A BUNDLE OF FACTS ABOUT BUNDLES:
FIBRATIONS, TRIVIALISATIONS,
SECTIONS, SHEAVES & RECONSTRUCTIONS
(DDD '24/25 V [RRS])

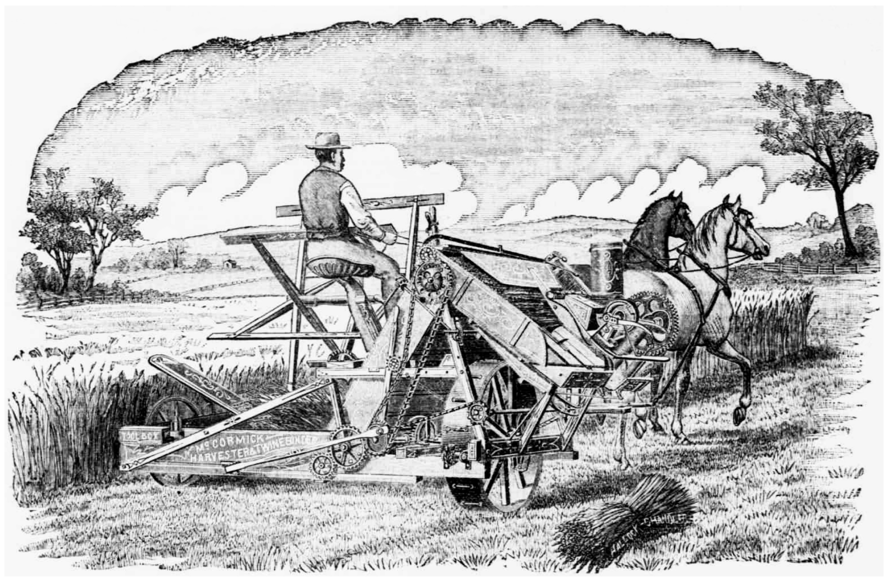


FIGURE 1. McCormick's sheaf-binder, an adaptation of John F. Appleby's invention of 1858, which revolutionised the harvesting process in Northern America at the end of the XIX century by substantially increasing its fficiency (an ad from 1884).

CONTENTS

1. General structures

In what follows, we review a (smooth) model of the configuration space of a field theory, combining the constituent elements of the phenomenological concept: a spacetime Σ and a space of internal degrees of freedom F (a vector space, an algebra module, a group torsor, a manifold endowed with a group action *etc.*) attached to each of its points into a single geometric object, which captures the underlying ideas:

- of an identification, over each point of the spacetime, of the globally fixed *type* F of a physical species inhabiting Σ, leading to a local model $O \times F$ of the configuration space over the laboratory floor $O \in \mathscr{T}(\Sigma)$ ($\mathscr{T}(\Sigma)$ is a topology of Σ);

- of an invertible (and smoothly so, in the present smooth paradigm) transcription law for any pair $O_i \times F,\ i \in \{1,2\}\}$ of local (laboratory) models of the configuration space over their nonempty spacetime intersection $O_1 \cap O_2 \neq \varnothing$, ensuring objectivisation of local measurements of phenomena involving the physical species modelled by that configuration space;
- of a local field profile $\phi \in C^\infty(O,F)$, representing an assignment of the internal degrees of freedom to each point in O.

The model is provided by

Definition 1. A **fibre bundle** is a quadruple

$$(E,B,F,\pi_E)$$

composed of smooth manifolds

- E, termed the **total space**;
- B, termed the **base**;
- F, termed the **typical fibre**,

and a smooth surjection

$$\pi_E\ :\ E \longrightarrow B\,,$$

termed the **base projection**, for which there exists an open cover $\mathcal{O}_B = \{O_i\}_{i\in I}$ of the base B together with the corresponding family of diffeomorphisms

$$\tau_i\ :\ \pi_E^{-1}(O_i) \xrightarrow{\cong} O_i \times F\,,$$

termed **local trivialisations**. The latter are assumed to compose the following commutative diagrams (for every $i \in I$)

$$\begin{array}{ccc} \pi_E^{-1}(O_i) & \xrightarrow{\tau_i} & O_i \times F \\ & {\scriptstyle \pi_E} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\ & O_i & \end{array}\,,$$

and give rise to maps

$$g_{ij}\ :\ O_{ij} \longrightarrow \mathrm{Aut}(F)$$

determined, for every pair $(i,j) \in I^{\times 2}$ such that $O_{ij} \equiv O_i \cap O_j \neq \varnothing$ and some subgroup $\mathrm{Aut}(F) \subseteq \mathrm{Diff}(F,F)$ of automorphisms[1] of the typical fibre, by the composition of diffeomorphisms

$$\tau_{ij} := \tau_i \circ \tau_j^{-1}\restriction_{O_{ij}\times F}\ :\ O_{ij} \times F \circlearrowleft\ :\ (x,f) \longmapsto \big(x, g_{ij}(x)(f)\big)\,,$$

with $O_{ij} \times F \longrightarrow F\ :\ (x,f) \longmapsto g_{ij}(x)(f)$ smooth. A cover with the above property is termed **trivialising** for E, and the g_{ij} are termed **transition maps**. Their common codomain $\mathrm{Aut}(F)$ is called the **structure group** of E.

We shall represent a fibre bundle by the following diagram

$$\begin{array}{ccc} F & \rightsquigarrow & E \\ & & \downarrow {\scriptstyle \pi_E} \\ & & B \end{array}\,,$$

in which the special ontological status (non-canonicity) of the embedding of the typical fibre in the total space is signalled by the wiggly arrow.

[1] A choice of a *proper* subgroup $\mathrm{Aut}(F) \subsetneq \mathrm{Diff}(F,F)$ is often eployed to encode the existence of an extra structure on the typical fibre, preserved by $\mathrm{Aut}(F)$. Examples of such structures include: a linear structure ($\mathrm{Aut}(F) \subset \mathrm{GL}(F)$), an action of a group G ($\mathrm{Aut}(F) \subset \mathrm{Diff}_{\mathrm{G}}(F)$), a metric structure ($\mathrm{Aut}(F) \subset \mathrm{Isom}(F,g)$), *etc.*

The preimage of a point $x \in B$ along the base projection,

$$\pi_E^{-1}(\{x\}) \equiv E_x$$

is termed the **fibre** of E **over** x.

