical constructions straight towards the goal of our investigation: the fibre bundles that enter the gauging procedure (or, equivalently, the mixing construction) for group-like symmetries and their generalisations. For this purpose, we shall consider another example of a canonical geometrisation of a simple algebraic structure, only to abstract from our considerations a definition of a new class of fibre bundles.

One of the most natural procedures one can perform in a given linear space is a choice of a basis $\{e_i\}_{i\in\overline{1,D}}$, $D = \dim_{\mathbb{K}} V$, *i.e.*, a choice of an isomorphism

$$\mathbb{K}^{\times D} \xrightarrow{\cong} V \;:\; (v^1, v^2, \ldots, v^D) \longmapsto v^i \triangleright e_i\,.$$

This construction has a somewhat nontrivial counterpart in the theory of vector bundles, which we consider next.

Definition 1. Let $(\mathbb{V}, B, \mathbb{K}^{\times r}, \pi_{\mathbb{V}})$ be a vector bundle of rank $r \in \mathbb{N}^\times$, with local trivialisations $\tau_i \;:\; \pi_{\mathbb{V}}^{-1}(O_i) \xrightarrow{\cong} O_i \times \mathbb{K}^{\times r}$ associated with an open cover $\mathcal{O}_B = \{O_i\}_{i\in I}$ of its base B. The **frame bundle** of $\mathbb{V}$ is the fibre bundle

$$\big(\mathsf{F}_{\mathrm{GL}}\mathbb{V}, B, \mathrm{GL}(r;\mathbb{K}), \pi_{\mathsf{F}_{\mathrm{GL}}\mathbb{V}}\big)$$

with the following components:

- the total space
$$\mathsf{F}_{\mathrm{GL}}\mathbb{V} \coloneqq \bigsqcup_{x\in B} \mathrm{Iso}_{\mathbb{K}}\big(\mathbb{K}^{\times r}, \mathbb{V}_x\big)$$
with a smooth structure induced along trivialisations $\{\tau_i\}_{i\in I}$ and with a fibre $(\mathsf{F}_{\mathrm{GL}}\mathbb{V})_x \equiv \mathrm{Iso}_{\mathbb{K}}(\mathbb{K}^{\times r}, \mathbb{V}_x)$ given by the set of all bases $\beta_x \;:\; \mathbb{K}^{\times r} \xrightarrow{\cong} \mathbb{V}_x$ of the fibre $\mathbb{V}_x$ of $\mathbb{V}$;
- the typical fibre $\mathrm{GL}(r;\mathbb{K}) \equiv \mathrm{Aut}_{\mathbb{K}}(\mathbb{K}^{\times r})$;
- the base projection $\pi_{\mathsf{F}_{\mathrm{GL}}\mathbb{V}} \;:\; \mathsf{F}_{\mathrm{GL}}\mathbb{V} \longrightarrow B \;:\; (\beta_x, x) \longmapsto x$.

Inverses $\mathsf{F}\tau_i$ of the maps

$$\mathsf{F}\tau_i^{-1} \;:\; O_i \times \mathrm{GL}(r;\mathbb{K}) \xrightarrow{\cong} \pi_{\mathsf{F}_{\mathrm{GL}}\mathbb{V}}^{-1}(O_i) \;:\; (x,\chi) \longmapsto \big(\tau_i^{-1}\big(x, \chi(\cdot)\big), x\big)$$

induce on $\mathsf{F}_{\mathrm{GL}}\mathbb{V}$ a strong pullback topology from the product (subspace) topology on $O_i \times \mathrm{GL}(r;\mathbb{K})$ (the topology on $\mathrm{GL}(r;\mathbb{K})$ being that of a subspace of the vector space $\mathbb{K}(r) \cong \mathbb{K}^{\times r^2}$), *i.e.* one in which a subset $\mathcal{U} \subset \mathsf{F}_{\mathrm{GL}}\mathbb{V}$ is open iff it satisfies the condition

$$\forall_{i\in I} \;:\; \mathsf{F}\tau_i\big(\mathcal{U} \cap \pi^{-1}_{\mathsf{F}_{\mathrm{GL}}\mathbb{V}}(O_i)\big) \in \mathscr{T}\big(O_i \times \mathrm{GL}(r;\mathbb{K})\big)\,.$$

In this topology, the $\mathsf{F}\tau_i$ are homeomorphic local trivialisations with the associated transition maps

$$g_{ij}^{\mathsf{F}_{\mathrm{GL}}\mathbb{V}} \equiv \mathrm{Hom}_{\mathbb{K}}\big(\mathbb{K}^{\times r}, g_{ij}(\cdot)\big) \;:\; \mathcal{O}_{ij} \longrightarrow \mathrm{End}_{\mathbb{K}}\big(\mathrm{GL}(r;\mathbb{K})\big) \;:\; x \longmapsto \mathrm{Hom}_{\mathbb{K}}\big(\mathbb{K}^{\times r}, g_{ij}(x)\big)\,,$$

where

$$\mathrm{Hom}_{\mathbb{K}}\big(\mathbb{K}^{\times r}, g_{ij}(x)\big) \;:\; \mathrm{GL}(r;\mathbb{K}) \circlearrowleft \;:\; \chi \longmapsto g_{ij}(x) \circ \chi\,.$$

The smooth structure on the bundle is induced along the homeomorphisms $\mathsf{F}\tau_i$ from the product smooth structure on the local model $O_i \times \mathrm{GL}(r;\mathbb{K})$, trivial on the second factor and that obtained through intersection with the atlas $\widehat{\mathscr{A}_B}$ on the first (base) manifold B. Relative to this structure, the local trivialisations $\mathsf{F}\tau_i$ are tautologically smooth, as is the base projection, with local restrictions $\pi_{\mathsf{F}_{\mathrm{GL}}\mathbb{V}}\restriction_{\pi^{-1}_{\mathsf{F}_{\mathrm{GL}}\mathbb{V}}(O_i)} \equiv \mathrm{pr}_1 \circ \mathsf{F}\tau_i$ which glue over the intersections O_{ij}.

The above definition calls for a few comments. First of all, note the existence of a natural right action—fibre by fibre—of the group $\mathrm{GL}(r;\mathbb{K})$ on the total space $\mathsf{F}_{\mathrm{GL}}\mathbb{V}$, given by

$$r \;:\; \mathsf{F}_{\mathrm{GL}}\mathbb{V} \times \mathrm{GL}(r;\mathbb{K}) \longrightarrow \mathsf{F}_{\mathrm{GL}}\mathbb{V} \;:\; \big((\beta_x, x), \chi\big) \longmapsto (\beta_x \circ \chi, x) \equiv (\beta_x, x) \triangleleft \chi\,.$$

This action is manifestly free due to invertibility of elements of the fibre $\mathrm{Iso}_{\mathbb{K}}(\mathbb{K}^{\times r}, \mathbb{V}_x)$. Moreover, it is transitive over every point $x \in B$—indeed, for every pairy $\beta_{x\,1}, \beta_{x\,2} \in \mathrm{Iso}_{\mathbb{K}}\big(\mathbb{K}^{\times r}, \mathbb{V}_x\big)$, we have an identity

$$\beta_{x\,2} \equiv \beta_{x\,1} \circ \big(\beta_{x\,1}^{-1} \circ \beta_{x\,2}\big)\,,$$

but $\beta_{x\,1}^{-1} \circ \beta_{x\,2} \in \mathrm{End}_{\mathbb{K}}(\mathbb{K}^{\times r})$ is invertible, with the inverse $\beta_{x\,2}^{-1} \circ \beta_{x\,1}$, and so we may write

$$(\beta_{x\,2}, x) = (\beta_{x\,1}, x) \triangleleft \big(\beta_{x\,1}^{-1} \circ \beta_{x\,2}\big)\,.$$

We conclude that $\mathrm{Iso}_{\mathbb{K}}(\mathbb{K}^{\times r}, \mathbb{V}_x)$ is a $\mathrm{GL}(r;\mathbb{K})$-torsor. A choice of an element $\beta_{x\,*} \in \mathrm{Iso}_{\mathbb{K}}(\mathbb{K}^{\times r}, \mathbb{V}_x)$ determines a *noncanonical* ($\mathrm{GL}(r;\mathbb{K})$-equivariant) isomorphism

$$\mathrm{Iso}_{\mathbb{K}}\big(\mathbb{K}^{\times r}, \mathbb{V}_x\big) \xrightarrow{\cong} \mathrm{GL}(r;\mathbb{K}) \;:\; \beta_x \longmapsto \beta_{x\,*}^{-1} \circ \beta_x\,.$$

From the above, we readily infer that the $\mathsf{F}\tau_i^{-1}$ are bijective: They assign to elements of the set $\{x\} \times \mathrm{GL}(r;\mathbb{K})$, $x \in O_i$ those of $\mathrm{Iso}_{\mathbb{K}}(\mathbb{K}^{\times r}, \mathbb{V}_x) \times \{x\}$ in a manifestly injective manner. Hence, they are invertible, specifically

$$\mathsf{F}\tau_i \;:\; \pi^{-1}_{\mathsf{F}_{\mathrm{GL}}\mathbb{V}}(O_i) \longrightarrow O_i \times \mathrm{GL}(r;\mathbb{K}) \;:\; (\beta_x, x) \longmapsto \big(x, \mathrm{pr}_2 \circ \tau_i\beta_x\big)\,,$$

which enables us to use them to induce a topology on $\mathsf{F}_{\mathrm{GL}}\mathbb{V}$ in the way described. Their identification as local trivialisations bases on the following direct calculation:

$$\begin{aligned}
\mathsf{F}\tau_i \circ \mathsf{F}\tau_j^{-1} \quad &: \quad \mathcal{O}_{ij} \times \mathrm{GL}(r;\mathbb{K}) \circlearrowleft \\
&: \quad (x, \chi) \longmapsto \mathsf{F}\tau_i\big(\tau_j^{-1}\big(x, \chi(\cdot)\big), x\big) \equiv \mathsf{F}\tau_i\big(\tau_i^{-1} \circ \tau_i \circ \tau_j^{-1}\big(x, \chi(\cdot)\big), x\big) \\
&\qquad = \mathsf{F}\tau_i\big(\tau_i^{-1}\big(x, g_{ij}(x) \circ \chi(\cdot)\big), x\big) \equiv \mathsf{F}\tau_i \circ \mathsf{F}\tau_i^{-1}\big(x, g_{ij}(x) \circ \chi(\cdot)\big) \\
&\qquad = \big(x, g_{ij}(x) \circ \chi(\cdot)\big)\,,
\end{aligned}$$

in which $g_{ij} \in C^\infty(\mathcal{O}_{ij}, \mathrm{GL}(r;\mathbb{K}))$ are transition maps of $\mathbb{V}$. The calculation demonstrates the smooth character of the $\mathsf{F}\tau_i \circ \mathsf{F}\tau_j^{-1}$.

