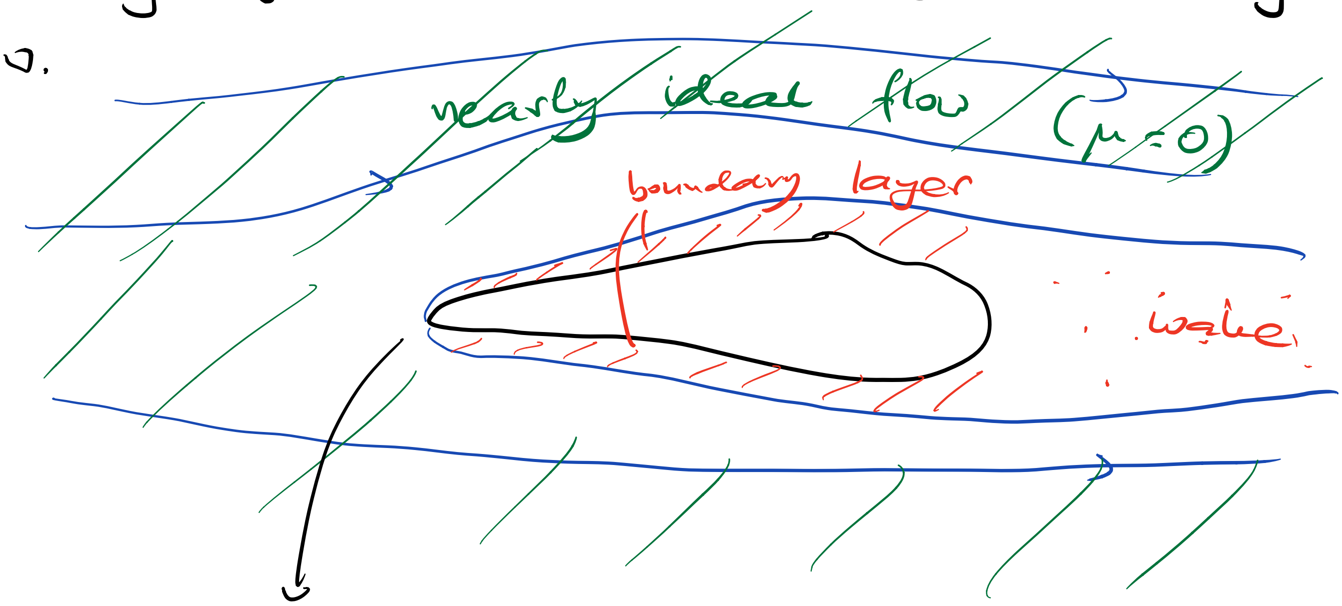
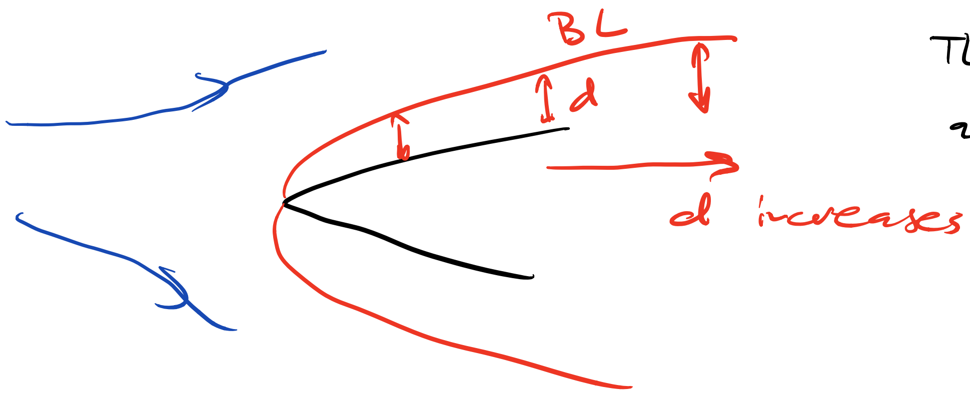
THE EKMAN LAYER

Boundary layers arise near a body in a nearly ideal flow.

