Duality, Descent & Defects I

LECTURE IV

2024/25


¡oï!
AN ACTIVE & RESILIENT INFINITESIMAL
– A MOSS PIGLET (THE ULTIMATE SUCKLING)

In what follows, we consider the restriction of $\operatorname{Ker} Tt$ to the identity bisection $Id(M) \subset \mathcal{G}$, & obtain

Prop. 64. For any Lie groupoid $Gr = (M, \mathcal{G}, s, t, Id, Inv, m)$, there exists a vector space isomorphism

$$\iota_L : \Gamma(Id^* \operatorname{Ker} Tt) \xrightarrow{\cong} \mathfrak{X}_L(\mathcal{G}).$$

Proof: For any $L \in \mathfrak{X}_L(\mathcal{G})$ & $g \in \mathcal{G}$, we compute

$$L(g) \equiv L\left(\ell_g\left(Id_{s(g)}\right)\right) = T_{Id_{s(g)}} \ell_g \left(L\left(Id_{s(g)}\right)\right).$$

Thus, L is uniquely determined by its values on $Id(M)$ (recall that s is a surjection).

HERE,

$$\mathcal{L}(\mathrm{Id}_{s(g)}) \in (\mathrm{Ker}\,\mathrm{T}t)_{\mathrm{Id}_{s(g)}} \equiv (\mathrm{Id}^{*}\mathrm{Ker}\,\mathrm{T}t)_{s(g)},$$

WHERE WE WORK in THE STANDARD MODEL

$$\begin{array}{ccc}
\mathrm{Id}^{*}\mathrm{Ker}\,\mathrm{T}t \equiv M\ {}_{\mathrm{Id}}\!\times_{\pi_{\mathrm{T}\mathcal{G}}\restriction} \mathrm{Ker}\,\mathrm{T}t & \xrightarrow{\ \mathrm{pr}_2\ } & \mathrm{Ker}\,\mathrm{T}t \\
\Big\downarrow {\scriptstyle \pi_{\mathrm{Id}^{*}\mathrm{Ker}\mathrm{T}t} \equiv \mathrm{pr}_1} & \circlearrowleft & \Big\downarrow {\scriptstyle \pi_{\mathrm{T}\mathcal{G}}\restriction} \\
M & \xrightarrow[\ \mathrm{Id}\ ]{} & \mathcal{G}
\end{array}$$

WE ARRIVE at

$$\Sigma_{\cdot} : \mathfrak{X}_{L}(\mathcal{G}) \longrightarrow \Gamma(\mathrm{Id}^{*}\mathrm{Ker}\,\mathrm{T}t) : \mathcal{L} \longmapsto (\mathrm{id}_M, \mathcal{L}(\mathrm{Id}_{\cdot})) \overset{\text{!!!}}{=} \Sigma_{\mathcal{L}}.$$

CONVERSELY, for ANY $\Sigma \equiv (\mathrm{id}_M, \tilde{\Sigma}) \in \Gamma(\mathrm{Id}^* \mathrm{Ker}\, Tt)$, WE DEFINE

$$L_\cdot : \Gamma(\mathrm{Id}^* \mathrm{Ker}\, Tt) \longrightarrow \Gamma(T\mathcal{G}) : \Sigma \longmapsto T_{\mathrm{Id}_{s(\cdot)}} \ell_\cdot \left(\underset{\underset{(\mathrm{Ker}\, Tt)_{\mathrm{Id}_{s(\cdot)}}}{\cap}}{\tilde{\Sigma}(s(\cdot))} \right)$$

& CONVINCE OURSELVES THAT THE IMAGE of $L_\cdot$ LIES in $\mathfrak{X}_L(\mathcal{G})$.

WE COMPUTE

$$L_{\Sigma_L}(g) = T_{\mathrm{Id}_{s(g)}} \ell_g \left(\tilde{\Sigma}_L(s(g)) \right) \equiv T_{\mathrm{Id}_{s(g)}} \ell_g \left(L(\mathrm{Id}_{s(g)}) \right)$$

$$= L(g) \quad \&$$

$$\Sigma_{L_\Sigma}(m) \equiv (m, L_\Sigma(\mathrm{Id}_m)) \equiv \left(m, T_{\mathrm{Id}_{s(\mathrm{Id}_m)}} \ell_{\mathrm{Id}_m} \left(\tilde{\Sigma}(s(\mathrm{Id}_m))\right)\right)$$

$$= (m, \tilde{\Sigma}(m)) \equiv \Sigma(m),$$ WHICH SHOWS THAT $L_\cdot = \Sigma_\cdot^{-1}$. □

WE SET $\mathscr{L}_L \equiv L_\cdot$.