A **subbundle** of a given bundle (E, B, F, π_E) is a fibre bundle (S, B, X, π_S) with a total space S embedded in E and $\pi_S = \pi_E \restriction_S$.

A **morphism** between fibre bundles $(E_\alpha, B_\alpha, F_\alpha, \pi_{E_\alpha})$, $\alpha \in \{1, 2\}$ (aka a **bundle map**) is a pair (Φ, f) of smooth maps which render the following diagram commutative:

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \downarrow{\scriptstyle \pi_{E_1}} & & \downarrow{\scriptstyle \pi_{E_2}} \\ B_1 & \xrightarrow[f]{} & B_2 \end{array} .$$

Example 1. (1) A **trivial bundle** is a quadruple represented by the diagram

$$\begin{array}{ccc} F & \hookrightarrow & B \times F \\ & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ & & B \end{array} ,$$

e.g., the 2-torus $\mathbb{T}^2 \equiv \mathbb{S}^1 \times \mathbb{S}^1 \longrightarrow \mathbb{S}^1$, or the cylinder $\mathbb{S}^1 \times \mathbb{R} \longrightarrow \mathbb{S}^1$.

(2) The **Möbius band** as a *non*trivial bundle over $\mathbb{S}^1$ with typical fibre $\mathbb{R}$.

(3) The **Hopf fibration** $(\mathbb{S}^3, \mathbb{S}^2, \mathbb{S}^1, h)$, with the base projection π_{H} readily expressible in the global coordinates $(z_1, z_2) \equiv ((x_0, x_1), (x_2, x_3))$ on the ambient $\mathbb{C}^{\times 2} \equiv \mathbb{R}^{\times 4} \supset \{ (x_0, x_1, x_2, x_3) \in \mathbb{R}^{\times 4} \mid x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \} \equiv \mathbb{S}^3$ as $\pi_{\mathrm{H}}(z_1, z_2) = (2 z_1 \overline{z}_2, |z_1|^2 - |z_2|^2) \in \mathbb{S}^2 \subset \mathbb{R}^{\times 3}$.

(4) Let (E, B_2, F, π_E) be a fibre bundle with local trivialisations $\tau_i : \pi_E^{-1}(O_i) \xrightarrow{\cong} O_i \times F$ over a trivialising cover $O_{B_2} = \{O_i\}_{i \in I}$ of the base B_2, and let $f : B_1 \longrightarrow B_2$ be a smooth map. The quadruple

$$(f^* E \equiv B_1 \times_{B_2} E, B_1, F, \mathrm{pr}_1),$$

is a fibre bundle, termed the **pullback bundle**, with

- total space given by the fibred product[2]

$$\begin{array}{ccc} B_1 \times_{B_2} E & \xrightarrow{\mathrm{pr}_2} & E \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \pi_E} \\ B_1 & \xrightarrow[f]{} & B_2 \end{array} ,$$

  endowed with a subspace topology, which is induced from the product topology on $B_1 \times E \supset B_1 \times_{B_2} E$;
- base projection $\pi_{f^* E} \equiv \mathrm{pr}_1 \restriction_{B_1 \times_{B_2} E} : B_1 \times_{B_2} E \longrightarrow B_1$ given as the (suitably restricted, and manifestly surjective) canonical projection to the first cartesian component;
- typical fibre identical with the typical fibre of E and the fibre over a point $x \in B_1$ in the base given by $\{x\} \times \pi_E^{-1}(\{f(x)\}) \equiv E_{f(x)}$;
- local trivialisations

$$\tau_i^{f^*} := \big(\mathrm{id}_{f^{-1}(O_i)} \times (\mathrm{pr}_2 \circ \tau_i)\big) \restriction_{\mathrm{pr}_1^{-1}(f^{-1}(O_i))} : \mathrm{pr}_1^{-1}\big(f^{-1}(O_i)\big) \xrightarrow{\cong} f^{-1}(O_i) \times F$$

[2]See [Sus23].

associated with the (pullback) trivialising cover $f^*O_{B_2} \equiv \{f^{-1}(O_i)\}_{i\in I}$ (its openness follows from continuity of f).

The trivialisations are, clearly, well defined in virtue of the identities

$$\mathrm{pr}_1\restriction^{-1}_{B_1\times_{B_2}E}\big(f^{-1}(O_i)\big) = \mathrm{pr}_2\restriction^{-1}_{B_1\times_{B_2}E}\big(\pi_E^{-1}(O_i)\big) \equiv \mathrm{pr}_2^{-1}\big(\tau_i^{-1}(O_i\times F)\big)\,,$$

implied by the commutativity of the above diagram. Moreover, the smoothness of the structure maps: π_{f^*E} and $\tau_i^{f^*}$ follows from the submersivity of π_E. Upon denoting ι_{f^*} : $B_1\times_{B_2}E \hookrightarrow B_1\times E$, we may write the maps as superpositions of smooth mappings

$$\pi_{f^*E} \equiv \mathrm{pr}_1\circ\iota_{f^*}\,, \qquad \tau_i^{f^*} \equiv \big(\mathrm{id}_{f^{-1}(O_i)}\times(\mathrm{pr}_2\circ\tau_i)\big)\circ\iota_{f^*}\restriction_{\mathrm{pr}_1^{-1}\left(f^{-1}(O_i)\right)}\,.$$

We readily derive the corresponding transition maps of the pullback bundle in the form:

$$g_{ij}^{f^*} \equiv g_{ij}\circ f\restriction_{f^{-1}(O_{ij})} \;:\; f^{-1}(O_{ij}) \longrightarrow \mathrm{Aut}(F)\,.$$

A moment's thought on the diffeomorphic model of the fibre bundle in a local trivialisation leads to the conclusion that the base projection is a surjective submersion. A simple yet physically signicant consequence of this fact is captured by

Theorem 1. Let $M_A,\ A\in\{1,2\}$ be smooth manifolds, and let the smooth map $f : M_1 \longrightarrow M_2$ be submersive at $x\in M_1$. There exists a neighbourhood $O_{f(x)}\subset M_2$ of the point $f(x)$, on which we find a well-defined smooth map $\sigma : O_{f(x)} \longrightarrow M_1$ with the following properties

$$f\circ\sigma = \mathrm{id}_{O_{f(x)}} \qquad\wedge\qquad \sigma\circ f(x) = x\,. \tag{1}$$

The map is termed a **local section** of f through x.