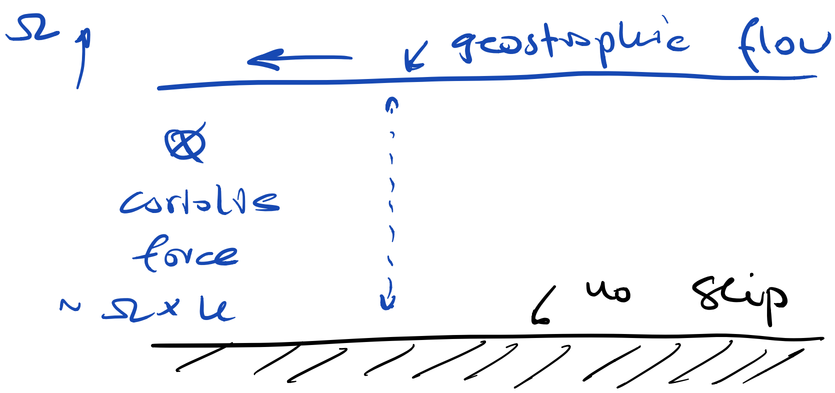




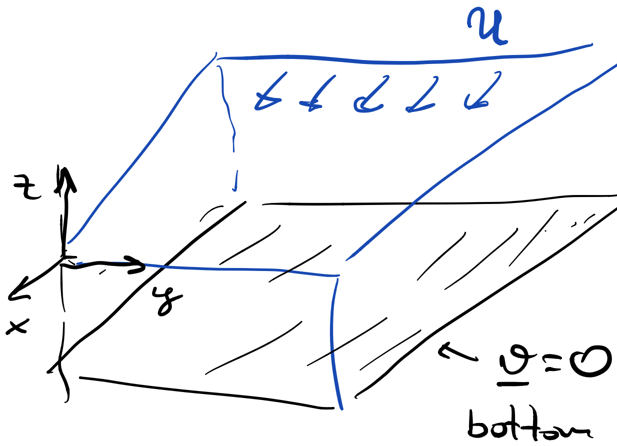
The B.L. thickens along the flow direction.

Ekman boundary layer in a rotating system

- Fluid in geostrophic flow
- $Ro = \dfrac{u}{2L\Omega} \lesssim 1$ Coriolis force important for the formation of boundary layers



Ekman layer solution



Top boundary:
geostrophic flow

$$-2\,\underline{\Omega} \times \underline{v} = \frac{1}{\rho}\nabla p^*$$

with

$$v_x = \mathcal{U} \;;\; v_y = 0$$

$$p^* = -2\rho\,\Omega\,\mathcal{U}\,y \qquad \text{(like last week)}$$

Bottom boundary:

$$\underline{v} = 0 \qquad \text{(no slip)}$$

Symmetries:

- solution invariant in x, y
- we are looking for a solution $\underline{v} = \underline{v}(z)$

Mass conservation: $\dfrac{\partial v_z}{\partial z} = 0 \;\rightarrow\; v_z = \text{const} = 0$
because $v_z(0) = 0$

- The transition flow is horizontal and independent of x, y but varies with z.

- N-S equations simplify because

$$(\underline{u}\cdot\underline{\nabla})\,\underline{u} = \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y}\right)\underline{v}(z) = 0 \quad \text{vanishes}$$

we include viscosity & the Coriolis force

- steady case

N-S equations $\quad 2\underline{\Omega}\times\underline{v} - \frac{1}{\rho}\nabla p^* + \nu\nabla^2\underline{v} = 0$

$\left(\underline{\Omega} = \Omega\,\hat{\underline{e}}_z\right) \qquad\qquad \nabla^2 \to \nabla_z^2$

components:

$$\begin{cases} 0 = 2\Omega v_y - \frac{1}{\rho}\frac{\partial p^*}{\partial x} + \nu\frac{\partial^2}{\partial z^2}v_x \\ 0 = -2\Omega v_x - \frac{1}{\rho}\frac{\partial p^*}{\partial y} + \nu\frac{\partial^2}{\partial z^2}v_y \\ 0 = -\frac{1}{\rho}\frac{\partial p^*}{\partial z} \end{cases}$$