THE INDUCTION of LIE BRACKET on $\Gamma(\mathrm{Id}^*\mathrm{Ker}\,\mathrm{T}t)$ BASES ON

PROP. 65. $$[\mathfrak{X}_L(\mathcal{G}), \mathfrak{X}_L(\mathcal{G})]_{\Gamma(\mathrm{T}\mathcal{G})} \subset \mathfrak{X}_L(\mathcal{G})$$

PROOF: FIRST, NOTE THAT ANY TWO LI VECTOR FIELDS ARE SECTIONS of THE TANGENT BUNDLE $\mathrm{Ker}\,\mathrm{T}t \equiv \mathrm{T}\left(\bigsqcup_{m\in M} t^{-1}(\{m\})\right)$ of THE SUBMANIFOLD $\bigsqcup_{m\in M} t^{-1}(\{m\}) \subset \mathcal{G}$, WHICH IMPLIES

$$[\mathfrak{X}_L(\mathcal{G}), \mathfrak{X}_L(\mathcal{G})]_{\Gamma(\mathrm{T}\mathcal{G})} \subset \Gamma(\mathrm{Ker}\,\mathrm{T}t).$$ In PARTICULAR,

$$\forall L_1, L_2 \in \mathfrak{X}_L(\mathcal{G})\ \forall m\in M\ \forall h\in t^{-1}(\{m\}):$$

$$[L_1, L_2]_{\Gamma(\mathrm{T}\mathcal{G})}(h) \equiv [L_1, L_2]_{\Gamma(\mathrm{T}t^{-1}(\{m\}))}(h).$$

NOW, THE RESTRICTIONS $L_A\restriction_{t^{-1}(\{s(g)\})}$ & $L_A\restriction_{t^{-1}(\{t(g)\})}$ ARE ℓ_g-RELATED, $A\in\{1,2\}$

AS A RESULT of LEFT-INVARIANCE of THE L_A. HENCE, THEIR LIE BRACKETS ARE ALSO ℓ_g-RELATED,

$$T_h\ell_g\left([L_1,L_2]_{\Gamma(Tg)}(h)\right) = T_h\ell_g\left([L_1,L_2]_{\Gamma(Tt^{-1}(\{s(g)\}))}(h)\right)$$

$$= [L_1,L_2]_{\Gamma(Tt^{-1}(\{t(g)\}))}\left(\ell_g(h)\right) \equiv [L_1,L_2]_{\Gamma(Tg)}(g.h). \quad \square$$

THIS LEADS to

DEF. 66. THE LIE BRACKET on $\Gamma(\mathrm{Id}^*\mathrm{Ker}\,Tt)$ IS

$$[\cdot,\cdot]_{\Gamma(\mathrm{Id}^*\mathrm{Ker}\,Tt)} : \Gamma(\mathrm{Id}^*\mathrm{Ker}\,Tt)\times\Gamma(\mathrm{Id}^*\mathrm{Ker}\,Tt) \longrightarrow \Gamma(\mathrm{Id}^*\mathrm{Ker}\,Tt)$$

$$: (\Sigma_1,\Sigma_2) \longmapsto \iota_L^{-1}\left([\iota_L(\Sigma_1),\iota_L(\Sigma_2)]_{\Gamma(Tg)}\right).$$

WE IDENTIFY THE CANDIDATE for THE ANCHOR in

PROP. 67. THE MAP

$$(\mathrm{id}_M, L(\mathcal{I}d)) \cong \Sigma \qquad \mathrm{T}_{\mathcal{I}d}\, s\circ L(\mathcal{I}d) \cong$$

$$\alpha \; : \Gamma(\mathcal{I}d^* \ker Tt) \longrightarrow \Gamma(TM) : \Sigma \longmapsto \mathrm{T}_{\mathcal{I}d}\, s\circ L_\Sigma(\mathcal{I}d)$$