Last, we note that the maps $\mathsf{F}\tau_i^{-1}$ (and so also the local trivialisations $\mathsf{F}\tau_i$) intertwine (*i.e.*, are equivariant relative to) the right actions of the group $\mathrm{GL}(r;\mathbb{K})$: the regular one $\wp$ on the second cartesian factor in their domain and the above-defined r on their codomain. Indeed, we compute directly, for $\gamma \in \mathrm{GL}(r;\mathbb{K})$ arbitrary,

$$\begin{aligned}
\mathsf{F}\tau_i^{-1} \circ (\mathrm{id}_{O_i} \times \wp_\gamma)(x, \chi) \quad &= \quad \mathsf{F}\tau_i^{-1}(x, \chi \circ \gamma) = \big(\tau_i^{-1}\big(x, \chi \circ \gamma(\cdot)\big), x\big) \equiv \big(\tau_i^{-1}\big(x, \chi(\cdot)\big) \circ \gamma(\cdot), x\big) \\
&\equiv \quad \big(\tau_i^{-1}\big(x, \chi(\cdot)\big), x\big) \triangleleft \gamma \equiv r_\gamma \circ \mathsf{F}\tau_i^{-1}(x, \chi)\,.
\end{aligned}$$

Consequently, the local trivialisations are compatible with the structure of a $\mathrm{GL}(r;\mathbb{K})$-torsor on the fibre of the frame bundle, noted previously.

The above is a perfect springboard for an abstraction given in the next section.

2. Principal bundles: axiomatics, elementary properties and morphisms

We start with the fundamental

Definition 2. Let G be a Lie group. A **principal bundle** with **structure group** G is a pair

$$\big((\mathsf{P},B,\mathrm{G},\pi_{\mathsf{P}}),r\big)$$

composed of

- a fibre bundle $(\mathsf{P},B,\mathrm{G},\pi_{\mathsf{P}})$;
- a right action $r\ :\ \mathsf{P}\times\mathrm{G}\longrightarrow\mathsf{P}$

with the following properties:

- the action r is free and fibrewise transitive, and represented by the commutative diagram

$$\begin{array}{ccc} \mathsf{P}\times\mathrm{G} & \xrightarrow{\ r\ } & \mathsf{P} \\ {\scriptstyle \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathsf{P}}} \\ \mathsf{P} & \xrightarrow[\ \pi_{\mathsf{P}}\ ]{} & B \end{array}\ ;$$

- the local trivialisations

$$\tau_i\ :\ \pi_{\mathsf{P}}^{-1}(O_i)\xrightarrow{\ \cong\ }O_i\times\mathrm{G}\,,\quad i\in I\,,$$

  associated with a trivialising open cover $\mathscr{O}=\{O_i\}_{i\in I}$ of the base B, are G-equivariant with respect to the right actions: r on the domain, and the regular one $\wp$ on the second cartesian factor of the codomain,

$$\widetilde{\wp}^i\equiv\mathrm{id}_{O_i}\times\wp\ :\ (O_i\times\mathrm{G})\times\mathrm{G}\longrightarrow O_i\times\mathrm{G}\ :\ \big((x,g),h\big)\longmapsto(x,g\cdot h)\,,$$

  i.e., they satisfy the conditions

$$\tau_i\circ r_g=\widetilde{\wp}^i_g\circ\tau_i\,,\qquad i\in I,\ g\in\mathrm{G}\,.$$

A **principal subbundle** of a principal bundle $(\mathsf{P},B,\mathrm{G},\pi_{\mathsf{P}}),r)$ is a subbundle $(\mathsf{P}_{\mathrm{H}},B,\mathrm{H},\pi_{\mathsf{P}}\restriction_{\mathsf{P}_{\mathrm{H}}})$ of that fibre bundle with the fibre given by a Lie subgroup $\mathrm{H}\subset\mathrm{G}$, and the defining action of H on its total space induced from r through restriction.

A **morphism of principal bundles** $(\mathsf{P}_\alpha,B_\alpha,\mathrm{G}_\alpha,\pi_{\mathsf{P}_\alpha}),r^\alpha)$, $\alpha\in\{1,2\}$ is a triple (Φ,f,φ) composed of a bundle map

$$(\Phi,f)\ :\ (\mathsf{P}_1,B_1,\mathrm{G}_1,\pi_{\mathsf{P}_1})\longrightarrow(\mathsf{P}_2,B_2,\mathrm{G}_2,\pi_{\mathsf{P}_2})$$

and a Lie-group homomorphism φ, the three being related as in the following commutative diagram:

$$\begin{array}{ccccc} \mathsf{P}_1\times\mathrm{G}_1 & \xrightarrow{\ r^1\ } & \mathsf{P}_1 & \xrightarrow{\ \pi_{\mathsf{P}_1}\ } & B_1 \\ {\scriptstyle \Phi\times\varphi}\big\downarrow & & \big\downarrow{\scriptstyle \Phi} & & \big\downarrow{\scriptstyle f} \\ \mathsf{P}_2\times\mathrm{G}_2 & \xrightarrow[\ r^2\ ]{} & \mathsf{P}_2 & \xrightarrow[\ \pi_{\mathsf{P}_2}\ ]{} & B_2 \end{array}\ . \tag{1}$$

Example 1.

(1) A trivial principal bundle with structure group G over B, *i.e.*, the trivial bundle

$$(B\times\mathrm{G},B,\mathrm{G},\mathrm{pr}_1)\,,$$

with the defining action given by the right regular action of G on the second cartesian factor of the total space.

(2) The frame bundle of a vector bundle $\mathbb{V}$ modelled on $\mathbb{K}^{\times r}$, *i.e.*, the fibre bundle

$$\big(\mathsf{F}_{\mathrm{GL}}\mathbb{V}, B, \mathrm{GL}(r;\mathbb{K}), \pi_{\mathsf{F}_{\mathrm{GL}}\mathbb{V}}\big)$$

with the action of $\mathrm{GL}(r;\mathbb{K})$ detailed in Section 1. In particular, we have the **tangent frame bundle** over a smooth manifold M (of dimension n):

$$\big(\mathsf{F}_{\mathrm{GL}}\mathsf{T}M, M, \mathrm{GL}(n;\mathbb{R}), \pi_{\mathsf{F}_{\mathrm{GL}}\mathsf{T}M}\big).$$

(3) The **Hopf fibration**—once again, but this time viewed as

$$\big(\mathrm{SU}(2) \equiv \mathbb{S}^3, \mathbb{S}^2, \mathrm{U}(1), \pi\big),$$

in which the base projection takes the form

$$\pi \;:\; \mathrm{SU}(2) \longrightarrow \mathbb{S}^2 \subset \mathbb{R}^{\times 3} \cong \mathbb{C} \times \mathbb{R} \;:\; \begin{pmatrix} z_1 & \overline{z}_2 \\ -z_2 & \overline{z}_1 \end{pmatrix} \longmapsto \big(2z_1 \cdot \overline{z}_2, |z_1|^2 - |z_2|^2\big),$$

and the defining action of the structure group $\mathrm{U}(1)$ is given by

$$\begin{array}{rcl} r. & : & \mathrm{SU}(2) \times \mathrm{U}(1) \longrightarrow \mathrm{SU}(2) \\ & : & \left(\begin{pmatrix} z_1 & \overline{z}_2 \\ -z_2 & \overline{z}_1 \end{pmatrix}, u\right) \longmapsto \begin{pmatrix} z_1 \cdot u & \overline{z}_2 \cdot u^{-1} \\ -z_2 \cdot u & \overline{z}_1 \cdot u^{-1} \end{pmatrix} \equiv \begin{pmatrix} z_1 & \overline{z}_2 \\ -z_2 & \overline{z}_1 \end{pmatrix} \cdot \begin{pmatrix} u & 0 \\ 0 & \overline{u} \end{pmatrix}, \end{array}$$

where in the last equality the group $\mathrm{U}(1)$ appears as a subgroup of the Lie group $\mathrm{SU}(2)$.

Definition 3. Let $((\mathsf{P}, B, \mathrm{G}, \pi_{\mathsf{P}}), r)$ be a principal bundle, and consider the corresponding fibred product

$$\mathsf{P} \times_B \mathsf{P} \coloneqq \{\ (p_1, p_2) \in \mathsf{P} \times \mathsf{P} \quad | \quad \pi_{\mathsf{P}}(p_1) = \pi_{\mathsf{P}}(p_2)\ \}.$$

The **division map** of P is the map

$$\phi_{\mathsf{P}} \;:\; \mathsf{P} \times_B \mathsf{P} \longrightarrow \mathrm{G}$$

determined (uniquely) by the condition

$$\forall_{(p_1,p_2) \in \mathsf{P} \times_B \mathsf{P}} \;:\; p_2 = p_1 \triangleleft \phi_{\mathsf{P}}(p_1, p_2).$$

Remark 1. The smoothness of the division map is best seen in a local trivialisation. Indeed, let $p_1, p_2 \in (\mathsf{P})_x$, $x \in O_i$, $i \in I$, where $p_\alpha = \tau_i^{-1}(x, g_\alpha)$, $\alpha \in \{1, 2\}$ for some $g_\alpha \in \mathrm{G}$. We then have the equality

$$p_2 = \tau_i^{-1}(x, g_2) \equiv \tau_i^{-1}\big(x, g_1 \cdot \big(g_1^{-1} \cdot g_2\big)\big) = \tau_i^{-1}(x, g_1) \triangleleft \big(g_1^{-1} \cdot g_2\big) \equiv p_1 \triangleleft \big(g_1^{-1} \cdot g_2\big),$$

from which a local presentation of ϕ_{P} ensues:

$$\phi_{\mathsf{P}}(p_1, p_2) \equiv g_1^{-1} \cdot g_2 = \mathrm{m}\big(\mathrm{Inv} \circ \mathrm{pr}_2 \circ \tau_i(p_1), \mathrm{pr}_2 \circ \tau_i(p_2)\big),$$

given as a superposition of smooth maps (m is the binary operation on G), and hence smooth.