$\Rightarrow p^*$ independent of z, so equal to $p^* = -2\rho\Omega U y$
(top boundary)

The eqs. of motion:

$$\begin{cases} \nu\frac{\partial^2}{\partial z^2}v_x = -2\Omega v_y \\ \nu\frac{\partial^2}{\partial z^2}v_y = -2\Omega(U - v_x) \end{cases} \qquad v_y = -\frac{\nu}{2\Omega}\partial_z^2 v_x$$

$$\frac{\partial^4}{\partial z^4}(U - v_x) = -\frac{4\Omega^2}{\nu^2}(U - v_x)$$

$\to$ solution: linear combination of 4 terms with e^{kz}

$$k^4 = -\frac{4\Omega^2}{\nu^2}$$

Define $\quad \delta = \sqrt{\frac{\nu}{\Omega}} \qquad\qquad \delta \sim$ length

$$\kappa^4 = -\frac{4}{\delta^4} \qquad \kappa = \pm\frac{1 \pm i}{\delta}$$

$\mathrm{Re}(\kappa) > 0$ no good

only $\kappa = -\frac{1+i}{\delta}$ and $\kappa = -\frac{1-i}{\delta}$

The general solution is

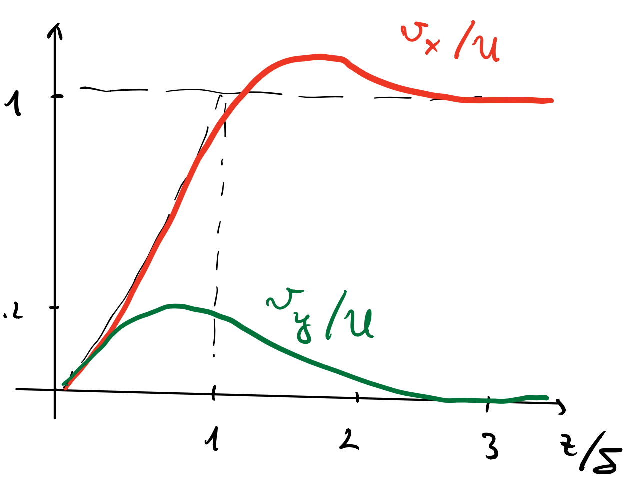
$$\begin{cases} U - v_x = A e^{-(1+i)z/\delta} + B e^{-(1-i)z/\delta} \\ v_y = i\left(A e^{-(1+i)z/\delta} - B e^{-(1-i)z/\delta}\right) \end{cases}$$

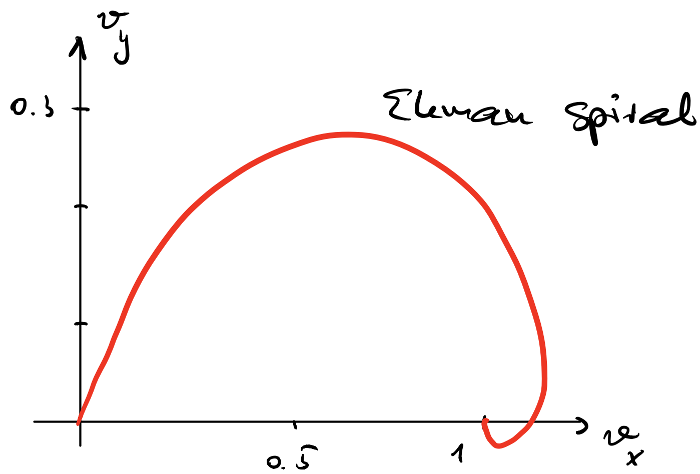
$v_x(z=0) = v_y(z=0) = 0$, so $A = B = \frac{U}{2}$

The final solution:

$$\begin{cases} v_x = U\left(1 - e^{-z/\delta}\cos z/\delta\right) \\ v_y = U e^{-z/\delta}\sin z/\delta \end{cases}$$

- δ is a measure of the thickness of the Ekman layer.
- δ independent of U.