IS A LIE-ALGEBRA HOMOMORPHISM. (THUS, ESSENTIALLY, $\alpha \equiv Ts$)

PROOF: CONSIDER VECTOR FIELDS

$$L \in \mathfrak{X}_L(\mathcal{G}) \quad \& \quad \alpha(\Sigma_L)$$

WE FIND, for ANY $g \in \mathcal{G}$,

$$\mathrm{T}_g s(L(g)) = \mathrm{T}_g s \circ \mathrm{T}_{\mathcal{I}d_{s(g)}} \ell_g (L(\mathcal{I}d_{s(g)})) = \mathrm{T}_{\mathcal{I}d_{s(g)}} (s\circ \ell_g)(L(\mathcal{I}d_{s(g)}))$$

$$= \mathrm{T}_{\mathcal{I}d_{s(g)}} s(L(\mathcal{I}d_{s(g)})) \equiv \alpha(\Sigma_L)(s(g)),$$

WHICH MEANS THAT THE FIELDS ARE **s**-RELATED, WHENCE

– for ANY $L_1, L_2 \in \mathfrak{X}_L(G)$ –

$$T_g s\left([L_1, L_2]_{\Gamma(TG)}(g)\right) = \left[\alpha(\Sigma_{L_1}), \alpha(\Sigma_{L_2})\right]_{\Gamma(TM)}(s(g))$$

$$\underset{\text{AS ABOVE}}{\equiv} T_{\mathrm{Id}_{s(g)}} s\left([L_1, L_2]_{\Gamma(TG)}(\mathrm{Id}_{s(g)})\right)$$

$$\equiv T_{\mathrm{Id}_{s(g)}} s\left([L_{\Sigma_{L_1}}, L_{\Sigma_{L_2}}]_{\Gamma(TG)}(\mathrm{Id}_{s(g)})\right)$$

by **Def. 66**

$$= T_{\mathrm{Id}_{s(g)}} s\left(L_{[\Sigma_{L_1}, \Sigma_{L_2}]_{\Gamma(\mathrm{Id}^* \ker Tt)}}(\mathrm{Id}_{s(g)})\right)$$

$$\equiv \alpha\left([\Sigma_{L_1}, \Sigma_{L_2}]\right)(s(g)),$$ i.e.,

$$[\alpha(\Sigma_{L_1}),\alpha(\Sigma_{L_2})]_{\Gamma(TM)} = \alpha([\Sigma_{L_1},\Sigma_{L_2}])$$ by SURJECTIVITY of ι. $\square$

ALTOGETHER, THEN, WE ARRIVE AT

THM. 68. / DEF. $\big(\mathrm{Id}^*\mathrm{Ker}\,T t,\ \pi_{\mathrm{Id}^*\mathrm{Ker}\,Tt},\ \underbrace{\iota_L^{-1}\circ[\cdot,\cdot]_{\Gamma(TG)}\circ(\iota_L\times\iota_L)}_{=\,[\cdot,\cdot]_{\mathfrak{gr}_L}},\ \underbrace{\Gamma(Ts)\circ\iota_L}_{=:\,\alpha_{\mathfrak{gr}_L}}\big)$

$\equiv: \mathfrak{gr}_L \equiv \mathrm{Lie}_L(\mathrm{Gr})$

IS A LIE ALGEBROID.

IT IS CALLED THE (LEFT) TANGENT LIE ALGEBROID of Gr.

PROOF: AT THIS STAGE, IT REMAINS TO PROVE THE LEIBNIZ PROPERTY.

TO THIS END, CONSIDER $\Sigma = (\mathrm{id}_M,\tilde{\Sigma}) \in \Gamma(\mathrm{Id}^*\mathrm{Ker}\,Tt)$ &

$f \in C^\infty(M;\mathbb{R})$, & $f \triangleright \Sigma$.