The basic structural properties of ϕ_{P}, to be used amply in what follows, are summarised in

Proposition 1. The division map ϕ_{P} of a principal bundle $((\mathsf{P}, B, \mathrm{G}, \pi_{\mathsf{P}}), r)$ satisfies identities expressed by the following commutative diagrams:

(DM1) (the 1-cocycle condition)

$$\begin{array}{ccc} \mathsf{P} \times_B \mathsf{P} \times_B \mathsf{P} & \xrightarrow{(\phi_{\mathsf{P}} \circ \mathrm{pr}_{1,2}, \phi_{\mathsf{P}} \circ \mathrm{pr}_{2,3})} & \mathrm{G} \times \mathrm{G} \\ & \searrow^{\phi_{\mathsf{P}} \circ \mathrm{pr}_{1,3}} & \downarrow M \\ & & \mathrm{G} \end{array},$$

where $\mathrm{pr}_{i,j} \;:\; \mathsf{P} \times_B \mathsf{P} \times_B \mathsf{P} \longrightarrow \mathsf{P} \times_B \mathsf{P} \;:\; (p_1, p_2, p_3) \longmapsto (p_i, p_j)$, $(i, j) \in \{(1,2), (2,3), (1,3)\}$ is (the restriction of) the canonical projection; in other words,

$$\forall_{(p_1,p_2,p_3) \in \mathsf{P} \times_B \mathsf{P} \times_B \mathsf{P}} \;:\; \phi_{\mathsf{P}}(p_2, p_3) \circ \phi_{\mathsf{P}}(p_1, p_3)^{-1} \circ \phi_{\mathsf{P}}(p_1, p_2) = e;$$

this implies, in particular, skew symmetry

$$\begin{array}{ccc} \mathsf{P}\times_B\mathsf{P} & \xrightarrow{\tau_{\mathsf{P},\mathsf{P}}} & \mathsf{P}\times_B\mathsf{P} \\ \Big\downarrow{\scriptstyle \phi_{\mathsf{P}}} & & \Big\downarrow{\scriptstyle \phi_{\mathsf{P}}} \\ \mathrm{G} & \xrightarrow[\mathrm{Inv}]{} & \mathrm{G} \end{array} ,$$

where $\tau_{\mathsf{P},\mathsf{P}}\ :\ \mathsf{P}\times_B\mathsf{P}\ \circlearrowleft\ :\ (p_1,p_2)\longmapsto(p_2,p_1)$ is (the restriction of) the canonical transposition; in other words,

$$\forall_{(p_1,p_2)\in\mathsf{P}\times_B\mathsf{P}}\ :\ \phi_{\mathsf{P}}(p_2,p_1)=\phi_{\mathsf{P}}(p_1,p_2)^{-1}\,;$$

(DM2) (G-equivariance)

$$\begin{array}{ccccc} \mathsf{P}\times_B\mathsf{P} & \xleftarrow{(r\circ\mathrm{pr}_{1,3},\mathrm{pr}_2)} & (\mathsf{P}\times_B\mathsf{P})\times\mathrm{G} & \xrightarrow{\mathrm{id}_{\mathsf{P}}\times r} & \mathsf{P}\times_B\mathsf{P} \\ \Big\downarrow{\scriptstyle \phi_{\mathsf{P}}} & & \Big\downarrow{\scriptstyle \phi_{\mathsf{P}}\times\mathrm{id}_{\mathrm{G}}} & & \Big\downarrow{\scriptstyle \phi_{\mathsf{P}}} \\ \mathrm{G} & \xleftarrow[\ell\circ(\mathrm{Inv}\times\mathrm{id}_{\mathrm{G}})\circ\tau_{\mathrm{G},\mathrm{G}}]{} & \mathrm{G}\times\mathrm{G} & \xrightarrow[\wp]{} & \mathrm{G} \end{array} ,$$

or, in other words,

$$\forall_{(p_1,p_2)\in\mathsf{P}\times_B\mathsf{P},\ g_1,g_2\in\mathrm{G}}\ :\ \phi_{\mathsf{P}}(p_1\triangleleft g_1,p_2\triangleleft g_2)=g_1^{-1}\cdot\phi_{\mathsf{P}}(p_1,p_2)\cdot g_2\,.$$

Furthermore, the following identity holds true:

(2) $$(\mathrm{pr}_1,\phi_{\mathsf{P}})=(\mathrm{pr}_1,r)^{-1}$$

for the two smooth maps $\mathsf{P}\times_B\mathsf{P}\longrightarrow\mathsf{P}\times\mathrm{G}$, *i.e.*, we have, in particular

(3) $$\mathsf{P}\times_B\mathsf{P}\cong\mathsf{P}\times\mathrm{G}\,.$$

Proof: Obvious. □

Remark 2. The existence of a smooth inverse of the map $(\mathrm{pr}_1,\phi_{\mathsf{P}})$ permits us to meaningfully and rigorously identify the base B of the principal bundle with the smooth quotient P/G, in keeping with Godement's Criterion (Thm. I.21.). To this end, consider on $\mathsf{P}\ni p_1,p_2$ the equivalence relation defined as

$$p_1\sim p_2\qquad\Longleftrightarrow\qquad\exists_{g\in\mathrm{G}}\ :\ p_2=p_1\triangleleft g\,.$$

Its graph is quite simply

$$\mathscr{R}\equiv(\mathrm{pr}_1,r)(\mathsf{P}\times\mathrm{G})\equiv\mathsf{P}\times_B\mathsf{P}\subset\mathsf{P}\times\mathsf{P}\,.$$

It is, in particular, an embedded submanifold in $\mathsf{P}\times\mathsf{P}$ (as a fibred product of manifolds). But in view of the above, it is also a homeomorphic image of the (closed) topological space $\mathsf{P}\times\mathrm{G}$, hence it is closed (because (pr_1,r) is a closed map). Finally, note that $\mathrm{pr}_1\restriction_{\mathscr{R}}$ is manifestly submersive. Altogether, then, the quotient $\mathsf{P}/\sim$ carries a (unique) smooth structure with respect to which the quotient map

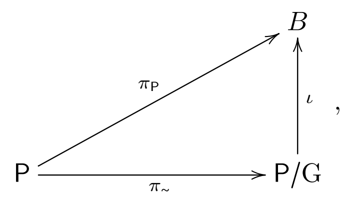
$$\pi_\sim\ :\ \mathsf{P}\longrightarrow\mathsf{P}/\sim$$

is a surjective submersion. However, from the definition of the equivalence relation considered, we read off the identity $\mathsf{P}/\sim\equiv\mathsf{P}/\mathrm{G}$, and it is intuitively clear that the submersion $\pi_\sim$ is but an elternative description of π_{P}.

A diffeomorphism $\mathsf{P}/\mathrm{G}\cong B$ can be written out explicitly. Consider a map

$$\iota\ :\ \mathsf{P}/\mathrm{G}\longrightarrow B\ :\ p\triangleleft\mathrm{G}\longmapsto\pi_{\mathsf{P}}(p)\,.$$

It is manifestly well-defined (as the defining action r preserves π_{P}-fibres). It is also smooth by Thm. V.2., as it closes the commutative diagram



with π_{P} and $\pi_\sim$ smooth, and the latter a surjective submersion.

Next, choose (arbirarily) a trivialising cover $\mathcal{O}_B = \{O_i\}_{i\in I}$ of B, and use the flat unital sections $\sigma_i \ : \ O_i \longrightarrow \mathsf{P} \ : \ x \in \tau_i^{-1}(x,e)$ induced from the corresponding local trivialisations τ_i of P to define smooth local maps

$$\jmath_i \coloneqq \pi_\sim \circ \sigma_i \ : \ O_i \longrightarrow \mathsf{P}/\mathrm{G} \ : \ x \longmapsto \sigma_i(x) \triangleleft \mathrm{G}\,.$$

These satisfy, at points $x \in O_{ij}$, identities

$$\jmath_j(x) = \sigma_j(x) \triangleleft \mathrm{G} \equiv \tau_j^{-1}(x,e) \triangleleft \mathrm{G} = \big(\tau_i^{-1}(x,e) \triangleleft g_{ij}(x)\big) \triangleleft \mathrm{G} \equiv \tau_i^{-1}(x,e) \triangleleft \mathrm{G} \equiv \jmath_i(x)\,,$$

which base on the assumed G-equivariance of the trivialisations. Thus, the $\jmath_i$ glue up to a globally smooth map

$$\jmath \ : \ B \longrightarrow \mathsf{P}/\mathrm{G}\,, \qquad \jmath\restriction_{O_i} = \jmath_i\,.$$

Finally, we check, for $x \in O_i$ and $p \equiv \tau_i^{-1}(x,g) \in \pi_{\mathsf{P}}^{-1}(O_i)$,

$$(\iota \circ \jmath)(x) = \iota\big(\jmath_i(x)\big) = \iota(\sigma_i(x) \triangleleft \mathrm{G}) \equiv \pi_{\mathsf{P}}\big(\sigma_i(x)\big) = x$$

and

$$\begin{aligned}(\jmath \circ \iota)\big(p \triangleleft \mathrm{G}\big) &\equiv \jmath\big(\pi_{\mathsf{P}}(p)\big) = \jmath_i(x) \equiv \sigma_i(x) \triangleleft \mathrm{G} \equiv \tau_i^{-1}\big(x,e\big) \triangleleft \mathrm{G} = \big(\tau_i^{-1}\big(x,g\big) \triangleleft g^{-1}\big) \triangleleft \mathrm{G} \\ &\equiv \big(p \triangleleft g^{-1}\big) \triangleleft \mathrm{G} = p \triangleleft \mathrm{G}\,,\end{aligned}$$

and conclude that $\jmath$ is a smooth inverse of the smooth map ι. The latter is the desired diffeomorphism

$$\iota \ : \ \mathsf{P}/\mathrm{G} \xrightarrow{\cong} B\,. \tag{4}$$

One can actually nail down the nature of the defining action r on the total space of the principal bundle upon invoking an equivalent definition of properness of an action expressed in terms of convergent (sub)sequences:

Proposition 2. An action $\lambda \ : \ \mathrm{G} \times X \longrightarrow X$ of a topological group G on a locally precompact[1] topological space $(X, \mathscr{T}(X))$ is proper iff the convergence of an arbitrary sequence of points

$$\lambda(g_\cdot, x_\cdot) \ : \ \mathbb{N} \longrightarrow M \ : \ n \longmapsto g_n \triangleright x_n\,,$$

defined in terms of a convergent sequence of points $x_\cdot \ : \ \mathbb{N} \longrightarrow X$ and an arbitrary sequence $g_\cdot \ : \ \mathbb{N} \longrightarrow \mathrm{G}$, implies the convergence of a subsequence of $g_\cdot$.