At middle latitudes $\delta \sim 55$ cm for $\nu = 1.5 \cdot 10^{-5} \frac{m^2}{s}$
(for the atmosphere)

Measured thickness ~ 1 km

why? Atmosphere is turbulent (not laminar)
and effective viscosity can be $\sim 10^6 \nu$

Ref: J. Pedlosky, Geophysical Fluid Dynamics, Springer 1987

Ekman upwelling & suction

If $u = (u_x, u_y, 0)$

the solution becomes:

$$(*) \quad \begin{cases} v_x = u_x\left(1 - e^{-z/\delta} \cos z/\delta\right) - u_y e^{-z/\delta} \sin z/\delta \\ v_y = u_y\left(1 - e^{-z/\delta} \cos z/\delta\right) + u_x e^{-z/\delta} \sin z/\delta \end{cases}$$

If the velocity components $u_x(x,y)$ and $u_y(x,y)$
change slowly with x and y on a large scale $L \gg \delta$
(*) is still valid because δ is independent of u.

- Slowly varying geostrophic flow generates a non-zero vertical flow $u_z(x,y)$.

$$\underline{\nabla} \cdot \underline{v} = \partial_x v_x + \partial_y v_y + \partial_z v_z = 0$$

$$\Rightarrow \partial_z v_z = -(\partial_x v_x + \partial_y v_y) = \underbrace{(\partial_x u_y - \partial_y u_x)}_{\omega_z \text{ geostrophic flow vorticity}} e^{-z/\delta} \sin z/\delta.$$

We have used the fact that $\partial_x u_x + \partial_y u_y = 0$.

Integrating over z with $v_z = 0$ for $z = 0$, we get:

$$v_z = \frac{1}{2}\delta\left(\partial_x u_y - \partial_y u_x\right)\left[1 - e^{-z/\delta}\left(\cos z/\delta + \sin z/\delta\right)\right]$$

Note $\quad v_z \sim \dfrac{\delta U}{L} \ll U$ because $\dfrac{\delta}{L} \ll 1$

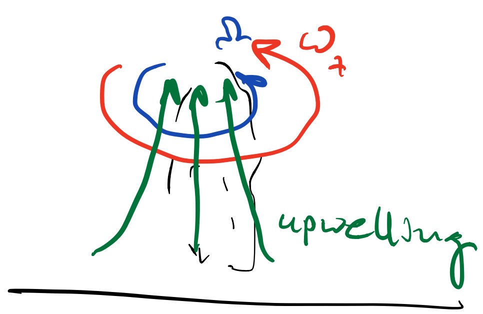
For $z \gg \delta$ there remains a vertical component

$$u_z = \frac{1}{2}\delta\,\omega_z$$

Note u_z is independent of z (Taylor-Proudman)