WE HAVE

$$l_L(f \triangleright \Sigma)(\cdot) \equiv T_{Id_{s(\cdot)}} l_{\cdot}\left(f(s(\cdot)) \triangleright \tilde{\Sigma}(s(\cdot))\right) \equiv s^*f(\cdot) \triangleright l_L(\Sigma)(\cdot),$$

& SO

$$[\Sigma_1, f \triangleright \Sigma_2]_{\mathfrak{gr}_L} \equiv l_L^{-1}\left([l_L(\Sigma_1), l_L(f \triangleright \Sigma_2)]_{\Gamma(T\mathcal{G})}\right)$$

$$= l_L^{-1}\left([l_L(\Sigma_1), s^*f \triangleright l_L(\Sigma_2)]_{\Gamma(T\mathcal{G})}\right)$$

$$= l_L^{-1}\left(s^*f \triangleright l_L\left(l_L^{-1}\left([l_L(\Sigma_1), l_L(\Sigma_2)]_{\Gamma(T\mathcal{G})}\right)\right) + l_L(\Sigma_1)(s^*f) \triangleright l_L(\Sigma_2)\right)$$

$$= l_L^{-1}\left(l_L\left(f \triangleright l_L^{-1}\left([l_L(\Sigma_1), l_L(\Sigma_2)]_{\Gamma(T\mathcal{G})}\right)\right) + l_L(\Sigma_1) \lrcorner\, s^*df \triangleright l_L(\Sigma_2)\right)$$

$$\equiv f \triangleright [\Sigma_1, \Sigma_2]_{\mathfrak{gr}_L} + l_L^{-1}\left(s^*(Ts \circ l_L(\Sigma_1) \lrcorner\, df) \triangleright l_L(\Sigma_2)\right)$$

$$\equiv f \triangleright [\Sigma_1, \Sigma_2]_{\mathfrak{gr}_L} + \alpha_{\mathfrak{gr}_L}(\Sigma_1)(f) \triangleright \Sigma_2 . \quad \square$$

In complete analogy, we prove/define

Thm. 69. / Def. $\Big(\mathrm{Id}^*\ker T s,\ \pi_{\mathrm{Id}^*\ker Ts} \overset{\cdot\cdot}{=} pr_1,\ \underbrace{l_R^{-1}\circ[\cdot,\cdot]_{\Gamma(T\mathcal{G})}\circ(l_R\times l_R)}_{=\,[\cdot,\cdot]_{\mathfrak{gr}_R}},\ \underbrace{\Gamma(Tt)\circ l_R}_{=\,\alpha_{\mathfrak{gr}_R}}\Big)$

$\trianglerighteq: \mathfrak{gr}_R \equiv \mathrm{Lie}_R(\mathrm{Gr})$ with $l_R(\mathrm{id}_M,\Sigma) = T_{\mathrm{Id}_{t(\cdot)}} p_\cdot(\Sigma(t(\cdot)))$

is a Lie algebroid.

It is called the (right) tangent Lie algebroid of Gr.

E.g., Ex. 70. $\mathrm{Lie}_L(\mathrm{Pair}(M)) \simeq \big(TM, M, \mathbb{R}^{\times \dim M}, \pi_{TM}, \mathrm{id}_{TM}, [\cdot,\cdot]_{\Gamma(TM)}\big)$

Ex. 71. $\mathrm{Lie}_R(G \ltimes_\lambda M) \simeq \big(M\times\mathfrak{g}, M, \mathbb{R}^{\times \dim G}, pr_1, -\mathcal{K}_\cdot, [\cdot,\cdot]_{\mathfrak{g}\ltimes_\lambda M}\big)$

$\trianglerighteq \mathfrak{g}\ltimes_\lambda M$ the action Lie algebroid with $f_1, f_2 : M \to \mathfrak{g}$

$$[\underbrace{f_1^A \triangleright t_A}_{=f_1}, \underbrace{f_2^B \triangleright t_B}_{=f_2}]_{\mathfrak{g}\ltimes_\lambda M} = \big(f_2^B \mathcal{K}_{t_B}(f_1^A) - f_1^B \mathcal{K}_{t_B}(f_2^A) - f_1^B f_2^C f_{BCA}\big) \triangleright t_A.$$

RUDIMENTS of THE MAURER–CARTAN CALCULUS:

Definition 72. *Let M be a smooth manifold. A* (**smooth**) **foliation** *on M is an integrable subbundle $\mathscr{F} \hookrightarrow$ TM, i.e., a subbundle with the* **Frobenius property**

$$[\Gamma(\mathscr{F}),\Gamma(\mathscr{F})] \subset \Gamma(\mathscr{F}) \,.$$

Definition 73. *Let X and M be smooth manifolds, $\mathscr{F}$ a foliation on X, and $\pi_{\mathbb{V}} : \mathbb{V} \longrightarrow M$ a rank-r real vector bundle. An* **$\mathscr{F}$-foliated differential 1-form on X with values in $\mathbb{V}$** *is a vector-bundle morphism*

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\ \eta\ } & \mathbb{V} \\ {\scriptstyle \pi_{\mathrm{T}X}|_{\mathscr{F}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathbb{V}}} \\ X & \xrightarrow[\ h\ ]{} & M \end{array} \,.$$

Whenever $\mathbb{V}$ is endowed with a vector-bundle morphism $\alpha_{\mathbb{V}} : \mathbb{V} \longrightarrow$ TM, we call the 1-form **$\alpha_{\mathbb{V}}$-anchored** *if it satisfies the identity*

$$\alpha_{\mathbb{V}} \circ \eta = \mathrm{T}h \,.$$

A PRIME EXAMPLE of THE OBJECT DEFINED ABOVE IS . . .

Example 74. ([FS14]). *Fix a Lie groupoid $\mathscr{G}$ with the tangent Lie algebroids $\mathfrak{g}_H$, $H \in \{L, R\}$. The* **left-invariant Maurer–Cartan form on $\mathscr{G}$** *is the* $\ker \mathrm{T}t$*-foliated 1-form with values in* $\mathfrak{gr}_{\mathrm{L}}$ *given in*

$$\begin{array}{ccc} \ker \mathrm{T}t & \xrightarrow{\ \theta_{\mathrm{L}}\ } & \mathfrak{gr}_{\mathrm{L}} \\ {\scriptstyle \pi_{\mathrm{T}\mathscr{G}}|_{\ker \mathrm{T}t}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathfrak{gr}_{\mathrm{L}}} \equiv \mathrm{pr}_1} \\ \mathscr{G} & \xrightarrow[\ s\ ]{} & M \end{array}, \qquad \theta_{\mathrm{L}} = \left(s \circ \pi_{\mathrm{T}\mathscr{G}}(\cdot), \mathrm{T}_{\pi_{\mathrm{T}\mathscr{G}}(\cdot)} l_{\mathrm{Inv} \circ \pi_{\mathrm{T}\mathscr{G}}(\cdot)}(\cdot)\right),$$

i.e., $\theta_L(g)(v) = (s(g), T_g \ell_{g^{-1}}(v))$, $v \in (\ker Tt)_g$

Similarly, the **right-invariant Maurer–Cartan form on $\mathscr{G}$** *is the* $\ker \mathrm{T}s$*-foliated 1-form with values in* $\mathfrak{gr}_{\mathrm{R}}$ *given in*

$$\begin{array}{ccc} \ker \mathrm{T}s & \xrightarrow{\ \theta_{\mathrm{R}}\ } & \mathfrak{gr}_{\mathrm{R}} \\ {\scriptstyle \pi_{\mathrm{T}\mathscr{G}}|_{\ker \mathrm{T}s}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathfrak{gr}_{\mathrm{R}}} \equiv \mathrm{pr}_1} \\ \mathscr{G} & \xrightarrow[\ t\ ]{} & \mathscr{G} \end{array}, \qquad \theta_{\mathrm{R}} = \left(t \circ \pi_{\mathrm{T}\mathscr{G}}(\cdot), \mathrm{T}_{\pi_{\mathrm{TMorGr}}(\cdot)} r_{\mathrm{Inv} \circ \pi_{\mathrm{T}\mathscr{G}}(\cdot)}(\cdot)\right).$$

i.e., $\theta_R(g)(w) = (t(g), T_g r_{g^{-1}}(w))$, $w \in (\ker Ts)_g$

Thus, $\theta_{\mathrm{L}}(g) \in (\ker \mathrm{T}t)^*_g \otimes_{\mathbb{R}} (\ker \mathrm{T}t)_{\mathrm{Id}_{s(g)}}$ *and* $\theta_{\mathrm{R}}(g) \in (\ker \mathrm{T}s)^*_g \otimes_{\mathbb{R}} (\ker \mathrm{T}s)_{\mathrm{Id}_{t(g)}}$.