Proof: See MAWF '23/24 1.XXIV. □

We now readily establish

Proposition 3. The defining action of the structure group on the total space of a principal bundle is proper.

[1] A topological space is termed **locally precompact** if every point in that space belongs to some precompact set (*i.e.*, one whose closure is compact) which contains an open neighbourhood of the point.

Proof: Let $((\mathsf{P},B,\mathrm{G},\pi_{\mathsf{P}}),r)$ be a principal bundle. Consider sequences $p_\cdot : \mathbb{N} \longrightarrow \mathsf{P}$ and $g_\cdot : \mathbb{N} \longrightarrow \mathrm{G}$ with properties

$$\lim_{n\to\infty} p_n = p\,, \qquad \lim_{n\to\infty} (p_n \triangleleft g_n) = \widetilde{p}\,.$$

As a consequence of continuity of π_{P} and the fibrewise nature of r, we obtain the equality

$$\pi_{\mathsf{P}}(\widetilde{p}) \equiv \pi_{\mathsf{P}}\big(\lim_{n\to\infty} (p_n \triangleleft g_n)\big) = \lim_{n\to\infty} \pi_{\mathsf{P}}(p_n \triangleleft g_n) = \lim_{n\to\infty} \pi_{\mathsf{P}}(p_n) = \pi_{\mathsf{P}}\big(\lim_{n\to\infty} p_n\big) = \pi_{\mathsf{P}}(p)\,,$$

which leads—*via* Def. 3—to the relation

$$\widetilde{p} = p \triangleleft \phi_{\mathsf{P}}(p,\widetilde{p})\,.$$

Let $\pi_{\mathsf{P}}(p) \in O_i$, where O_i is an element of a trivialising open cover of B. There exists an index $N \in \mathbb{N}$ such that

$$\forall_{n\geq N} \ :\ p_n, p_n \triangleleft g_n \in \pi_{\mathsf{P}}^{-1}(O_i)\,,$$

and so we may consider subsequences $p_{N+\cdot}$ and $p_{N+\cdot} \triangleleft g_{N+\cdot}$ in the trivialisation $\tau_i : \pi_{\mathsf{P}}^{-1}(O_i) \xrightarrow{\cong} O_i \times \mathrm{G}$, in which

$$\tau_i(p_n) =: (x_n,\gamma_n)\,, \qquad \tau_i(p) =: (x,\gamma)\,,$$

that is

$$\lim_{n\to\infty} (x_n,\gamma_n) = (x,\gamma)\,,$$

and hence

$$\tau_i(p_n \triangleleft g_n) = \tau_i(p_n) \triangleleft g_n = (x_n,\gamma_n) \triangleleft g_n = (x_n,\gamma_n \cdot g_n)\,,$$

and also

$$\tau_i(\widetilde{p}) = \tau_i\big(p \triangleleft \phi_{\mathsf{P}}(p,\widetilde{p})\big) = \tau_i(p) \triangleleft \phi_{\mathsf{P}}(p,\widetilde{p}) = (x,\gamma) \triangleleft \phi_{\mathsf{P}}(p,\widetilde{p}) = \big(x,\gamma\cdot\phi_{\mathsf{P}}(p,\widetilde{p})\big)\,,$$

which further implies

$$\lim_{n\to\infty} (\gamma_n \cdot g_n) = \gamma \cdot \phi_{\mathsf{P}}(p,\widetilde{p})\,.$$

Continuity of group operations then yields

$$\lim_{n\to\infty} g_n \equiv \lim_{n\to\infty} \big(\gamma_n^{-1} \cdot (\gamma_n \cdot g_n)\big) = \gamma^{-1} \cdot \big(\gamma \cdot \phi_{\mathsf{P}}(p,\widetilde{p})\big) = \phi_{\mathsf{P}}(p,\widetilde{p})\,,$$

which concludes the proof. □

Continuing along these lines, in the direction of physical applications to come, we stumble upon

Corollary 1. Let $((\mathsf{P},B,\mathrm{G},\pi_{\mathsf{P}}),r)$ be a principal bundle, and M—a smooth manifold with a (left) action $\lambda : \mathrm{G}\times M \longrightarrow M$ of its structure group G. Consider the product manifold $\mathsf{P}\times M$. The diagonal action of G on the latter given by the formula

$$\widetilde{\lambda}_\cdot \ :\ \mathrm{G}\times(\mathsf{P}\times M) \longrightarrow \mathsf{P}\times M \ :\ \big(g,(p,x)\big) \longmapsto \big(r(p,g^{-1}),\lambda(g,m)\big)$$

(5)

is free and proper.

Proof: Obvious. □

Sometimes, we do not have at our disposal the "full package" of structures that enter the definition of a principal bundle, or follow from it. Therefore, it is worthwhile to pause and look for an alternative set of *constitutive* structures, whose existence implies that of a principal bundle. One such possibility is presented in

Proposition 4. Let P, B be smooth manifolds, and let G be a Lie group. Assume given a surjective submersion $\pi : \mathsf{P} \longrightarrow B$ and a smooth right action $r_\cdot : \mathsf{P} \times \mathrm{G} \longrightarrow \mathsf{P}$ of G on P. If r is free, its orbits coincide with level sets of π, and the map $\phi_{\mathsf{P}} : \mathsf{P} \times_B \mathsf{P} \longrightarrow \mathrm{G}$ determined uniquely by the condition

$$\forall_{(p_1,p_2)\in \mathsf{P}\times_B \mathsf{P}} \ : \ p_2 = r_{\phi_{\mathsf{P}}(p_1,p_2)}(p_1)$$

is smooth, then the quintuple

$$\big((\mathsf{P}, B, \mathrm{G}, \pi), r\big)$$

is a principal bundle.

<u>*Proof:*</u> In virtue of Thm. V.1, there exists an open cover $\mathscr{O} = \{O_i\}_{i\in I}$ of the manifold B, whose elements support smooth local sections $\sigma_i : O_i \longrightarrow \mathsf{P}$ of the submersion π. Accordingly, we may define the manifestly smooth maps

$$\tau_i^{-1} \ : \ O_i \times \mathrm{G} \longrightarrow \pi^{-1}(O_i) \ : \ (x,g) \longmapsto r_g\big(\sigma_i(x)\big)\,.$$

Using the map ϕ_{P}, we readily derive their (smooth) inverses:

$$\tau_i \ : \ \pi^{-1}(O_i) \longrightarrow O_i \times \mathrm{G} \ : \ p \longmapsto \big(\pi(p), \phi_{\mathsf{P}}\big(\sigma_i \circ \pi(p), p\big)\big)\,.$$

These are well-defined as

$$\pi\big(\sigma_i \circ \pi(p)\big) = (\pi \circ \sigma_i) \circ \pi(p) = \mathrm{id}_{O_i} \circ \pi(p) = \pi(p)\,,$$

and, indeed, satisfy the postulated identities:

$$\tau_i^{-1} \circ \tau_i(p) \quad = \quad \tau_i^{-1}\big(\pi(p), \phi_{\mathsf{P}}\big(\sigma_i \circ \pi(p), p\big)\big) = r_{\phi_{\mathsf{P}}(\sigma_i\circ\pi(p),p)}\big(\sigma_i \circ \pi(p)\big) \equiv p\,,$$

$$\begin{aligned}\tau_i \circ \tau_i^{-1}(x,g) \quad &= \quad \tau_i\big(r_g\big(\sigma_i(x)\big)\big) = \big(\pi \circ r_g \circ \sigma_i(x), \phi_{\mathsf{P}}\big(\sigma_i \circ \pi \circ r_g \circ \sigma_i(x), r_g \circ \sigma_i(x)\big)\big) \\ &= \quad \big(\pi \circ \sigma_i(x), \phi_{\mathsf{P}}\big(\sigma_i \circ \pi \circ \sigma_i(x), r_g \circ \sigma_i(x)\big)\big) = \big(x, \phi_{\mathsf{P}}\big(\sigma_i(x), r_g \circ \sigma_i(x)\big)\big) = (x,g)\,,\end{aligned}$$

the second of which follows from the fact that the action of G maps a level set of π to itself, with

$$\forall_{(p_1,p_2,g)\in \mathsf{P}^{\times 2}\times \mathrm{G}} \ : \ \phi_{\mathsf{P}}\big(p_1, r_g(p_2)\big) = \phi_{\mathsf{P}}(p_1,p_2)\cdot g\,.$$

They are also G-equivariant, as required, because[2]

$$\tau_i^{-1}\big((x,g) \triangleleft h\big) \equiv \tau_i^{-1}(x, g\cdot h) = r_{g\cdot h}\big(\sigma_i(x)\big) = r_h \circ r_g\big(\sigma_i(x)\big) = r_h\big(r_g \circ \sigma_i(x)\big) \equiv r_h \circ \tau_i^{-1}(x,g)\,.$$

The local trivialisations constructed above satisfy, at every point $x \in \mathcal{O}_{ij},\ i,j \in I$, the condition

$$\begin{aligned}\tau_i \circ \tau_j^{-1}(x,g) \quad &= \quad \tau_i\big(r_g \circ \sigma_j(x)\big) = \big(\pi \circ r_g \circ \sigma_j(x), \phi_{\mathsf{P}}\big(\sigma_i \circ \pi \circ r_g \circ \sigma_j(x), r_g \circ \sigma_j(x)\big)\big) \\ &= \quad \big(x, \phi_{\mathsf{P}}\big(\sigma_i(x), r_g \circ \sigma_j(x)\big)\big) = \big(x, \phi_{\mathsf{P}}\big(\sigma_i(x), \sigma_j(x)\big)\cdot g\big)\,,\end{aligned}$$

from which we read off the form of the structure maps

$$g_{ij} \ : \ \mathcal{O}_{ij} \longrightarrow \mathrm{G} \ : \ x \longmapsto \phi_{\mathsf{P}}\big(\sigma_i(x), \sigma_j(x)\big)\,.$$

This completes the identification of the postulated structure of a principal bundle with structure group G. □

Next, we discuss a useful criterion of trivialisibility of a principal bundle.

Proposition 5. There exists a one-to-one correspondence between smooth (local) sections of a principal bundle and its (local) trivialisations. In particular, a principal bundle is globally trivialisable iff it has a smooth global section.

[2] Recall that the inverse of a G-equivariant bijection is automatically G-equivariant.