Proof: The statement is of a local character, and so we may restrict our considerations to a neighbourhood $O_x\ni x$ which supports a local coordinate chart $\kappa_1 : O_x \xrightarrow{\cong} \mathcal{U}_1,\ \mathcal{U}_1\in\mathscr{T}(\mathbb{R}^{\times n_1}),\ n_1\equiv\dim M_1$ such that $\kappa_1(x)=0$, and to a neighbourhood $\widetilde{O}_{f(x)}\ni f(x)$ which supports a local coordinate chart $\kappa_2 : \widetilde{O}_{f(x)} \xrightarrow{\cong} \mathcal{U}_2,\ \mathcal{U}_2\in\mathscr{T}(\mathbb{R}^{\times n_2}),\ n_2\equiv\dim M_2$ such that $\kappa_2\circ f(x)=0$. The submersivity of f at x implies that the tangent map

$$\mathsf{T}_{\kappa(x)=0}\big(\kappa_2\circ f\circ\kappa_1^{-1}\big) \;:\; \mathsf{T}_{\kappa_1(x)=0}\mathbb{R}^{\times n_1}\equiv\mathbb{R}^{\times n_1} \longrightarrow \mathsf{T}_{\kappa_2\circ f(x)=0}\mathbb{R}^{\times n_2}\equiv\mathbb{R}^{\times n_2}$$

is an epimorphism between the $\mathbb{R}$-linear spaces. Let $V_1\subset\mathbb{R}^{\times n_1}$ be an arbitrary subspace mapped isomorphically to $\mathbb{R}^{\times n_2}$ by (the restriction of) $\mathsf{T}_{\kappa(x)}(\kappa_2\circ f\circ\kappa_1^{-1})$. Then, the tangent of the smooth map

$$F := \kappa_2\circ f\circ\kappa_1^{-1}\restriction_{\mathcal{U}_1\cap V_1} \;:\; \mathcal{U}_1\cap V_1 \longrightarrow \mathcal{U}_2\subset\mathbb{R}^{\times n_2}\,,$$

with a manifestly nonempty domain (note that V_1 is a subspace in $\mathbb{R}^{\times n_1}$, and $\mathcal{U}_1$ is a neighbourhood of the zero vector), is invertible. Indeed, in virtue of the identity $\mathsf{T}_0V_1\equiv V_1$, the domain of T_0F takes the form $\mathsf{T}_0\mathcal{U}_1\cap\mathsf{T}_0V_1\equiv\mathbb{R}^{\times n_1}\cap V_1 = V_1$, which means that T_0F is an isomorphism

$$\mathsf{T}_0F \equiv \mathsf{T}_0(\kappa_2\circ f\circ\kappa_1^{-1})|_{V_1}\,.$$

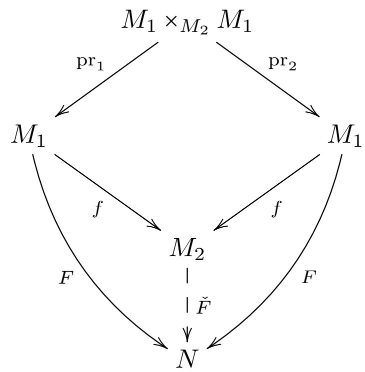
Invoking the Inverse-Function Theorem, we infer that $F\equiv\kappa_2\circ f\circ\kappa_1^{-1}\restriction_{\mathcal{U}_1\cap V_1}$ admits a desired (smooth) inverse $\kappa_1\circ\sigma\circ\kappa_2^{-1}\restriction_{F(\mathcal{U}_0)}$ on a certain neighbourhood $\mathcal{U}_0\subset F(\mathcal{U}_1\cap V_1)$ of the vector $0\equiv\kappa_2\circ f(x)$. The homeomorphic preimage $\kappa_2^{-1}(\mathcal{U}_0)$ of the latter can be chosen as the postulated neighbourhood of $f(x)$, on which there exists a smooth local section σ. □

Submersions enjoy a status in the smooth category akin to that of universal objects studied in the elementary course on (linear) algebra. This is illustrated in

Theorem 2 (Quasi-universality[3] of submersions)**.** Let $f : M_1 \longrightarrow M_2$ be a (smooth) surjective submersion. Furthermore, let N be a smooth manifold, and let $\check{F} : M_2 \longrightarrow N$ be an arbitrary map. The latter is smooth iff the composite map $\check{F}\circ f : M_1 \longrightarrow N$ has this property. In

[3]The reason why the standard notion of universality is qualified by the prefix 'quasi-' is that the class of objects for which a pair (M_2, f) plays the rôle of an initial object is defined in terms of the map f itself (through the condition of constancy on the fibres of the latter).

particular, to every smooth map $F : M_1 \longrightarrow N$ constant on fibres of f, there corresponds a unique $\check{F} \in C^\infty(M_2, N)$ with a property expressed—in conjunction with the said property of f—by the commutative diagram



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Proof: Whenever $\check{F}$ is smooth, so is $\check{F} \circ f$ as a superposition of smooth maps.

Conversely, let $\check{F} \circ f \in C^\infty(M_1, N)$. Due to surjectivity of f, an arbitrary point in M_2 can be written as $f(x)$ for some $x \in M_1$. Pick up a point $f(x) \in M_2$ together with its neighbourhood $O_{f(x)} \subset M_2$ such that there exists a local section $\sigma : O_{f(x)} \longrightarrow M_1$ of the map f, satisfying (1). We then obtain, in the notation of the proof of Thm. 1, the following identity:

$$\check{F} \restriction_{O_{f(x)}} \equiv \check{F} \circ \mathrm{id}_{O_{f(x)}} = (\check{F} \circ f) \circ \sigma ,$$

which ensures smoothness of $\check{F} \restriction_{O_{f(x)}}$ in consequence of the assumed smoothness of $\check{F} \circ f$ and the same property of the local section σ, stated in Thm. 1. The arbitrariness of our choice of $f(x)$ implies global smoothness of $\check{F}$.