... AND ... ACTION !

WE SAY THAT **Gr** ACTS on **X** along **M**

Definition 75. *Given a Lie groupoid* **Gr** $= (M, \mathscr{G}, s, t, \mathrm{Id}, \mathrm{Inv}, .)$, *a* ***right-Gr-module space*** *is a triple* (X, μ, ρ) *composed of*

- *a smooth manifold* X;
- *a smooth map* $\mu : X \longrightarrow M$, *called the* ***momentum*** (**of the action**);
- *a smooth map*

$$\rho : X\,{}_{\mu}\!\times_{t} \mathscr{G} \longrightarrow X : (x, g) \longmapsto \rho(x, g) \equiv \rho_g(x) \equiv x \blacktriangleleft g\,,$$

termed the ***action*** (**map**)

subject to the relations (in force whenever the expressions are well-defined):

(GrM1) $\mu(x \blacktriangleleft g) = s(g)$;
(GrM2) $x \blacktriangleleft \mathrm{Id}_{\mu(x)} = x$;
(GrM3) $(x \blacktriangleleft g) \blacktriangleleft h = x \blacktriangleleft (g.h)$.

A **left-Gr-module space** *is defined similarly (with the rôles of the source and target maps in the definition interchanged).*

A **bi-Gr-module space** *is a pair of* **Gr***-module-space structures on a single smooth manifold: a left one and a right one, which commute with one another in an obvious manner.*

A (right) action ρ *is termed* ***free*** *iff the following implication obtains:*

$$x \blacktriangleleft g = x \qquad \Longrightarrow \qquad g = \mathrm{Id}_{\mu(x)}\,,$$

so that, in particular, the ***isotropy group*** $\mathscr{G}_m = s^{-1}(\{m\}) \cap t^{-1}(\{m\})$ *of* $m \in M$ *acts freely (in the usual sense) on the fibre* $\mu^{-1}(\{m\})$.

The (right) action ρ *is termed* ***transitive*** *iff for any two points* $x, x' \in X$ *there exists an arrow* $g \in \mathscr{G}$ *such that* $x' = x \blacktriangleleft g$.

Let $\mathbf{Gr}_A$, $A \in \{1, 2\}$ *be a pair of Lie groupoids and let* (X_A, μ_A, ρ^A) *be the respective right-*$\mathbf{Gr}_A$*-module spaces. A* ***morphism*** *between the latter is a pair* (Θ, Φ) *consisting of a smooth manifold map* $\Theta : X_1 \longrightarrow \sigma_2$ *together with a functor* $\Phi : \mathbf{Gr}_1 \longrightarrow \mathbf{Gr}_2$ *for which the following diagrams commute*

$$\begin{array}{ccc} X_1 & \xrightarrow{\ \Theta\ } & \sigma_2 \\ {\scriptstyle \mu_1}\downarrow & & \downarrow{\scriptstyle \mu_2} \\ M_1 & \xrightarrow{\ \Phi\ } & M_2 \end{array}\,,$$

$$\begin{array}{ccc} X_1\,{}_{\mu_1}\!\times_{t_1} \mathscr{G}_1 & \xrightarrow{\ \Theta \times \Phi\ } & \sigma_2\,{}_{\mu_2}\!\times_{t_2} \mathscr{G}_2 \\ {\scriptstyle \rho^1}\downarrow & & \downarrow{\scriptstyle \rho^2} \\ X_1 & \xrightarrow{\ \Theta\ } & \sigma_2 \end{array}\,.$$

IN ANY **Gr**-MODULE SPACE (X, μ, ρ), THERE ARISE ORBITS, WHICH ARE CLASSES OF POINTS $x_1, x_2 \in X$ W.R.T. EQUIVALENCE RELATION:

$$x_1 \sim_\rho x_2 \overset{\text{ex}}{\underset{\text{def}}{\Longleftrightarrow}} \exists g \in \mathcal{G} : x_2 = x_1 \triangleleft g \quad \text{RESP.} \quad x_2 = g \triangleright x_1 .$$