<u>*Proof:*</u> To a local section $\sigma \;:\; \mathcal{O} \longrightarrow \pi_{\mathsf{P}}^{-1}(\mathcal{O}) \subset \mathsf{P},\ \mathcal{O} \in \mathscr{T}(B)$, we associate a local trivialisation

$$\tau_\sigma \;:\; \pi_{\mathsf{P}}^{-1}(\mathcal{O}) \longrightarrow \mathcal{O} \times \mathrm{G} \;:\; p \longmapsto \big(\pi_{\mathsf{P}}(p), \phi_{\mathsf{P}}\big(\sigma \circ \pi_{\mathsf{P}}(p), p\big)\big)$$

with all the desired properties, *i.e.*, (smoothly) invertible,

$$\tau_\sigma^{-1} \;:\; \mathcal{O} \times \mathrm{G} \longrightarrow \pi_{\mathsf{P}}^{-1}(\mathcal{O}) \;:\; (x, g) \longmapsto \sigma(x) \triangleleft g\,,$$

and G-equivariant,

$$\begin{aligned} \tau_\sigma(p \triangleleft g) \quad &\equiv \quad \big(\pi_{\mathsf{P}}(p \triangleleft g), \phi_{\mathsf{P}}\big(\sigma \circ \pi_{\mathsf{P}}(p \triangleleft g), p \triangleleft g\big)\big) = \big(\pi_{\mathsf{P}}(p), \phi_{\mathsf{P}}\big(\sigma \circ \pi_{\mathsf{P}}(p), p \triangleleft g\big)\big) \\ &= \quad \big(\pi_{\mathsf{P}}(p), \phi_{\mathsf{P}}\big(\sigma \circ \pi_{\mathsf{P}}(p), p\big) \cdot g\big) = \big(\pi_{\mathsf{P}}(p), \phi_{\mathsf{P}}\big(\sigma \circ \pi_{\mathsf{P}}(p), p\big)\big) \triangleleft g \equiv \tau_\sigma(p) \triangleleft g\,. \end{aligned}$$

Conversely, to a local trivialisation $\tau \;:\; \pi_{\mathsf{P}}^{-1}(\mathcal{O}) \xrightarrow{\cong} \mathcal{O} \times \mathrm{G}$, we associate a local section

$$\sigma_\tau \;:\; \mathcal{O} \longrightarrow \pi_{\mathsf{P}}^{-1}(\mathcal{O}) \;:\; x \longmapsto \tau^{-1}(x, e)\,.$$

The two assignments are mutually inverse. Indeed, on the one hand,

$$\forall_{x \in \mathcal{O}} \;:\; \sigma_{\tau_\sigma}(x) = \tau_\sigma^{-1}(x, e) = \sigma(x) \triangleleft e = \sigma(x)\,,$$

and on the other—

$$\forall_{p \in \pi_{\mathsf{P}}^{-1}(\mathcal{O})} \;:\; \tau_{\sigma_\tau}(p) = \big(\pi_{\mathsf{P}}(p), \phi_{\mathsf{P}}\big(\sigma_\tau \circ \pi_{\mathsf{P}}(p), p\big)\big) = \big(\pi_{\mathsf{P}}(p), \phi_{\mathsf{P}}\big(\tau^{-1}\big(\pi_{\mathsf{P}}(p), e\big), p\big)\big)\,,$$

but since

$$p \equiv \tau^{-1}\big(\pi_{\mathsf{P}}(p), e\big) \triangleleft \phi_{\mathsf{P}}\big(\tau^{-1}\big(\pi_{\mathsf{P}}(p), e\big), p\big)\,,$$

so that

$$\begin{aligned} \tau(p) \quad &= \quad \tau\big(\tau^{-1}\big(\pi_{\mathsf{P}}(p), e\big) \triangleleft \phi_{\mathsf{P}}\big(\tau^{-1}\big(\pi_{\mathsf{P}}(p), e\big), p\big)\big) = \tau \circ \tau^{-1}\big(\pi_{\mathsf{P}}(p), e\big) \triangleleft \phi_{\mathsf{P}}\big(\tau^{-1}\big(\pi_{\mathsf{P}}(p), e\big), p\big) \\ &= \quad \big(\pi_{\mathsf{P}}(p), e\big) \triangleleft \phi_{\mathsf{P}}\big(\tau^{-1}\big(\pi_{\mathsf{P}}(p), e\big), p\big) = \big(\pi_{\mathsf{P}}(p), \phi_{\mathsf{P}}\big(\tau^{-1}\big(\pi_{\mathsf{P}}(p), e\big), p\big)\big)\,, \end{aligned}$$

we arrive at

$$\tau_{\sigma_\tau}(p) = \tau(p)\,.$$

□

Remark 3. It deserves to be stressed that the statement in the above proposition distinguishes principal bundles among fibre bundles. In order to appreciate this constatation, it suffices to remark that every vector bundle admits a global section, to wit, the zero section, but not every one is globally trivialisable, see, *e.g.*, the tangent bundle of the unkempt $\mathbb{S}^2$.

We conclude our traverse across rudiments of the theory of principal bundles with a derivation of a convenient local presentation of those morphisms of principal bundles which cover the identity diffeomorphism $f = \mathrm{id}_B$ of the base and preserve the structure group.

Theorem 1 (The Clutching Theorem for Vertical Morphisms of Principal Bundles)**.** Every morphism $(\Phi, \mathrm{id}_B, \mathrm{id}_{\mathrm{G}})$ between principal bundles $\big((\mathsf{P}_\alpha, B, \mathrm{G}, \pi_{\mathsf{P}_\alpha}), r^\alpha\big),\ \alpha \in \{1, 2\}$ with the respective local trivialisations $\tau_i^\alpha \;:\; \pi_{\mathsf{P}_\alpha}^{-1}(O_i) \xrightarrow{\cong} O_i \times \mathrm{G}$ (associated with a common trivialising cover $\mathscr{O} = \{O_i\}_{i \in I}$) and transition maps $g_{ij}^\alpha \;:\; \mathcal{O}_{ij} \longrightarrow \mathrm{G}$), described by the commutative diagram

$$\begin{array}{ccc} \mathsf{P}_1 & \xrightarrow{\;\Phi\;} & \mathsf{P}_2 \\ {\scriptstyle \pi_{\mathsf{P}_1}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathsf{P}_2}} \\ B & \underset{\mathrm{id}_B}{=\!=\!=} & B \end{array}\,,$$

gives rise to a family $\{h_i\}_{i\in I}$ of smooth maps

$$h_i \;:\; O_i \longrightarrow \mathrm{G}\,, \quad i \in I$$

with the property

(6) $$\forall_{x\in\mathcal{O}_{ij}} \;:\; g^2_{ij}(x) = h_i(x)\cdot g^1_{ij}(x)\cdot h_j(x)^{-1}\,.$$

Conversely, every such family determines uniquely a morphism of the above kind.

Proof: Assume given a morphism $(\Phi, \mathrm{id}_B, \mathrm{id}_{\mathrm{G}})$. The map Φ is fixed uniquely by the values it takes on the flat unital sections $\sigma^1_i \equiv \sigma_{\tau^1_i}$, $i \in I$ induced from the local trivialisations of its domain along the lines of Prop. 5. Indeed, in virtue of the assumed G-equivariance of the trivialisations, every point in the fibre $\mathsf{P}_{1\,x}$ can be written as

$$\tau^{1\,-1}_i(x,g) = \sigma^1_i(x) \triangleleft g\,,$$

and so, by the assumed G-equivariance of Φ, we obtain

$$\Phi\big(\tau^{1\,-1}_i(x,g)\big) = \Phi\big(\sigma^1_i(x)\triangleleft g\big) = \Phi\big(\sigma^1_i(x)\big)\triangleleft g\,.$$

We now define the manifestly smooth maps

$$h_i := \mathrm{pr}_2\circ\tau^2_i\circ\Phi\circ\sigma^1_i \;:\; O_i \longrightarrow \mathrm{G}$$

and check that they obey the anticipated gluing law: On one hand,

$$\tau^{2\,-1}_j\big(x, h_j(x)\big) = \tau^{2\,-1}_i\big(x, g^2_{ij}(x)\cdot h_j(x)\big)\,,$$

and on the other—

$$\begin{aligned}\tau^{2\,-1}_j\big(x,h_j(x)\big) &= \Phi\circ\sigma^1_j(x) \equiv \Phi\big(\tau^{1\,-1}_j(x,e)\big) = \Phi\big(\tau^{1\,-1}_i\big(x,g^1_{ij}(x)\big)\big) = \Phi\big(\tau^{1\,-1}_i(x,e)\big)\triangleleft g^1_{ij}(x)\\ &\equiv \Phi\circ\sigma^1_i(x)\triangleleft g^1_{ij}(x) = \tau^{2\,-1}_j\big(x,h_i(x)\big)\triangleleft g^1_{ij}(x) = \tau^{2\,-1}_j\big(x,h_i(x)\cdot g^1_{ij}(x)\big)\,,\end{aligned}$$

which altogether reproduces the postulated relation owing to the bijectivity of the local trivialisations

Let, next, $((\mathsf{P}_\alpha, B, \mathrm{G}, \pi_{\mathsf{P}_\alpha}), r^\alpha)$, $\alpha \in \{1,2\}$ be principal bundles with the respective local trivialisations $\tau^\alpha_i \;:\; \pi^{-1}_{\mathsf{P}^\alpha}(O_i) \xrightarrow{\cong} O_i \times \mathrm{G}$ and transition maps $g^\alpha_{ij} \;:\; \mathcal{O}_{ij} \longrightarrow \mathrm{G}$. Given a family $h_i \;:\; O_i \longrightarrow \mathrm{G}$ of maps, as described in the statement of the theorem, we define locally smooth maps

$$\Phi_i \;:\; \pi^{-1}_{\mathsf{P}^1}(O_i) \longrightarrow \pi^{-1}_{\mathsf{P}^2}(O_i) \;:\; \tau^{1\,-1}_i(x,g) \longmapsto \tau^{2\,-1}_i\big(x, h_i(x)\cdot g\big)\,.$$

These satisfy, at every point $(x,g) \in \mathcal{O}_{ij}\times\mathrm{G}$, the identities

$$\begin{aligned}\Phi_j\big(\tau^{1\,-1}_i(x,g)\big) &= \Phi_j\big(\tau^{1\,-1}_j\big(x, g^1_{ji}(x)\cdot g\big)\big) = \tau^{2\,-1}_j\big(x, h_j(x)\cdot g^1_{ji}(x)\cdot g\big)\\ &= \tau^{2\,-1}_i\big(x, g^2_{ij}(x)\cdot h_j(x)\cdot g^1_{ji}(x)\cdot g\big) = \tau^{2\,-1}_i\big(x, h_i(x)\cdot g\big) \equiv \Phi_i\big(\tau^{1\,-1}_i(x,g)\big)\,.\end{aligned}$$

This implies that the Φ_i ar erestrictions of a globally smooth map

$$\Phi \;:\; \mathsf{P}^1 \longrightarrow \mathsf{P}^2\,, \qquad \Phi\restriction_{\pi^{-1}_{\mathsf{P}^1}(O_i)} = \Phi_i\,,$$

manifestly fibre-preserving and G-equivariant (owing to commutativity of the left and right regular actions of G on itself, and the assumed G-equivariance of the trivialisations). □

Our lightning *tour d'horizon* of the theory of principal bundles is crowned with the following proposition—a straightforward consequence of (the constructive proof of) the above theorem.