Finally, let us address the question of existence and uniqueness of a map $\check{F} \in C^\infty(M_2, N)$, assumed to obey

$$F = \check{F} \circ f .$$

First of all, note that any two such maps coincide on the set $f(M_1)$, which is the same as M_2 by surjectivity of f. Hence, there is at most one map F. Invoking surjectivity of f once more, we postulate F in the manifestly smooth form

$$\check{F} : M_2 \longrightarrow N : f(x) \longmapsto F \circ \sigma\big(f(x)\big) \equiv F(x) ,$$

in which σ is an *arbitrary* section as in Thm. 1. That the definition makes sense is ensured by the assumed constancy of F on fibres of f—indeed, $F(x)$ does not depend on the choice of a representative of the fibre $f^{-1}(\{f(x)\})$, *i.e.*, it does not depend on the choice of a section σ, whose existence is guaranteed by the theorem). The desired identity

$$\check{F} \circ f(x) = F(x)$$

now follows by definition. □

When referred to the base projection of a fibre bundle, the above analysis of elementary properties of surjective submersions leads us to distinguish certain sets of mappings, indicated in

Remark 1. The set of local sections of a fibre bundle (E, B, F, π_E) is denoted as

$$\Gamma_{\mathrm{loc}}(E) ,$$

whereas the set of its global sections is denoted as

$$\Gamma(E) .$$

Definition 1 provides us with just the desired rigorous geometric rendering of the nebular physical concept of a smooth distribution, over a given spacetime B, of internal degrees of freedom F (amenable to further structurisation). The existence of local trivialisations paves the way towards encoding the information about the structure of the bundle in the *locally* smooth transition maps g_{ij}. Exactly how rich and comprehensive that information is, we state in

Theorem 3 (The Clutching Theorem)**.** Transition maps of a fibre bundle (E, B, F, π_E) with a trivialising cover $\mathcal{O}_B = \{O_i\}_{i\in I}$ of its base B satisfy the **1-cocycle condition**

$$\forall_{i,j,k\in I,\ x\in O_{ijk}}\ :\ g_{ij}(x)\circ g_{kj}(x)^{-1}\circ g_{ki}(x) = \mathrm{id}_F\,. \tag{2}$$

Conversely, let $\mathcal{O}_B = \{O_i\}_{i\in I}$ be an open cover of a smooth manifold B, and let F be an arbitrary smooth manifold with an automorphism group $\mathrm{Aut}(F) \subseteq \mathrm{Diff}^\infty(F)$ (see previous remarks). An arbitrary family of maps

$$g_{ij}\ :\ O_{ij} \longrightarrow \mathrm{Aut}(F)\,,\quad i,j\in I$$

inducing smooth maps

$$O_{ij}\times F \longrightarrow F\ :\ (x,f)\longmapsto g_{ij}(x)(f)$$

and satisfying the above condition determines a fibre bundle with transition maps, associated with $\mathcal{O}_B$, given by the g_{ij}. Whenever these latter maps come from a fibre bundle over B (as its transition maps), the induced bundle is (canonically) isomorphic with the original (inducing) one.

Proof: The first part of the theorem is a direct consequence of the following equality, true for every triple $(i,j,k)\in\langle I^{\times 3}\rangle$ and $(x,f)\in O_{ijk}\times F$:

$$\begin{aligned}(x,f) &\equiv (\mathrm{id}_{O_{ijk}}\times\mathrm{id}_F)(x,f) = \big((\tau_i\circ\tau_j^{-1})\circ(\tau_k\circ\tau_j^{-1})^{-1}\circ(\tau_k\circ\tau_i^{-1})\big)(x,f)\\ &= \big(x, g_{ij}(x)\circ g_{kj}(x)^{-1}\circ g_{ki}(x)(f)\big)\,.\end{aligned}$$

The point of departure of the second part is the construction of the disjoint union $\bigsqcup_{i\in I}(O_i\times F)$ (in the catregory of sets), on which we define a relation

$$(x,f,i)\sim_{g_{\cdot\cdot}}(y,g,j)\qquad\Longleftrightarrow\qquad\begin{cases} y = x\in O_{ij}\\ g = g_{ji}(x)(f)\end{cases}\,,$$

The 1-cocycle condition satisfied by the transition maps implies that this is an equivalence relation. Indeed, for $i=j=k$, we obtain

$$g_{ii}(x)\equiv g_{ii}(x)\circ g_{ii}(x)^{-1}\circ g_{ii}(x) = \mathrm{id}_F\,,$$

and so $\sim_{g_{\cdot\cdot}}$ is reflexive. , This further results in skew symmetry of the g_{ij},

$$g_{ji}(x)\circ g_{ij}(x) = g_{ji}(x)\circ g_{ii}(x)^{-1}\circ g_{ij}(x) = \mathrm{id}_F\,,$$

which translates into symmetricity of $\sim_{g_{\cdot\cdot}}$. iFinally, a suitably rewriting of the 1-cocycle condition:

$$g_{ij}(x)\circ g_{jk}(x) = g_{ij}(x)\circ g_{kj}(x)^{-1} = g_{ki}(x)^{-1} = g_{ik}(x)\,,$$

demonstrates its transitivity. Therefore, we may pass from $\bigsqcup_{i\in I}(O_i\times F)$ to the set of equivalence classes

$$\mathscr{R}_{g_{\cdot,\cdot}} := \Big(\bigsqcup_{i\in I}(O_i\times F)\Big)/_{g_{\cdot\cdot}}\,,$$

on which we define a map

$$\pi_{\mathscr{R}_{g_{\cdot,\cdot}}}\ :\ \mathscr{R}_{g_{\cdot,\cdot}}\twoheadrightarrow B\ :\ [(x,f,i)]_{\sim_{g_{\cdot\cdot}}}\longmapsto x\,.$$

Note that every class $[(x,f,i)]_{\sim_{g_{\cdot\cdot}}}$ contains exactly one representative with a given index, $(x,f,i)\in O_i\times F\times\{i\}$, because—by definition—

$$(y,g,i)\in[(x,f,i)]_{\sim_{g_{\cdot\cdot}}}\quad\Longrightarrow\quad(y,g) = \big(x, g_{ii}(x)(f)\big) = \big(x,\mathrm{id}_F(f)\big) = (x,f)\,.$$

Since, furthermore,

$$\forall_{(x,f)\in O_i\times F}\ :\ (x,f,i)\in[(x,f,i)]_{\sim_{g_{\cdot\cdot}}}\,,$$

we establish a bijection

$$[\tau_i]\ :\ \pi^{-1}_{\mathscr{R}_{g_{\cdot,\cdot}}}(O_i)\xrightarrow{\cong} O_i\times F\ :\ [(x,f,i)]_{\sim_{g_{\cdot\cdot}}}\longmapsto(x,f)\,.$$