GIVEN A POINT $x \in X$, WE SPEAK OF ITS ISOTROPY GROUP

$$\rho(\mathcal{G})_x = \{ g \in \mathcal{G} \mid x \triangleleft g = x \ \ \text{RESP.} \ \ g \triangleright x = x \} .$$

E.G.,

Example 76 . *The object manifold* M *of a Lie groupoid* $\mathbf{Gr} = (M, \mathscr{G}, s, t, \mathrm{Id}, \mathrm{Inv}, .)$ *carries a natural structure of a left-*$\mathbf{Gr}$*-module space given by*

$$\left(M, \mathrm{id}_M, \mathrm{Aim}^{\mathscr{G}}\right),$$

where the action map reads

$$\mathrm{Aim}^{\mathscr{G}} := t \circ \mathrm{pr}_1 : \mathscr{G}\,{}_s\!\times_{\mathrm{id}_M} M \longrightarrow M : \left(g, s(g)\right) \longmapsto t(g) \equiv \mathrm{Aim}^{\mathscr{G}}_g\left(s(g)\right) .$$

Example 77. *We may regard $\mathscr{G}$ as a left-$\mathbf{Gr}^{\times 2}$-module space for the* ***product groupoid***

$$\mathbf{Gr}^{\times 2} \quad : \quad \mathrm{Mor}\big(\mathbf{Gr}^{\times 2}\big) = \mathscr{G}^{\times 2} \underset{t^{\times 2} := t \times t}{\overset{s^{\times 2} := s \times s}{\rightrightarrows}} M^{\times 2} = \mathrm{Ob}\big(\mathbf{Gr}^{\times 2}\big) \ ,$$

with the composition map defined as

$$(g_1, g_2) \bullet (h_1, h_2) := (g_1.h_1, g_2.h_2) \,,$$

the identity map

$$\mathrm{Id}_{(m_1, m_2)} = (\mathrm{Id}_{m_1}, \mathrm{Id}_{m_2}) \,,$$

and the inversion map

$$\mathrm{Inv}(g_1, g_2) = \big(\mathrm{Inv}(g_1), \mathrm{Inv}(g_2)\big) \,.$$

The left-$\mathbf{Gr}^{\times 2}$-module space

$$(\mathscr{G}, \mu_{\mathrm{c}}, \mathrm{c})$$

is the triple composed of the smooth manifold $\mathscr{G}$, the momentum

$$\mu_{\mathrm{c}} \equiv (t, s) : \mathscr{G} \longrightarrow \mathrm{Ob}\big(\mathbf{Gr}^{\times 2}\big) \,,$$

defining the set

$$\mathrm{Mor}\big(\mathbf{Gr}^{\times 2}\big)_{s^{\times 2}} \times_{\mu_{\mathrm{c}}} \mathscr{G} := \big\{ \, \big((h_1, h_2), g\big) \in \mathscr{G}^{\times 3} \quad | \quad s^{\times 2}(h_1, h_2) = \mu_{\mathrm{c}}(g) \, \big\} \,,$$

and the action

$$\mathrm{c} : \mathrm{Mor}\big(\mathbf{Gr}^{\times 2}\big)_{s^{\times 2}} \times_{\mu_{\mathrm{c}}} \mathscr{G} \longrightarrow \mathscr{G} : \big((h_1, h_2), g\big) \longmapsto h_1.g.h_2^{-1} \,,$$

to be referred to as the **self-conjugation** *in what follows.*

Note also that the above left-$\mathbf{Gr}^{\times 2}$-module structure on $\mathscr{G}$ is, in fact, a combination of two 'chiral' $\mathbf{Gr}$-module structures on the same manifold: the left-$\mathbf{Gr}$-module structure

$$(\mathscr{G}, t, l \equiv .) \,,$$

with the **left-fibred action**

(45) $$l(h, g) \equiv l_h(g) = h.g \,,$$

and the right-$\mathbf{Gr}$-module structure

(46) $$(\mathscr{G}, s, r \equiv .) \,,$$

with the **right-fibred action**

(47) $$r(g, h) \equiv r_h(g) = g.h \,.$$

The two make $\mathscr{G}$ into a prototypical bi-$\mathbf{Gr}$-module.