Proposition 6. Let $\mathbf{Bun}_{\mathrm{G}}(B)$ be the category of principal bundles with base B and structure group G. Its subcategory

$$\mathbf{Bun}_{\mathrm{G}}(B)/B$$

composed of all objects of $\mathbf{Bun}_{\mathrm{G}}(B)$ and those morphisms between them which cover the identity diffeomorphism $f = \mathrm{id}_B$ is a groupoid.

Proof: In the light of Thm. 1, it is sufficient to carry out a proof in the local picture, in which an arbitrary morphism $\Phi : \mathsf{P}_1 \longrightarrow \mathsf{P}_2$ covering $f = \mathrm{id}_B$ is represented by a family of smooth maps $h_i : O_i \longrightarrow \mathrm{G},\ i \in I$. By the same theorem, the corresponding family $\{\widetilde{h}_i := \mathrm{Inv} \circ h_i\}_{i \in I}$ determines a morphism $\mathsf{P}_2 \longrightarrow \mathsf{P}_1$, which inverts Φ by construction. □

3. The Ehresmann–Atiyah groupoid of P, and the Ehresmann bibundle

Weinstein's clever case for Lie-groupoidal models of symmetry suggests that whatever steps we take towards or within a field theory with a *group*-like symmetry gauged, we should always seek to retrace these steps in a formulation amenable to a direct generalisation to the Lie-groupoidal setting. The obvious point of departure of any such rephrasal is the standard 'groupoidification' of a Lie group G, in which the latter is viewed as a Lie groupoid $\mathrm{G} \Longrightarrow \bullet$ over a singleton $\bullet$.

In the present setting, in which we are dealing with an 'auxiliary' object $(\mathsf{P}, B, \mathrm{G}, \pi_{\mathsf{P}})$, there are two basic structures that are to be reformulated, to wit,

- the defining *global* action r of the symmetry agent G on the space of local gauges P, and
- base-dependent automorphisms of the latter space P, among which those preserving fibres (*i.e.*, covering the identity diffeomorphism of the base) will ultimately model autoequivalences of a physical theory with the global symmetry G rendered local.

Being base-independent and fibre-preserving, the former are simpler to grasp, and so we begin our reformulation from them. The very general idea is to replace the original (right) action of the symmetry *group* G with an action of the symmetry *groupoid* $\mathrm{G} \Longrightarrow \bullet$, and to put P, accordingly, in the rôle of a (right) module of the latter Lie groupoid, as introduced in Def. IV.75. In what follows, we denote the symmetry groupoid as $\vec{\mathrm{G}}$ for the sake of brevity.

Proposition 7. Every principal bundle $(\mathsf{P}, B, \mathrm{G}, \pi_{\mathsf{P}}), r)$ carries a canonical structure of a (right) $\vec{\mathrm{G}}$-module, with momentum

$$\mu_{\mathsf{P}} \equiv \bullet \ :\ \mathsf{P} \dashrightarrow \bullet$$

and action

$$\varrho_{\mathsf{P}} \equiv r \ :\ \mathsf{P}\,{}_{\mu_{\mathsf{P}}}\!\times_t \mathrm{Mor}(\vec{\mathrm{G}}) \equiv \mathsf{P} \times \mathrm{G} \longrightarrow \mathsf{P}$$

The action preserves the fibres of the base projecion π_{P} and the composite map

$$(\mathrm{pr}_1, \varrho_{\mathsf{P}}) \ :\ \mathsf{P}\,{}_{\mu_{\mathsf{P}}}\!\times_t \mathrm{Mor}(\vec{\mathrm{G}}) \longrightarrow \mathsf{P}\,{}_{\pi_{\mathsf{P}}}\!\times_{\pi_{\mathsf{P}}} \mathsf{P}$$

is a diffeomorphism.

Proof: A trivial translation of the formerly established properties of P. □

The above exercise leads us to the following abstraction:

Definition 4. [Moe91, Sec. 1.2] Let $\mathscr{G} \Longrightarrow M$ be a Lie groupoid. A **right principal $\mathscr{G}$-bundle** is a quintuple $(\check{\mathrm{P}}, \Sigma, \pi_{\check{\mathrm{P}}}, \mu, \varrho)$ composed of a pair of smooth manifolds:

- the **total space** $\check{\mathrm{P}}$ of the bundle;
- its **base** Σ,

and a triple of smooth maps:

- a surjective submersion $\pi_{\check{\mathrm{P}}} : \check{\mathrm{P}} \to \Sigma$, termed the **bundle projection**;
- the **moment map** $\mu : \check{\mathrm{P}} \to M$;
- the **action (map)** $\varrho : \check{\mathrm{P}}\,{}_\mu\!\times_t \mathscr{G} \to \check{\mathrm{P}}$

with the following properties:

(PGr1) $(\check{\mathrm{P}}, \mu, \varrho)$ is a right $\mathscr{G}$-module space;

(PGr2) $\pi_{\check{\mathrm{P}}}$ is $\mathscr{G}$-invariant in the sense made precise by the following commutative diagram (in which pr_1 is the canonical projection)

$$\begin{array}{ccc} \check{\mathrm{P}}\,{}_\mu\!\times_t\mathscr{G} & \xrightarrow{\ \varrho\ } & \check{\mathrm{P}} \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \pi_{\check{\mathrm{P}}}} \\ \check{\mathrm{P}} & \xrightarrow[\pi_{\check{\mathrm{P}}}]{} & \Sigma \end{array} \;;$$

(PGr3) the map

$$(\mathrm{pr}_1,\varrho)\ :\ \check{\mathrm{P}}\,{}_\mu\!\times_t\mathscr{G} \longrightarrow \check{\mathrm{P}}\,{}_{\pi_{\check{\mathrm{P}}}}\!\times_{\pi_{\check{\mathrm{P}}}}\check{\mathrm{P}} \equiv \check{\mathrm{P}}^{[2]}\,,\ (p,g) \longmapsto (p, p \blacktriangleleft g)$$

is a diffeomorphism, so that $\mathscr{G}$ acts freely and transitively on $\pi_{\check{\mathrm{P}}}$-fibres. The smooth inverse of (pr_1,ϱ) takes the form

$$(\mathrm{pr}_1,\varrho)^{-1} =: (\mathrm{pr}_1,\phi_{\check{\mathrm{P}}})\,, \qquad\qquad \phi_{\check{\mathrm{P}}}\ :\ \check{\mathrm{P}}^{[2]} \longrightarrow \mathscr{G}$$

and $\phi_{\check{\mathrm{P}}}$ is called the **division map**.

The Lie groupoid $\mathscr{G}$ is termed the **structure groupoid** of $\check{\mathrm{P}}$.

We shall represent a right principal $\mathscr{G}$-bundle by the simplified (non-commutative) diagram

$$(7)\qquad \begin{array}{ccccc} \check{\mathrm{P}} & & & & \mathrm{Mor}\mathbf{Gr} \\ {\scriptstyle \pi_{\check{\mathrm{P}}}}\downarrow & \searrow^{\mu} & & \swarrow\!\!\swarrow & \\ \Sigma & & \mathrm{Ob}\mathbf{Gr} & & \end{array}\ ,$$

in which the additional structure is implicit.

Let $(\check{\mathrm{P}}_A,\Sigma,\pi_{\check{\mathrm{P}}_A},\mu_A,\varrho^A)$, $A\in\{1,2\}$ be a pair of right principal $\mathscr{G}$-bundles over a common base Σ. A **morphism**[3] between the two bundles is a morphism $(\Theta,\mathrm{Id}_{\mathbf{Gr}})$ between the corresponding right $\mathscr{G}$-modules $(\check{\mathrm{P}}_A,\mu_A,\varrho^A)$ which maps $\pi_{\check{\mathrm{P}}_1}$-fibres to $\pi_{\check{\mathrm{P}}_2}$-fibres.

Left principal $\mathscr{G}$-bundles $(\check{\mathrm{P}},\Sigma,\pi_{\check{\mathrm{P}}},\mu,\lambda)$ (and morphisms between them) are defined analogously. The corresponding diagrams take the self-explanatory form

$$\begin{array}{ccccc} \mathrm{Mor}\mathbf{Gr} & & & & \check{\mathrm{P}} \\ & \searrow\!\!\searrow & & \swarrow^{\mu} & \downarrow{\scriptstyle \pi_{\check{\mathrm{P}}}} \\ & & \mathrm{Ob}\mathbf{Gr} & & \Sigma \end{array}\ ,$$

Customarily, principal $\mathscr{G}$-bundles are taken to be right $\mathscr{G}$-modules, and so whenever the term is used without a qualifier, it is to be understood that we are dealing with a right principal $\mathscr{G}$-bundle.

