Duality, Descent & Defects I

LECTURE III

2024/25

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PROP. 51. LET $\mathrm{Gr} = (M, \mathcal{G}, s, t, \mathrm{Id}, \mathrm{Inv}, m)$ BE A LIE GROUPOID, & LET $m \in M$ BE ARBITRARY. THE ISOTROPY GROUP $\mathcal{G}_m$ of m ACTS SMOOTHLY, FREELY & PROPERLY from THE RIGHT on THE s-FIBRE $s^{-1}(\{m\})$.

PROOF: THE ACTION of INTEREST IS

$$\underline{P} : s^{-1}(\{m\}) \times \mathcal{G}_m \longrightarrow s^{-1}(\{m\}) : (g, h) \longmapsto m(g, h).$$

THIS MAKES SENSE as $t(h) = m \equiv s(g)$. SMOOTHNESS of $\underline{P}$ IS INHERITED from m. ITS FREENESS IS IMPLIED by

$$\underline{P}_h(g) = g \Longleftrightarrow h = g^{-1} \cdot g = \mathrm{Id}_{s(g)} = \mathrm{Id}_m.$$

FINALLY, CONSIDER A CONVERGENT SEQUENCE $g_\cdot : \mathbb{N} \longrightarrow s^{-1}(\{m\})$, with $g \equiv \lim_{n\to\infty} g_n \in S^{-1}(\{m\})$, & a sequence $h_\cdot : \mathbb{N} \to \mathfrak{g}_m$ s.t. the product sequence $\mu_\cdot : \mathbb{N} \to S^{-1}(\{m\}) : n \mapsto P_{h_n}(g_n)$ converges to $\tilde{\mu} \equiv \lim_{n\to\infty} \mu_n$. Taking into account continuity of $\mathbf{m}$ & $\mathbf{Inv}$ (implied by smoothness of these maps), we establish the identity

$$\lim_{n\to\infty} h_n \equiv \lim_{n\to\infty}\left(\left(g_n^{-1}\cdot g_n\right)\cdot h_n\right) = \lim_{n\to\infty}\left(g_n^{-1}\cdot \mu_n\right) = \left(\lim_{n\to\infty} g_n\right)^{-1}\cdot \lim_{n\to\infty}\mu_n$$

$$= g^{-1}\cdot\tilde{\mu},$$

which documents convergence of $h_\cdot$. $\square$

COR. 52. THE SPACE OF ORBITS $s^{-1}(\{m\})/\mathcal{G}_m$ CARRIES THE STRUCTURE OF A SMOOTH MANIFOLD S.T. THE ORBIT PROJECTION $\pi : s^{-1}(\{m\}) \longrightarrow s^{-1}(\{m\})/\mathcal{G}_m : g \longmapsto g \cdot \mathcal{G}_m$ IS A SURJECTIVE SUBMERSION.

PROOF: FOLLOWS DIRECTLY FROM **THM. 21**. $\square$

DEF. 53. LET $\mathbf{Gr} = (M, \mathcal{G}, s, t, \mathrm{Id}, \mathrm{Inv}, m)$ BE A LIE GROUPOID, & LET $m \in M$ BE ARBITRARY. THE ORBIT OF m IS THE SUBSET

$$\mathcal{G} \triangleright m := t(s^{-1}(\{m\})) \left(\equiv \{ x \in M \mid \exists\, g \in s^{-1}(\{m\}) \cap t^{-1}(\{x\}) \} \right) \subset M.$$

PROP. 54. THE RELATION on M:

$$m_1 \underset{\mathcal{G}}{\sim} m_2 \overset{\text{ex}}{\underset{\text{def}}{\iff}} \exists\, g \in s^{-1}(\{m_1\}) \cap t^{-1}(\{m_2\})$$

IS AN EQUIVALENCE RELATION.

PROOF: TRIVIAL. $\square$

PROP. 55. THE ORBIT of AN ARBITRARY POINT $m \in M$ in THE OBJECT MANIFOLD M of A LIE GROUPOID IS AN IMMERSED SUBMANIFOLD of M.