In the next step, we endow $\mathscr{R}_{g.,.}$ with the quotient topology, declaring as open an arbitrary subset $\mathcal{O} \subset \mathscr{R}_{g.,.}$ whose preimage along the (canonical) projection

$$\pi_\sim \; : \; \bigsqcup_{i \in I} (O_i \times F) \longrightarrow \mathscr{R}_{g.,.} \tag{3}$$

is open in $\widetilde{\mathscr{R}}_{g.,.}$ in the disjoint-sum topology[4] of the spaces $O_i \times F,\ i \in I$, each of which carries the usual product topology. In the said quotient topology, the bijections $[\tau_i]$ are homeomorphisms, and the projection $\pi_{\mathscr{R}_{g.,.}}$ is continuous (tautologically). The topology is hausdorff. Indeed, whenever $[(x_1, f_1, i_1)]_{\sim_{g..}} \neq [(x_2, f_2, i_2)]_{\sim_{g..}}$, we encounter a disjunction: Either $x_2 \neq x_1$, in which case we separate points x_1 and x_2 by taking the respective opens $O_1 \subset O_{i_1}$ and $O_2 \subset O_{i_2}$ in the hausdorff (by assumption) base B, whereupon we take manifestly disjoint neighbourhoods $\pi_\sim(O_\alpha \times F \times \{i_\alpha\}),\ \alpha \in \{1,2\}$ of the corresponding two classes in $\mathscr{R}_{g.,.}$ ($\pi_\sim$ identifies points in the fibre over a point in the base!), or $x_2 = x_1$ with $i_2 = i_1$ (the equality $x_2 = x_1$ makes it possible for us to *choosec* $i_2 = i_1$, potentially at the expense of changing f_2), and so we may separate points f_1 and f_2 by putting them in the respective opens $\mathcal{U}_1$ and $\mathcal{U}_2$ in the hausdorff (also by assumption) fibre F, whereupon we form disjoint (open) neighbourhoods $\pi_\sim(O_{i_1} \times \mathcal{U}_\alpha \times \{i_1\}),\ \alpha \in \{1,2\}$ of the two classes $\mathscr{R}_{g.,.}$ (this time round, (nontrivial) identifications only pertain to points in $\bigsqcup_{i \in I}(O_i \times F)$ with (cover) indices *different* from i_1).

At this stage, we may construct an atlas on the topological space defined above. To this end, we fix atlases on the O_i through restriction of an arbitrary atlas on the base B. In this way, we obtain local (coordinate) charts $\xi_{i,A} \; : \; O_{i,A} \xrightarrow{\cong} \mathcal{U}_{i,A},\ A \in J_i$ on subsets $O_{i,A} \in \mathscr{T}(O_i)$ (in subspace topology) modelled on the respective $\mathcal{U}_{i,A} \in \mathscr{T}(\mathbb{R}^{\times n}),\ n = \dim B$, alongside an atlas on the typical fibre $\zeta_\alpha \; : \; \mathcal{V}_\alpha \xrightarrow{\cong} \mathcal{W}_\alpha,\ \alpha \in K$ with subsets $\mathcal{V}_\alpha \in \mathscr{T}(F)$ modelled on the respective $\mathcal{W}_\alpha \in \mathscr{T}(\mathbb{R}^{\times m}),\ m = \dim F$. We then define an atlas on $\mathscr{R}_{g.,.}$ as the set of local charts

$$\begin{aligned} \kappa_{i,A,\alpha} \quad &: \quad \mathcal{Q}_{i,A,\alpha} \equiv \pi_\sim(O_{i,A} \times \mathcal{V}_\alpha) \xrightarrow{\cong} \mathcal{U}_{i,A} \times \mathcal{W}_\alpha \subset \mathbb{R}^{\times n+m} \\ &: \quad [(x,f,i)]_{\sim_{g..}} \longmapsto \big(\xi_{i,A}(x), \zeta_\alpha(f)\big), \end{aligned}$$

which are well-defined as—in the light of our fomer conclusions—there exists, for each class $[(x,f,i)]_{\sim_{g..}}$, a unique representative $(x,f) \in O_{i,A} \times \mathcal{V}_\alpha \subset O_i \times F$ with a fixed index $i \in I$. In the intersection of their domains, we find coordinate transformations

$$\begin{aligned} \kappa_{i,A,\alpha\, j,B,\beta} \equiv \kappa_{i,A,\alpha} \circ \kappa_{j,B,\beta}^{-1} \quad &: \quad \kappa_{j,B,\beta}\big(O_{i,A\, j,B} \times \mathcal{V}_{\alpha\beta}\big) \xrightarrow{\cong} \kappa_{i,A,\alpha}\big(O_{i,A\, j,B} \times \mathcal{V}_{\alpha\beta}\big) \\ &: \quad \big(\xi_{j,B}(x), \zeta_\beta(f)\big) \longmapsto \big(\xi_{i,A}(x), \zeta_\alpha \circ g_{ij}(x)(f)\big) \\ &\equiv \Big(\xi_{i,A} \circ \xi_{j,B}^{-1}\big(\xi_{j,B}(x)\big), \zeta_\alpha \circ \big(g_{ij} \circ \xi_{j,B}^{-1}\big)\big(\xi_{j,B}(x)\big) \circ \zeta_\beta^{-1}\big(\zeta_\beta(f)\big)\Big). \end{aligned}$$

Bearing in mind that the $\xi_{i,A} \circ \xi_{j,B}^{-1}$ are coordinate transformations of the (refined) atlas on B, smooth by assumption, and that the $g_{ij} \circ \xi_{j,B}^{-1}$ are local presentations of transition maps, likewise smooth by assumption (in the previously considered sense), and—finally—that the $\zeta_\alpha \circ g_{ij}(x) \circ \zeta_\beta^{-1}$ are local (coordinate) presentations of the automorphisms $g_{ij}(x)$ of the fibre F, also smooth by assumption, we conclude that the transformations $\kappa_{i,A,\alpha\, j,B,\beta}$ are smooth. Hence, the local charts $\kappa_{i,A,\alpha}$ endow $\mathscr{R}_{g.,.}$ with the structure of a smooth manifold. Relative to it, the maps $[\tau_i]$ are (tautologically) diffeomorphisms (as is the base projection $\pi_{\mathscr{R}_{g.,.}} \restriction_{\mathcal{Q}_{i,A,\alpha}} \equiv \mathrm{pr}_1 \circ [\tau_i]$), and so they induce on $\mathscr{R}_{g.,.}$ the structure of a fibre bundle.