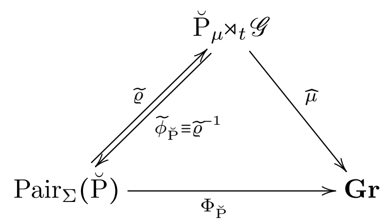
Remark 4. It is worth noting that the definition of a principal $\mathscr{G}$-bundle can be viewed as a structural relation between three Lie groupoids. Indeed, we may rephrase it as a statement of existence, for a given surjective submersion $\pi_{\check{\mathrm{P}}}:\check{\mathrm{P}}\to\Sigma$ and a (smooth) map $\mu:\check{\mathrm{P}}\to M$, of

- an action Lie groupoid $\check{\mathrm{P}}\,{}_\mu\!\rtimes_t\mathscr{G}$ with $\check{\mathrm{P}}$ as the object manifold and $\check{\mathrm{P}}\,{}_\mu\!\times_t\mathscr{G}$ as the arrow manifold, with pr_1 as the source map, a smooth map $\varrho:\check{\mathrm{P}}\,{}_\mu\!\times_t\mathscr{G}\to\check{\mathrm{P}}$ as the target map, $(\mathrm{id}_{\check{\mathrm{P}}},\mathrm{Id}\circ\mu)$ as the unit map, $(\varrho,\mathrm{Inv}\circ\mathrm{pr}_2)$ as the inverse map, and $(\varrho(p,g),h).(p,g) = (p,g.h)$ as the multiplication map,
- a Lie-groupoid morphism $\Phi_{\check{\mathrm{P}}}:\mathrm{Pair}_\Sigma(\check{\mathrm{P}})\to\mathbf{Gr}$ with μ as the object component and a smooth map $\phi_{\check{\mathrm{P}}}:\check{\mathrm{P}}\times_\Sigma\check{\mathrm{P}}\to\mathscr{G}$ as the morphism component,

such that the Lie-groupoid morphism $\widetilde{\varrho}:\check{\mathrm{P}}\,{}_\mu\!\rtimes_t\mathscr{G}\to\mathrm{Pair}_\Sigma(\check{\mathrm{P}})$, with the object component $\mathrm{id}_{\check{\mathrm{P}}}$ and the morphism component (pr_1,ϱ), is invertible, with the inverse $\widetilde{\phi}_{\check{\mathrm{P}}}$ given by ($\mathrm{id}_{\check{\mathrm{P}}}$ on objects, and)

[3] In Ref. [MM03, Sec. 5.7], these morphisms were termed "equivariant maps".

$(\mathrm{pr}_1, \phi_{\breve{\mathrm{P}}})$ on morphisms. Thus, the entire information on $\breve{\mathrm{P}}$ is neatly encoded in the following commutative diagram in the category of Lie groupoids:



,

in which the Lie-groupoid morphism $\widehat{\mu}$ has μ as the object component and pr_2 as the morphism component.

Proposition 8. Let $(\breve{\mathrm{P}}, \Sigma, \pi_{\breve{\mathrm{P}}}, \mu, \varrho)$ be a principal $\mathscr{G}$-bundle. The orbispace

$$\breve{\mathrm{P}}//\mathscr{G} = \{\ p \blacktriangleleft \mathscr{G} \mid p \in \breve{\mathrm{P}}\ \}$$

is a smooth manifold, canonically diffeomorphic with Σ,

$$\breve{\mathrm{P}}//\mathscr{G} \cong \Sigma\,.$$

Proof: The orbispace is a quotient of $\breve{\mathrm{P}}$ by an equivalence relation on $\breve{\mathrm{P}} \ni p_1, p_2$ defined as

$$p_1 \sim p_2 \qquad \Longleftrightarrow \qquad \exists_{g \in \mathscr{G}}\ :\ p_2 = p_1 \blacktriangleleft g\,.$$

Its graph takes the form

$$(\mathrm{pr}_1, \varrho)\big(\breve{\mathrm{P}}_\mu{\times}_t\mathscr{G}\big) = \breve{\mathrm{P}}^{[2]} \subset \breve{\mathrm{P}} \times \breve{\mathrm{P}}\,,$$

and so it is an embedded submanifold in $\breve{\mathrm{P}}{\times}\breve{\mathrm{P}}$ (as a fibred square of $\breve{\mathrm{P}}$). Furthermore, since (pr_1, ϱ) is closed (as a homeomorphism), it is also closed. Hence, by Godement's Criterion (Thm. I.21.), the orbispace is smooth and the canonical projection

$$\breve{\pi}_\sim\ :\ \breve{\mathrm{P}} \longrightarrow \breve{\mathrm{P}}//\mathscr{G}\ :\ p \longmapsto p \blacktriangleleft \mathscr{G}$$

is a surjective submersion.

The proof of the second part of the proposition develops along the lines of the reasoning presented in Rem. 2: We start with the manifestly smooth map

$$\iota\ :\ \breve{\mathrm{P}}//\mathscr{G} \longrightarrow \Sigma\ :\ p \blacktriangleleft \mathscr{G} \longmapsto \pi_{\breve{\mathrm{P}}}(p)\,,$$

and look for its smooth inverse. To this end, we consider an open cover $\mathcal{O}_\Sigma = \{O_i\}_{i \in I}$ of Σ whose elements support the respective smooth sections $\breve{\sigma}_i\ :\ O_i \longrightarrow \breve{\mathrm{P}}$ of the surjective submersion $\pi_{\breve{\mathrm{P}}}$.́These we employ in the definition of smooth local maps

$$\jmath_i := \breve{\pi}_\sim \circ \breve{\sigma}_i\ :\ O_i \longrightarrow \breve{\mathrm{P}}//\mathscr{G}\,,$$

subsequently proven to glue up to a globally smooth map

$$\jmath\ :\ \Sigma \longrightarrow \breve{\mathrm{P}}//\mathscr{G}\,, \qquad \jmath\restriction_{O_i} \equiv \jmath_i\,.$$

Indeed, for any $x \in O_{ij}$, we obtain the equality

$$\jmath_j(x) \equiv \breve{\pi}_\sim\big(\breve{\sigma}_j(x)\big) = \breve{\pi}_\sim\big(\breve{\sigma}_i(x) \blacktriangleleft \phi_{\breve{\mathrm{P}}}\big(\sigma_i(x), \sigma_j(x)\big)\big) = \breve{\pi}_\sim\big(\breve{\sigma}_i(x)\big) \equiv \jmath_i(x)\,.$$

Last, we check the identities

$$\iota \circ \jmath = \mathrm{id}_\Sigma\,, \qquad \jmath \circ \iota = \mathrm{id}_{\breve{\mathrm{P}}//\mathscr{G}}$$

through the following direct computations:

$$\big(\iota \circ \jmath\big)(x) \equiv \iota\big(\breve{\sigma}_i(x) \blacktriangleleft \mathscr{G}\big) \equiv \pi_{\breve{\mathrm{P}}}\big(\breve{\sigma}_i(x)\big) = x\,,$$

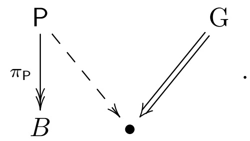
in which $x \in O_i$, and

$$\big(\jmath \circ \iota\big)\big(p \blacktriangleleft \mathscr{G}\big) \equiv \jmath\big(\pi_{\breve{\mathrm{P}}}(p)\big) \equiv \breve{\sigma}_i\big(\pi_{\breve{\mathrm{P}}}(p)\big) \blacktriangleleft \mathscr{G} = \big(p \blacktriangleleft \phi_{\breve{\mathrm{P}}}\big(p, \breve{\sigma}_i\big(\pi_{\breve{\mathrm{P}}}(p)\big)\big)\big) \blacktriangleleft \mathscr{G} = p \blacktriangleleft \mathscr{G}\,,$$

in which $p \in \pi_{\breve{\mathrm{P}}}^{-1}(O_i)$. □

Thus, just to summarise, the defining action of the structure group G on (the total space of) the principal bundle P is neatly captured by the diagram

(8)



Passing to B-dependent automorphisms of P, we recall Ex. II.36. in conjunction with Ex. IV.80. to conclude that we should be looking at bisections of the pair groupoid $\mathrm{Pair}(\mathsf{P})$ covering those of the base pair groupoid $\mathrm{Pair}(B)$, not any bisections, though, but those compatible with the defining G-action on P, *i.e.*, G-equivariant ones, where the relevant action on the arrow manifold $\mathsf{P}\times\mathsf{P}$ of the former pair groupoid is the diagonal one. Speaking in terms of the newly introduced structure of a right $\vec{\mathrm{G}}$-module on P, we readily see that we are after a refinement of the canonical left $\mathrm{Pair}(\mathsf{P})$-module structure on P, which commutes with the aforementioned right $\vec{\mathrm{G}}$-module structure. In order to rigorously pin down the refinement, we invoke

Proposition 9. Let $(M_\alpha,\lambda_\alpha)$, $\alpha\in\{1,2\}$ be G-manifolds equipped with the respective free and proper G-actions λ_α. Every G-equivariant map $f_{12}\in C^\infty_{\mathrm{G}}(M_1,M_2)$ canonically induces a smooth map

$$[f_{12}]\ :\ M_1//\mathrm{G}\longrightarrow M_2//\mathrm{G}\ :\ [m]_{\sim_1}\longmapsto[f_{12}(m)]_{\sim_2}\,,$$

written in terms of the G-orbits $[m]_{\sim_1}\equiv\mathrm{G}\rhd_{(1)}m$ and $[f_{12}(m)]_{\sim_2}\equiv\mathrm{G}\rhd_{(2)}f_{12}(m)$ for $m\in M_1$.

Maps thus induced compose in a 'functorial' manner: Let (M_3,λ_3) be another G-manifold endowed with a free and proper G-action λ_3, and let $f_{23}\in C^\infty_{\mathrm{G}}(M_1,M_2)$. Then,

$$[f_{23}\circ f_{12}]=[f_{23}]\circ[f_{12}]\,.$$

Moreover, we have

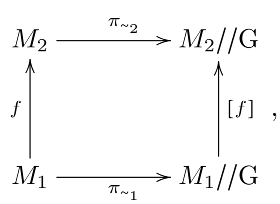
$$[\mathrm{id}_{M_1}]=\mathrm{id}_{M_1//\mathrm{G}}\,.$$

<u>*Proof:*</u> The map $[f]$ is well-defined as

$$[f(g\rhd_{(1)}m)]_{\sim_2}=[g\rhd_{(2)}f(m)]_{\sim_2}\equiv[f(m)]_{\sim_2}\,,$$

owing to the assumed G-equivariance of f.

Its smoothness follows from Thm. V.2., referred to the commutative diagram



in which

- $\pi_{\sim_\alpha}\ :\ M_\alpha\longrightarrow M_\alpha//\mathrm{G}\ :\ m_\alpha\longmapsto\mathrm{G}\rhd_{(\alpha)}m_\alpha$ are the canonical quotient maps, both smooth in virtue of Thm. I.21.;
- $\pi_{\sim_1}$ is a surjective submersion by the same Thm. I.21.;
- $\pi_{\sim_2}\circ f$ is smooth as a superposition of smooth maps.