PROOF: FIX $m \in M$ & CONSIDER A MAP

$$\iota : s^{-1}(\{m\})/\mathcal{G}_m \longrightarrow M : g \cdot \mathcal{G}_m \longmapsto t(g).$$

IT IS MANIFESTLY WELL-DEFINED AS $t(g \cdot h) = t(g)$. MOREOVER,

ι CLOSES THE COMMUTATIVE DIAGRAM

$$\begin{tikzcd} & & M \\ s^{-1}(\{m\}) \arrow[rru, "t"] \arrow[rr, "\pi"'] & & s^{-1}(\{m\})/\mathcal{G}_m \arrow[u, "\iota"'] \end{tikzcd},$$

in WHICH π IS A SURJECTIVE SUBMERSION & t IS SMOOTH. THEREFORE, ι IS SMOOTH in VIRTUE of THE THEOREM on QUASI-UNIVERSALITY of SURJECTIVE SUBMERSIONS.

LET $g_A \in s^{-1}(\{m\}), A \in \{1,2\}$ BE s.t. $t(g_1) = t(g_2)$. THEN, $g_1^{-1} \cdot g_2$ IS WELL-DEFINED, & $s(g_1^{-1} \cdot g_2) = s(g_2) = m \equiv s(g_1) = t(g_1^{-1}) \equiv t(g_1^{-1} \cdot g_2)$, WHICH IMPLIES $g_1^{-1} \cdot g_2 \in \mathcal{G}_m$. THUS, $g_2 \equiv g_1 \cdot (g_1^{-1} \cdot g_2) \in g_1 \cdot \mathcal{G}_m$,

& so $g_1 \cdot G_m = g_2 \cdot G$. WE CONCLUDE THAT ι IS INJECTIVE.

NEXT, NOTE THAT $t\restriction_{s^{-1}(\{m\})} \equiv \iota \circ \pi$, HENCE DUE TO SUBMERSIVITY OF π, WE FIND $\mathrm{rk}\,\iota = \mathrm{rk}\, t\restriction_{s^{-1}(\{m\})} = \mathrm{const}$, i.e., ι IS A SMOOTH INJECTION OF CONSTANT RANK. ITS IMAGE

$$\iota\left(s^{-1}(\{m\})/G_m\right) = t\left(s^{-1}(\{m\})\right) \equiv G\circ m$$

IS AN IMMERSED SUBMANIFOLD OF M IN VIRTUE OF THE GLOBAL-RANK THEOREM [Lee 2012, Thm 5.12]. $\square$

DEF. 56. THE SPACE

$$M/\!\!/\mathcal{G} := \{ \mathcal{G} \triangleright m \mid m \in M \}$$

of ORBITS of A LIE GROUPOID $\mathbf{Gr} = (M, \mathcal{G}, s, t, \mathrm{Id}, \mathrm{Inv}, m)$, ENDOWED with QUOTIENT TOPOLOGY IS CALLED THE ORBISPACE of $\mathbf{Gr}$.

THE DECOMPOSITION of M into CONNECTED COMPONENTS of (PAIRWISE DISJOINT) ORBITS IS CALLED THE CHARACTERISTIC FOLIATION of $\mathbf{Gr}$.

OFF on THE TANGENT:

Definition 57. *Let M be a smooth manifold. A* (**real**) **Lie algebroid over** M (**of rank** $N \in \mathbb{N}^\times$) *is a quintuple* $(\mathcal{E}, M, \mathbb{R}^{\times N}, \pi_{\mathcal{E}}, \alpha_{\mathcal{E}}, [\cdot,\cdot]_{\mathcal{E}})$ *composed of*

- *a vector bundle* $(\mathcal{E}, M, \mathbb{R}^{\times N}, \pi_{\mathcal{E}})$ *(of rank* $N \in \mathbb{N}^\times$*);*
- *a vector-bundle morphism*

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\alpha_{\mathcal{E}}"] \arrow[d, "\pi_{\mathcal{E}}"'] & \mathrm{T}M \arrow[d, "\pi_{\mathrm{T}M}"] \\ M \arrow[r, equal, "\mathrm{id}_M"'] & M \end{tikzcd} ,$$

  termed the **anchor** (**map**)*;*
- *a binary operation* $[\cdot,\cdot]_{\mathcal{E}} : \Gamma(\mathcal{E}) \times \Gamma(\mathcal{E}) \longrightarrow \Gamma(\mathcal{E})$,

satisfying the following conditions:

- $[\cdot,\cdot]_{\mathcal{E}}$ *is a Lie bracket;*
- $\forall_{\varepsilon_1,\varepsilon_2 \in \Gamma(\mathcal{E})} \forall_{f \in C^\infty(M;\mathbb{R})} : [\varepsilon_1, f \triangleright \varepsilon_2]_{\mathcal{E}} = f \triangleright [\varepsilon_1, \varepsilon_2] + \alpha_{\mathcal{E}}(\varepsilon_1)(f) \triangleright \varepsilon_2$ *(the* **Leibniz** PROPERTY*).*

E.g.,

Ex. 58. A LIE ALGEBROID OVER $M = \{*\}$ IS A LIE ALGEBRA.

Example 59. *The* **tangent Lie algebroid of** M *is the canonical structure of a Lie algebroid on the tangent bundle* $\pi_{TM} : TM \longrightarrow M$ *with the identity anchor* $\alpha_{TM} = \mathrm{id}_{TM}$*, and the standard Lie bracket* $[\cdot,\cdot]_{TM}$ *of vector fields on* M.

Ex. 60. GIVEN $M \in \mathrm{Man}$ & $\omega \in \Omega^2(M)$, THERE EXISTS A CANONICAL STRUCTURE of A LIE ALGEBROID on

$$\begin{array}{c} \mathcal{E} = TM \times \mathbb{R} \equiv TM \times_M (M \times \mathbb{R}) \\ \pi_{TM} \circ pr_1 \big\downarrow \\ M \end{array}$$

with $\alpha_{\mathcal{E}} = pr_1$

$$[(X,f),(Y,g)] = \left([X,Y]_{\Gamma(TM)}, X(g) - Y(f) + \omega(X,Y)\right)$$

$X, Y \in \Gamma(TM)$; $f, g \in C^\infty(M;\mathbb{R})$

IFF $d\omega = 0$.

Ex. 61. THE LIE ALGEBROID of THE LIE GROUPOID, WHICH WE DISCUSS BELOW.

Prop. 62. IN EVERY LIE ALGEBROID, THE ANCHOR INDUCES A LIE-ALGEBRA HOMOMORPHISM on SECTIONS.

Proof: CONSIDER ARBITRARY $X, Y, Z \in \Gamma(\mathcal{E})$ & $f \in C^\infty(M, \mathbb{R})$. WE CALCULATE, with the HELP of THE JACOBI & LEIBNIZ IDENTITIES,

$$[[X,Y]_\mathcal{E}, f \triangleright Z]_\mathcal{E} \overset{(L)}{=} f \triangleright [[X,Y]_\mathcal{E}, Z] + \alpha_\mathcal{E}([X,Y]_\mathcal{E})(f) \triangleright Z$$

$$(J) \parallel$$

$$[[X, f \triangleright Z]_\mathcal{E}, Y]_\mathcal{E} - [[Y, f \triangleright Z]_\mathcal{E}, X]_\mathcal{E}$$

$$\overset{L}{=} [f \triangleright [X,Z]_\mathcal{E} + \alpha_\mathcal{E}(X)(f) \triangleright Z, Y]_\mathcal{E} - [f \triangleright [Y,Z]_\mathcal{E} + \alpha_\mathcal{E}(Y)(f) \triangleright Z, X]_\mathcal{E}$$

$$= f \triangleright \left([[X,Z]_\mathcal{E},Y]_\mathcal{E} - [[Y,Z]_\mathcal{E},X]_\mathcal{E} \right) - \alpha_\mathcal{E}(Y)(f) \triangleright [X,Z]_\mathcal{E} + \alpha_\mathcal{E}(X)(f) \triangleright [Z,Y]_\mathcal{E}$$

$$- \alpha_\mathcal{E}(Y) \circ \alpha_\mathcal{E}(X)(f) \triangleright Z + \alpha_\mathcal{E}(X)(f) \triangleright [Y,Z]_\mathcal{E} - \alpha_\mathcal{E}(Y)(f) \triangleright [Z,X]_\mathcal{E}$$

$$+ \alpha_\mathcal{E}(X) \circ \alpha_\mathcal{E}(Y)(f) \triangleright Z ,$$ WHENCE

$$\left(\alpha_\mathcal{E}([X,Y]_\mathcal{E}) - [\alpha_\mathcal{E}(X), \alpha_\mathcal{E}(Y)]_{\Gamma(TM)} \right)(f) \triangleright Z = 0 .$$