We complete the proof by demonstrating the equivalence of the two structures of a fibre bundle: the one on a given fibre bundle (E, B, F, π_E), with local trivialisations $\tau_i \; : \; \pi_E^{-1}(O_I) \xrightarrow{\cong} O_i \times F,\ i \in I$, and the onre obtained through the above reconstruction from the latter's transition maps g_{ij}. For that, we consider the local mappings

$$\iota_i := [\tau_i]^{-1} \circ \tau_i \; : \; \pi_E^{-1}(O_i) \xrightarrow{\cong} O_i \times F \xrightarrow{\cong} \pi_{\mathscr{R}_{g.,.}}^{-1}(O_i), \quad i \in I. \tag{4}$$

[4] The standard topology on a disjoint sum is composed of sets whose preimages along *all* canonical injections $O_j \times F \longrightarrow \bigsqcup_{i \in I}(O_i \times F)$ are open.

Note that these local diffeomorphisms satisfy, at every point $y \in \pi_E^{-1}(O_{ij})$, corresponding to some $x \in O_{ij}$ and $f \in F$ through the formula $y = \tau_i^{-1}(x,f)$, the relation

$$\begin{aligned} \iota_j(y) &\equiv \iota_j\big(\tau_i^{-1}(x,f)\big) \equiv [\tau_j]^{-1} \circ \tau_j \circ \tau_i^{-1}(x,f) = [\tau_j]^{-1}\big(x, g_{ji}(x)(f)\big) = [\tau_i]^{-1} \circ [\tau_i] \circ [\tau_j]^{-1}\big(x, g_{ji}(x)(f)\big) \\ &= [\tau_i]^{-1}\big(x, g_{ij}(x) \circ g_{ji}(x)(f)\big) = [\tau_i]^{-1}(x,f) \equiv [\tau_i]^{-1} \circ \tau_i \circ \tau_i^{-1}(x,f) \equiv \iota_i(y)\,, \end{aligned}$$

and so they are restrictions of a global diffeomorphism

$$\iota \; : \; E \xrightarrow{\cong} \mathscr{R}_{g_{\cdot,\cdot}}\,,$$

given by

$$\iota\restriction_{\pi_E^{-1}(O_i)} = \iota_i\,.$$

The diffeomorphism fits into the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\;\iota\;} & \mathscr{R}_{g_{\cdot,\cdot}} \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_{\mathscr{R}_{g_{\cdot,\cdot}}}} \\ B & \underset{\mathrm{id}_B}{=\!=\!=} & B \end{array}\,,$$

which permits us to identify ι as the postulated bundle isomorphism. □

2. The canonical example, and an abstraction

A geometrically natural—even *canonical* (in the sense of it being fully determined by the smooth structure on its base)—example of a fibre bundle is the tangent bundle over a given smooth manifold $(M, \widehat{\mathscr{A}})$. There exist several equivalent definitions of this bundle, each emphasising a different structural property. Below, we recall one of them, which bases on the fundamental theorem just proved.

Definition 2. Let $(M, \widehat{\mathscr{A}})$ be a smooth manifold of dimension $n \in \mathbb{N}^\times$, with atlas $\widehat{\mathscr{A}} = \{\kappa_i\}_{i \in I}$ associated with an open cover $\mathcal{O}_M = \{O_i\}_{i \in I}$. The **tangent bundle** over M is the smooth manifold $(\mathsf{T}M, \mathsf{T}\widehat{\mathscr{A}})$ constructed as follows:

- the underlying set is that of equivalence classes

$$\mathsf{T}M := \Big(\bigsqcup_{i \in I} O_i \times \mathbb{R}^{\times n}\Big) / \sim_{\mathsf{D}t_{\cdot\cdot}}$$

of the relation

$$(x,v,i) \sim (y,w,j) \qquad \Longleftrightarrow \qquad \left\{ \begin{array}{c} y = x \in O_{ij} \\ w = \mathsf{D}\big(\kappa_j \circ \kappa_i^{-1}\big)\big(\kappa_i(x)\big)(v) \equiv \mathsf{D}t_{ji}\big(\kappa_i(x)\big)(v) \end{array} \right. ,$$

which comes together with the map

$$\pi_{\mathsf{T}M} \; : \; \mathsf{T}M \longrightarrow M \; : \; [(x,v,i)]_{\sim_{\mathsf{D}t_{\cdot\cdot}}} \longmapsto x\,,$$

called the **canonical projection** (**on the base of the tangent bundle**), whose level set

$$\mathsf{T}_x M := \pi_{\mathsf{T}M}^{-1}\big(\{x\}\big)$$

carries the name of the **tangent space** at point x;

- the topology of the set $\mathsf{T}M$ is the quotient one, induced along the surjective projection

$$\pi_\sim \; : \; \bigsqcup_{i \in I} O_i \times \mathbb{R}^{\times n} \longrightarrow \Big(\bigsqcup_{i \in I} O_i \times \mathbb{R}^{\times n}\Big) / \sim_{\mathsf{D}t_{\cdot\cdot}} \; : \; (x,v,i) \longmapsto [(x,v,i)]_{\sim_{\mathsf{D}t_{\cdot\cdot}}}$$

from the disjoint-sum topology for the family of spaces $O_i \times \mathbb{R}^{\times n}$ indexed by I, with canonical injections

$$\jmath_i \;:\; O_i \times \mathbb{R}^{\times n} \longrightarrow \bigsqcup_{j \in I} O_j \times \mathbb{R}^{\times n} \;:\; (x, v) \longmapsto (x, v, i)\,, \quad i \in I\,,$$

whose domains carry product topology

$$\mathscr{T}(\mathsf{T}M) \coloneqq \left\{ \mathcal{O} \subset \mathsf{T}M \quad \middle| \quad \forall_{i \in I} \;:\; \jmath_i^{-1}\big(\pi_\sim^{-1}(\mathcal{O})\big) \in \mathscr{T}\big(O_i \times \mathbb{R}^{\times n}\big) \right\}\,;$$