PROP. 78. A LEFT ACTION $\lambda : \mathcal{G}\,{}_s\!\times_\mu X \to X$ OF A LIE GROUPOID $\mathcal{G} \rightrightarrows M$ ON A MANIFOLD X CANONICALLY INDUCES A LEFT ACTION $\Lambda : \mathbb{B} \times X \to X$ OF THE GROUP OF BISECTIONS $\mathbb{B}$ OF $\mathcal{G}$ ON X.

AN ANALOGOUS STATEMENT IS TRUE FOR RIGHT ACTIONS.

PROOF: WE PRESENT A PROOF FOR A LEFT ACTION

$$\lambda : \mathcal{G}\,{}_s\!\times_\mu X \longrightarrow X,$$

LEAVING THE TECHNICALLY SOMEWHAT MORE COMPLEX PROOF OF THE STATEMENT FOR A RIGHT ACTION AS AN EXERCISE FOR AN INTERESTED READER.

DEFINE A MAP

$$\lambda : \mathbb{B} \times X \longrightarrow \mathcal{G}\,{}_s{\times}_\mu\, X \longrightarrow X$$

$$: (\beta, x) \longmapsto (\beta(\mu(x)), x) \longmapsto \beta(\mu(x)) \blacktriangleright x$$

$$\equiv \beta \triangleright x$$

THE DEFINITION MAKES SENSE AS

$$s(\beta(\mu(x))) = (s \circ \beta)(\mu(x)) = id_M(\mu(x)) = \mu(x).$$

IT NOW SUFFICES to CHECK THE AXIOMS of AN ACTION...

FIRST, WE COMPUTE

$$\beta_2 \triangleright (\beta_1 \triangleright x) \equiv \beta_2 \triangleright (\beta_1(\mu(x)) \blacktriangleright x) \equiv \beta_2(\mu(\beta_1(\mu(x)) \blacktriangleright x)) \blacktriangleright (\beta_1(\mu(x)) \blacktriangleright x)$$

$$= \beta_2(\mu(\beta_1(\mu(x)) \blacktriangleright x)) \cdot \beta_1(\mu(x)) \blacktriangleright x$$

HERE, $\mu\left(\beta_1(\mu(x)) \triangleright x\right) = t\left(\beta_1(\mu(x))\right)$ by THE AXIOMS of A LEFT $\mathcal{G}$-MODULE, & SO

$$\beta_2 \triangleright (\beta_1 \triangleright x) = (\beta_2 \cdot \beta_1)(\mu(x)) \triangleright x \equiv (\beta_2 \cdot \beta_1) \triangleright x,$$ AS DESIRED.

MOREOVER,

$$\mathrm{Id} \triangleright x \equiv \mathrm{Id}_{\mu(x)} \triangleright x = x,$$

ALSO IN CONFORMITY with THE AXIOMATICS of A LEFT $\mathbb{B}$-SPACE. □

EXAMPLE 79.

THE ACTION of $\mathbb{B}$ INDUCED from THE LEFT CANONICAL ACTION

$$\mathrm{Aim}^{\mathcal{G}} : \mathcal{G}\;{}_s\!\times_{id_M} M \longrightarrow M$$

$$: (g, s(g)) \longmapsto t(g)$$

of $\mathcal{G}$ on M in THE ABOVE FASHION COINCIDES with

$$t_* : \mathbb{B} \times M \longrightarrow M.$$

$$: (\beta, m) \longmapsto \mathrm{Aim}^{\mathcal{G}}_{\beta(m)}(m)$$

$$\text{Ш}$$

$$t(\beta(m)) \equiv t_*\beta(m).$$

In PARTICULAR,

EXAMPLE 80. FOR $\mathcal{G} \rightrightarrows M = \mathrm{Pair}(M)$, WE OBTAIN

$$\mathrm{Am}^{\mathrm{Pair}(M)} \rightsquigarrow \mathrm{ev}$$

$$((m_2, m_1), m_1) \longmapsto m_2 \qquad ((f, \mathrm{id}_M), m) \longmapsto f(m)$$

$$\wr$$

$$(f, m) \longmapsto f(m).$$