The last part of the proposition follows straightforwardly from the definition of the induced map. □

Using the above result (and invoking Eq. (4)), we may next associate a smooth map $[\beta]\circ\iota^{-1}\ :\ B\cong\mathsf{P}//\mathrm{G}\longrightarrow(\mathsf{P}\times\mathsf{P})//\mathrm{G}$. Here, the existence of the smooth quotient $(\mathsf{P}\times\mathsf{P})//\mathrm{G}$ is ensured by the properness of the G-action on $\mathsf{P}\times\mathsf{P}$, a property inherited from the same property of r. In the next step, we note that all structure maps of $\mathrm{Pair}(\mathsf{P})$ are trivially G-equivariant, and so they also descend to the quotient $(\mathsf{P}\times\mathsf{P},\mathsf{P})//\mathrm{G} \equiv ((\mathsf{P}\times\mathsf{P})//\mathrm{G},\mathsf{P}//\mathrm{G})$. Clearly, then, dividing out the G-action gives rise to a new species of Lie groupoid: $(\mathsf{P}\times\mathsf{P})//\mathrm{G} \rightrightarrows B$ (upon identifying $\Sigma \equiv \mathsf{P}//\mathrm{G}$), with structure maps $[s]$, $[t]$, $[\mathrm{Id}]$, $[\mathrm{Inv}]$ and $[.]$. Moreover, we readily see that a (G-equivariant) bisection $\beta \equiv (\Phi,\mathrm{id}_{\mathsf{P}}) \in \mathrm{Bisec}_{\mathrm{G}}(\mathrm{Pair}(\mathsf{P}))$ necessarily satisfies

$$[s]\circ[\beta] = [s\circ\beta] = [\mathrm{id}_{\mathsf{P}}] = \mathrm{id}_{\mathsf{P}//\mathrm{G}}$$

and

$$[t]\circ[\beta] = [t\circ\beta] \in [\mathrm{Diff}_{\mathrm{G}}(\mathsf{P})] \subset \mathrm{Diff}(\Sigma)\,.$$

Thus, G-equivariant bisections of $\mathrm{Pair}(\mathsf{P})$ (which automatically cover diffeomorphisms of B in virtue of Prop. 9) induce bisections of $(\mathsf{P}\times\mathsf{P})//\mathrm{G} \rightrightarrows B$. We shall have more to say about the correpondence between the two groups of bisection later. Meanwhile, let us agree that the claim for fame of our new Lie groupoid has been defended convincingly, and give it a name:

Definition 5. Let $((\mathsf{P},B,\mathrm{G},\pi_{\mathsf{P}}),r)$ be a principal bundle. The **Ehresmann–Atiyah groupoid** (aka the **gauge groupoid**) of P is the Lie groupoid described by the diagram

$$\mathrm{At}(\mathsf{P})\times_B\mathrm{At}(\mathsf{P}) \xrightarrow{\ \mathsf{M}\ } \mathrm{At}(\mathsf{P}) \xrightarrow{\ \mathsf{J}\ } \mathrm{At}(\mathsf{P}) \underset{\mathsf{T}}{\overset{\mathsf{S}}{\rightrightarrows}} B\,, \qquad \mathsf{I} : B \to \mathrm{At}(\mathsf{P})$$

that is—with arrow manifold

$$\mathrm{At}(\mathsf{P}) \coloneqq (\mathsf{P}\times\mathsf{P})//\mathrm{G}$$

and structure maps

$$\mathsf{S} : \mathrm{At}(\mathsf{P}) \longrightarrow B : [(p_2,p_1)] \longmapsto \pi_{\mathsf{P}}(p_1)\,, \qquad \mathsf{T} : \mathrm{At}(\mathsf{P}) \longrightarrow B : [(p_2,p_1)] \longmapsto \pi_{\mathsf{P}}(p_2)\,,$$

$$\mathsf{J} : \mathrm{At}(\mathsf{P}) \longrightarrow \mathrm{At}(\mathsf{P}) : [(p_2,p_1)] \longmapsto [(p_1,p_2)]\,,$$

$$\mathsf{M} : \mathrm{At}(\mathsf{P})\times_B\mathrm{At}(\mathsf{P}) \longrightarrow \mathrm{At}(\mathsf{P}) : \big([(p_3,p_2)],[(p_2,p_1)]\big) \longmapsto [(p_3,p_1)]\,,$$

and

$$\mathsf{I} : B \longrightarrow \mathrm{At}(\mathsf{P}) : \sigma \longmapsto [(\tau_i^{-1}(\sigma,e),\tau_i^{-1}(\sigma,e)]\,,$$

where $\sigma \in O_i$ in the last definition.

Remark 5. The above object made its first appearance in Ehresmann's seminal papers [Ehr50, Ehr52] on principal bundles and theoir morphisms—there, it was called *groupoïde associé*. It reappeared in Atiyah's later work [Ati57] on connections.

The naturality of the object introduced above from the point of view of our physically motivated considerations is emphasised in

Proposition 10. The Ehresmann–Atiyah groupoid of a principal bundle $((\mathsf{P},B,\mathrm{G},\pi_{\mathsf{P}}),r)$ carries a canonical structure of a fibre-bundle object in the category of Lie groupoids, with total space $\mathrm{At}(\mathsf{P}) \rightrightarrows B$, base $\mathrm{Pair}(B)$ and typical fibre $\vec{\mathrm{G}}$. The base projection is given by the Lie-groupoid morphism π with morphism component $(\mathsf{T},\mathsf{S}) : \mathrm{At}(\mathsf{P}) \longrightarrow B\times B$ and object component id_B, all captured by the diagram

$$\begin{array}{ccc} \vec{\mathrm{G}} & \hookrightarrow & \mathrm{At}(\mathsf{P}) \\ & & \big\downarrow \pi \\ & & \mathrm{Pair}(B) \end{array}\,.$$

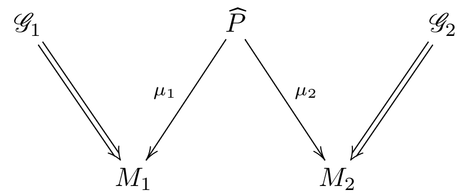
<u>*Proof:*</u> Obvious. □

Returning to the original challenge of encoding bundle automorphisms of P in a Lie-groupoidal structure, we first introduce a useful abstraction.

Definition 6. Let $\mathscr{G}_A \rightrightarrows M_A,\ A \in \{1,2\}$ be Lie groupoids. A $(\mathscr{G}_1, \mathscr{G}_2)$**-bibundle** is a manifold $\widehat{P}$ which carries the structure of a left $\mathscr{G}_1$-module $(\widehat{P}, \mu_1, \lambda_1 \equiv \blacktriangleright)$ and that of a right $\mathscr{G}_2$-module $(\widehat{P}, \mu_2, \rho_2 \equiv \blacktriangleleft)$, such that the two actions commute and each moment map is invariant with respect to the other action, *i.e.*, we have, for all $(g_1, p, g_2) \in \mathscr{G}_1\,{}_{s_1}\!\times_{\mu_1} \widehat{P}\,{}_{\mu_2}\!\times_{t_2} \mathscr{G}_2$,

- $(g_1 \blacktriangleright p) \blacktriangleleft g_2 = g_1 \blacktriangleright (p \blacktriangleleft g_2)$;
- $\mu_1(p \blacktriangleleft g_2) = \mu_1(p)$;
- $\mu_2(g_1 \blacktriangleright p) = \mu_2(p)$.

Whenever $(\widehat{P}, M_2, \mu_2, \mu_1, \lambda_1)$ is a (left) principal $\mathscr{G}_1$-bundle (with base $M_2 \cong \widehat{P}/\mathscr{G}_1$, by Godement's Criterion), and $(\widehat{P}, M_1, \mu_1, \mu_2, \rho_2)$ is a (right) principal $\mathscr{G}_2$-bundle (with base $M_1 \cong \widehat{P}/\mathscr{G}_2$, by the same 'criterion), we call $\widehat{P}$ a (**bi**)**principal** $(\mathscr{G}_1, \mathscr{G}_2)$**-bibundle**, and depict it by the following W-diagram:

(9) 

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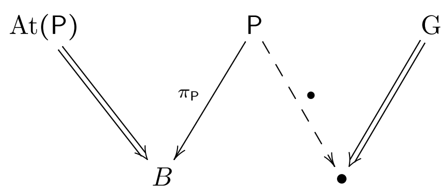
Our hitheto findings are now neatly summarised in

Theorem 2. Every principal bundle $\big((\mathsf{P}, B, \mathrm{G}, \pi_{\mathsf{P}}), r\big)$ carries a canonical structure of a biprincipal $(\mathrm{At}(\mathsf{P}), \mathrm{G})$-bibundle captured by the W-diagram



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<u>*Proof:*</u> The only thing which remains to be proven is the existence of a structure of a left principal $\mathrm{At}(\mathsf{P})$-bundle on P, with the corresponding (left) action commuting with that of the groupoid $\vec{\mathrm{G}}$ from the right, encoded by Diag. 8. The action in question is defined straighforwardly as

$$\lambda_{\mathsf{P}} \;:\; \mathrm{At}(\mathsf{P})_{\mathsf{S}}\!\times_{\pi_{\mathsf{P}}}\mathsf{P} \longrightarrow \mathsf{P} \;:\; \big([(p_2, p_1)], p_1\big) \longmapsto p_2\,,$$

and so smooth by quasi-universality of the submersion $\pi_\sim \;:\; \mathsf{P} \times \mathsf{P} \longrightarrow \mathrm{At}(\mathsf{P}) \;:\; (p_2, p_1) \longrightarrow [(p_2, p_1)]$ (itself smooth due to the properness of the G-action divided out, as discussed earlier). It trivially preserves the single $\bullet$-fibre P. Equally trivially, we establish the smooth inverse of the map

$$(\lambda_{\mathsf{P}}, \mathrm{pr}_2) \;:\; \mathrm{At}(\mathsf{P})_{\mathsf{S}}\!\times_{\pi_{\mathsf{P}}}\mathsf{P} \longrightarrow \mathsf{P}_\bullet\!\times_\bullet \mathsf{P} \equiv \mathsf{P} \times \mathsf{P} \;:\; \big([(p_2, p_1)], p_1\big) \longmapsto (p_2, p_1)$$

in the form

$$(\lambda_{\mathsf{P}}, \mathrm{pr}_2)^{-1} \;:\; \mathsf{P} \times \mathsf{P} \longrightarrow \mathrm{At}(\mathsf{P})_{\mathsf{S}}\!\times_{\pi_{\mathsf{P}}}\mathsf{P} \;:\; (p_2, p_1) \longmapsto \big([(p_2, p_1)], p_1\big)\,,$$

from which we read off the (smooth) division map $\phi_{\mathsf{P}} = \pi_\sim$. □

Remark 6. The strucure of the biprincipal $(\mathrm{At}(\mathsf{P}), \mathrm{G})$-bibundle on P was first contemplated by none other than Charles Ehresmann in [Ehr50, Ehr52].

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