- the smooth structure on the above topological space $\mathsf{T}M$ is defined by the (manifestly homeomorphic) maps

$$\mathsf{T}\kappa_i \;:\; \pi_{\mathsf{T}M}^{-1}(O_i) \xrightarrow{\;\cong\;} \mathcal{U}_i \times \mathbb{R}^{\times n} \;:\; [(x, v, i)]_{\sim_{\mathsf{D}t_{..}}} \longmapsto \big(\kappa_i(x), v\big)\,, \quad i \in I\,,$$

also known as **natural charts** (coefficients of the decomposition of a vector v in a basis (chosen arbitrarily) are promoted to the rank of global coordinates on $\mathbb{R}^{\times n}$), which, in turn, determine coordinate transformations

$$\begin{aligned} \mathsf{T}t_{ji} \coloneqq \mathsf{T}\kappa_j \circ (\mathsf{T}\kappa_i)^{-1}\restriction_{\kappa_i(O_{ij}) \times \mathbb{R}^{\times n}} \quad &: \quad \kappa_i(O_{ij}) \times \mathbb{R}^{\times n} \xrightarrow{\;\cong\;} \kappa_j(O_{ij}) \times \mathbb{R}^{\times n} \\ &: \quad \big(\kappa_i(x), v\big) \longmapsto \Big(\kappa_j(x), \mathsf{D}t_{ji}\big(\kappa_i(x)\big)(v)\Big)\,, \end{aligned}$$

obviously lower by one in the degree of smoothness with respect to those of the underlying manifold M, that is still $\infty - 1 = \infty$ in our case—thus, we are dealing here with a smooth action of the structure group $\mathrm{GL}(\mathbb{R}^{\times n}; \mathbb{R}) \ni \mathsf{D}t_{ji}(\kappa_i(x))$.

The equivalence class

$$V(x) \coloneqq [(x, v, i)]_{\sim_{\mathsf{D}t_{..}}} \in \mathsf{T}_x M \tag{5}$$

is called a **tangent vector** on $(M, \widehat{\mathscr{A}})$ (attached) at point $x \in M$.

The (smooth) map

$$\mathbf{0}_{\mathsf{T}M} \;:\; M \longrightarrow \mathsf{T}M \;:\; x \longmapsto [(x, \mathbf{0}^n, i)]_{\sim_{\mathsf{D}t_{..}}}$$

is referred to as the **zero section** of the tangent bundle $\mathsf{T}M$.

A linear structure, which we anticipate in a model of the "space of infinitesimal motions" or "space of velocities", is camouflaged in our definition as the judicious choice of the local model of the typical fibre $\mathbb{R}^{\times \dim M}$, with a natural interpretation in terms of local coordinates.

Proposition 1. For every pointt $x \in O_i \subset M$ in a smooth manifold $(M, \widehat{\mathscr{A}})$ of dimension $n \in \mathbb{N}^\times$, the map

$$\begin{aligned} \mathsf{T}_x\kappa_i \quad &: \quad \mathsf{T}_x M \xrightarrow{\;\mathsf{T}\kappa_i\;} \mathcal{U}_i \times \mathbb{R}^{\times n} \xrightarrow{\;\mathrm{pr}_2\;} \mathbb{R}^{\times n} \\ &: \quad [(x, v, i)]_{\sim_{\mathsf{D}t_{..}}} \longmapsto \big(\kappa_i(x), v\big) \longmapsto v\,, \end{aligned}$$

given as the supersposition of a local (natural) chart on $\mathsf{T}M$ with the canonical projection onto the second component of the cartesian product, is a bijection and, as such, canonically induces on $\mathsf{T}_x M$ the structure of an $\mathbb{R}$-linear space,

$$\mathsf{T}_x M \underset{\mathbb{R}-\mathrm{lin.}}{\cong} \mathbb{R}^{\times n}\,,$$

termed the **tangent space** of M at x.

Proof: The $\mathbb{R}$-linear structure referred to in the statement of the proposition is determined by the formula

$$\lambda_1 \triangleright [(x, v_1, i)]_{\sim_{\mathsf{D}t_{..}}} + \lambda_2 \triangleright [(x, v_2, i)]_{\sim_{\mathsf{D}t_{..}}} \coloneqq [(x, \lambda_1 \triangleright v_1 + \lambda_2 \triangleright v_2, i)]_{\sim_{\mathsf{D}t_{..}}}\,,$$

written for arbitrary $\lambda_1, \lambda_2 \in \mathbb{R}$. Owing to the $\mathbb{R}$-linear character of the equivalence relation in Def. 2, this structure is well-defined. Indeed, let $x \in O_{ij}$ and let $w_\alpha \coloneqq \mathsf{D}t_{ji}\big(\kappa_i(x)\big)(v_\alpha)$, $\alpha \in \{1, 2\}$, to obtain

$$\lambda_1 \triangleright [(x, w_1, j)]_{\sim_{\mathsf{D}t_{..}}} + \lambda_2 \triangleright [(x, w_2, j)]_{\sim_{\mathsf{D}t_{..}}} = [(x, \lambda_1 \triangleright w_1 + \lambda_2 \triangleright w_2, j)]_{\sim_{\mathsf{D}t_{..}}}$$

$$= \quad [\big(x, \lambda_1 \triangleright \mathsf{D}t_{ji}\big(\kappa_i(x)\big)(v_1) + \lambda_2 \triangleright \mathsf{D}t_{ji}\big(\kappa_i(x)\big)(v_2), j\big)]_{\sim_{\mathsf{D}t_{..}}}$$

$$= \quad \big[\big(x, \mathsf{D}t_{ji}\big(\kappa_i(x)\big)(\lambda_1 \triangleright v_1 + \lambda_2 \triangleright v_2), j\big)\big]_{\sim} = [(x, \lambda_1 \triangleright v_1 + \lambda_2 \triangleright v_2, i)]_{\sim_{\mathsf{D}t_{..}}}$$

$$\equiv \quad \lambda_1 \triangleright [(x, v_1, i)]_{\sim_{\mathsf{D}t_{..}}} + \lambda_2 \triangleright [(x, v_2, i)]_{\sim_{\mathsf{D}t_{..}}} .$$

□

From the above, we abstract a notion of a bundle with an additional linear structure on the typical fibre, which is 'propagated' coherently (in particular, smoothly) over the base. This we give in