In VIEW of ARBITRARINESS of Z & f, WE THUS OBTAIN THE DESIRED IDENTITY

$$\alpha_\mathcal{E}([X,Y]_\mathcal{E}) - [\alpha_\mathcal{E}(X), \alpha_\mathcal{E}(Y)]_{\Gamma(TM)} = 0$$

$$\forall\, X,Y \in \Gamma(\mathcal{E}) . \qquad \square$$

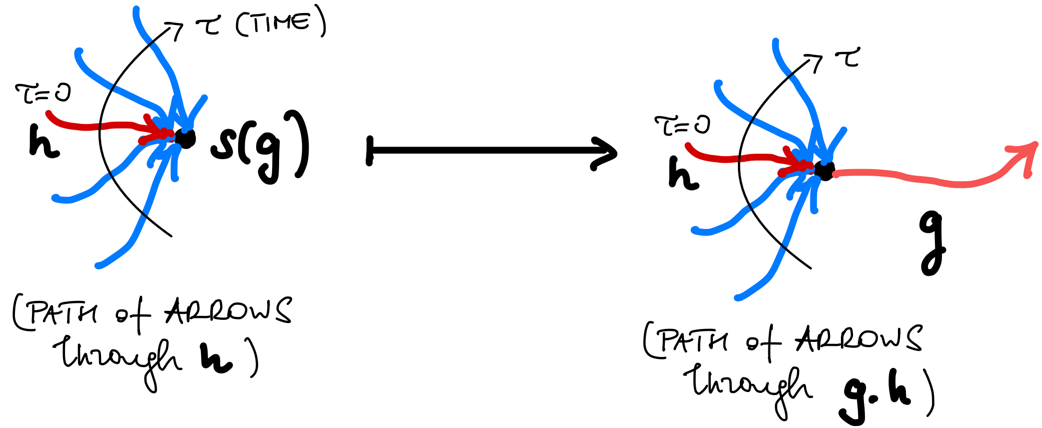
WE ARE NOW READY to APPROACH THE QUESTION of A DIFFERENTIAL CALCULUS on $\mathcal{G}$ COMPATIBLE with, SAY, LEFT-TRANSLATIONS of **DEF. 31**. RECALL

$$l_g : t^{-1}(\{s(g)\}) \longrightarrow t^{-1}(\{t(g)\})$$

to CONCLUDE THAT THE ONLY WAY to MAKE SENSE of THE NOTION of LEFT-INVARIANCE of A VECTOR FIELD on $\mathcal{G}$ IS to TAKE IT from THE DISTRIBUTION TANGENT to t-FIBRES. ACTUALLY, in VIRTUE of SUBMERSIVITY of t (IMPLYING $\mathrm{rk}\, t \equiv \dim M = \mathrm{const}$), THE DISTRIBUTION is REGULAR (by THE CONSTANT-RANK THEOREM for VECTOR BUNDLES), i.e., IT IS A VECTOR SUBBUNDLE $\mathrm{Ker}\, Tt \subset T\mathcal{G}$.

WE HAVE $T_h \ell_g : (\operatorname{Ker} Tt)_h \xrightarrow{\simeq} (\operatorname{Ker} Tt)_{g\cdot h}$

$t(h) = s(g)$



DEF. 63. LET $Gr = (M, \mathcal{G}, s, t, Id, Inv, m)$ BE A LIE GROUPOID.

A LEFT-INVARIANT VECTOR FIELD on $\mathcal{G}$ IS A SECTION

$$\mathcal{L} \in \Gamma(\operatorname{Ker} Tt) \subset \Gamma(T\mathcal{G})$$

with THE PROPERTY $\forall g \in \mathcal{G}\ \forall h \in t^{-1}(\{s(g)\}) : T_h \ell_g (\mathcal{L}(h)) = \mathcal{L}(g\cdot h)$.