Definition 3. Let $n \in \mathbb{N}$ and consider the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with the standard (euclidean) topology and smooth structure. A (smooth) **vector bundle of rank** r **over field** $\mathbb{K}$ is a fibre bundle $(\mathbb{V}, B, \mathbb{K}^{\times r}, \pi_{\mathbb{V}})$ with the following properties:

- $\forall_{x \in B} \;:\; \mathbb{V}_x \equiv \pi_{\mathbb{V}}^{-1}(\{x\}) \in \mathrm{Ob}\, \mathbf{Vect}_{\mathbb{K}}^{(<\infty)}$;
- restrictions of the diffeomorphisms (local trivialisations)

$$\mathrm{pr}_2 \circ \tau_i \restriction_{\mathbb{V}_x} \;:\; \mathbb{V}_x \xrightarrow{\cong} \mathbb{K}^{\times r}, \quad x \in B$$

  are $\mathbb{K}$-linear isomorphisms ,

with the maps that define the $\mathbb{K}$-linear structure on fibres of $\mathbb{V}$ smooth over B. By the latter, we mean that there exist maps:

- a smooth map

(6) $$\mathbb{A} \;:\; \mathbb{V} \times_B \mathbb{V} \longrightarrow \mathbb{V}$$

  modelled on the defining binary operation $A^r \;:\; \mathbb{K}^{\times r} \times \mathbb{K}^{\times r} \longrightarrow \mathbb{K}^{\times r}$ in the sense expressed by the commutative diagram

(7) $$\begin{array}{ccc} \pi_{\mathbb{V}}^{-1}(\mathcal{O}_i) \times_B \pi_{\mathbb{V}}^{-1}(\mathcal{O}_i) & \xrightarrow{\quad \mathbb{A} \quad} & \pi_{\mathbb{V}}^{-1}(\mathcal{O}_i) \\ \Big\downarrow{\scriptstyle \tau_i \times \tau_i} & & \Big\downarrow{\scriptstyle \tau_i} \\ \big(\mathcal{O}_i \times \mathbb{K}^{\times r}\big) \times_{\mathcal{O}_i} \big(\mathcal{O}_i \times \mathbb{K}^{\times r}\big) & \xrightarrow[(\mathrm{pr}_1, A^r \circ \mathrm{pr}_{2,4})]{} & \mathcal{O}_i \times \mathbb{K}^{\times r} \end{array} ;$$

- a family of diffeomorphisms:

(8) $$\mathbb{K}^{\times} \longrightarrow \mathrm{Diff}^k(\mathbb{V}) \;:\; \lambda \longmapsto \mathbb{L}_\lambda$$

  with $\mathbb{K}$-linear restrictions to fibres, augmented by the $\mathbb{K}$-linear map $\mathbb{L}_{0_{\mathbb{K}}}$, modelled on the defining action $\ell^r \;:\; \mathbb{K} \times \mathbb{K}^{\times r} \longrightarrow \mathbb{K}^{\times r}$ in the sense expressed by the commutative diagram

(9) $$\begin{array}{ccc} \pi_{\mathbb{V}}^{-1}(\mathcal{O}_i) & \xrightarrow{\quad \mathbb{L}_\lambda \quad} & \pi_{\mathbb{V}}^{-1}(\mathcal{O}_i) \\ \Big\downarrow{\scriptstyle \tau_i} & & \Big\downarrow{\scriptstyle \tau_i} \\ \mathcal{O}_i \times \mathbb{K}^{\times r} & \xrightarrow[\ell^r_\lambda]{} & \mathcal{O}_i \times \mathbb{K}^{\times r} \end{array} .$$

If $\mathbb{K} = \mathbb{R}$, we speak of a **real vector bundle**, whereas for $\mathbb{K} = \mathbb{C}$, we have a **complex vector bundle**.

The rank of the vector bundle is denoted as $\mathrm{rk}\,\mathbb{V}$. Whenever $\mathrm{rk}\,\mathbb{V} = 1$, we call $\mathbb{V}$ a **line bundle**, and customarily denote it as L,

$$\begin{array}{ccc} \mathbb{K} & \hookrightarrow & L \\ & & \downarrow \pi_L \\ & & B \end{array} .$$

The smooth map

$$\mathbf{0}_{\mathbb{V}} \;:\; B \longrightarrow \mathbb{V} \;:\; x \longmapsto \tau_i^{-1}(x, \mathbf{0}^r)\,, \quad x \in \mathcal{O}_i\,,$$

is termed the **zero section** of $\mathbb{V}$. It is a global section of $\mathbb{V}$ in the sense of Remark 1. It is also to be noted that $\Gamma(\mathbb{V})$ carries a (pointwise) structure of a module over the ring $C^\infty(B, \mathbb{K})$.

A **vector subbundle of rank** $s \leq r$ in a vector bundle $(\mathbb{V}, B, \mathbb{K}^{\times r}, \pi_{\mathbb{V}})$ is a subbundlea $(\mathbb{W}, B, \mathbb{K}^{\times s}, \pi_{\mathbb{V}} \restriction_{\mathbb{W}})$ of that fibre bundle with the following property: over every point $x \in B$ of the base, the fibre $\mathbb{W}_x \subset \mathbb{V}_x$ is a $\mathbb{K}$-linear subspace of $\mathbb{V}_x$.

A **morphism of vector bundlesh** (**over field** $\mathbb{K}$) $(\mathbb{V}_\alpha, B_\alpha, \mathbb{K}^{\times r_\alpha}, \pi_{\mathbb{V}_\alpha})$, $r_\alpha \in \mathbb{N}$, $\alpha \in \{1, 2\}$ is a bundle map

$$(\Phi, f) \;:\; (\mathbb{V}_1, B_1, \mathbb{K}^{\times r_1}, \pi_{\mathbb{V}_1}) \longrightarrow (\mathbb{V}_2, B_2, \mathbb{K}^{\times r_2}, \pi_{\mathbb{V}_2})\,,$$

with $\mathbb{K}$-linear restrictions

$$\Phi \restriction_{\mathbb{V}_{1\,x}} \;:\; \mathbb{V}_{1\,x} \longrightarrow \mathbb{V}_{2\,f(x)}\,. \tag{10}$$

The **rank** of (Φ, f) is the map

$$\mathrm{rk}\,(\Phi, f) \;:\; B_1 \longrightarrow \mathbb{N} \;:\; x \longmapsto \mathrm{rk}\big(\Phi \restriction_{\mathbb{V}_{1\,x}}\big)\,.$$

References

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