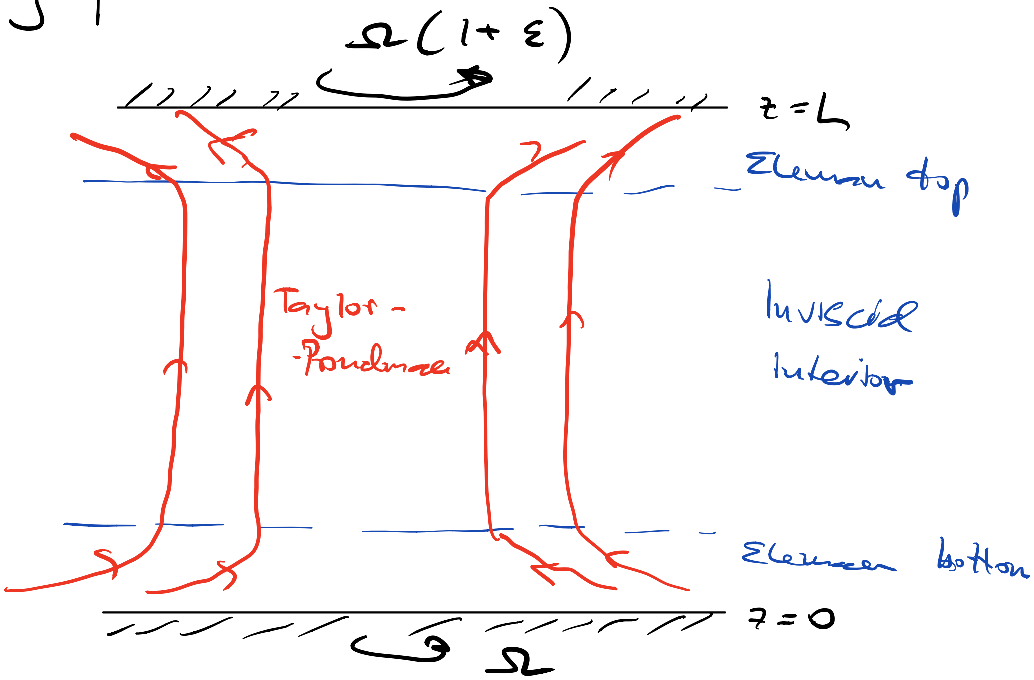
Note If $\omega_z > 0$ (of the same sign as global rotation Ω) the fluid wells up from the Ekman layer

Ex: low p cyclone



§§ Flow in differentially rotating boundaries

Two rotating plates



Three regions:

A) Boundary layers (top & bottom)

B) Interior flow (approx. ideal fluid)

B) Interior flow

(x,y,t) $\quad \underline{\Omega} = (0,0,\Omega)$ $\quad \underline{v} = (v_x, v_y, v_z)$

Geostrophic flow:

$$2\Omega v_y = \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$-2\Omega v_x = \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$0 = \frac{1}{\rho}\frac{\partial p}{\partial z} \quad \rightarrow \quad \frac{\partial v_z}{\partial z} = 0$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

$\underline{v}$ indep. of z.
(Taylor-Proudman)

① Bottom & top

Consider first the b.l. at $z=0$. N-S eq. take the form: with velocity field $\underline{u} = (u_x, u_y, u_z)$

$$-2\Omega u_y = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u_x}{\partial z^2}$$

$$2\Omega u_x = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \frac{\partial^2 u_y}{\partial z^2}$$

we assumed faster variations in z than in x, y

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \frac{\partial^2 u_z}{\partial z^2}$$

$$\frac{\partial}{\partial x}u_x + \frac{\partial}{\partial y}u_y + \frac{\partial}{\partial z}u_z = 0$$

u_z is much smaller than the horizontal velocity.

Repeating previous arguments, we can show

$$p = p(x,y)$$

so $\frac{\partial p}{\partial x}$ & $\frac{\partial p}{\partial y}$ same as for the interior flow,

We get:

$$-2\Omega(\underbrace{u_y}_{\text{B.L}} - \underbrace{v_y}_{\text{int.}}) = \nu \frac{\partial^2 u_x}{\partial z^2}$$

$$2\Omega(u_x - v_x) = \nu \frac{\partial^2 u_x}{\partial z^2}$$

They can be integrated. → Ekman profile

Vertical flow component at the edge of the Ekman layer:

$$u_z^E = \frac{1}{2}\delta\left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x}\right) = \frac{1}{2}\delta\,\omega_I$$

interior flow vorticity

If now the boundary is rotating with Ω_B with respect to the rotating frame, this generalizes to

$$u_z^E = \delta\left(\frac{1}{2}\omega_I - \Omega_B\right) \qquad (*)$$

If the upper boundary at $z=L$ has the ang. velocity Ω_T rel. to the rotating frame, there we have

$$u_z^E(x,y) = \delta\left(\Omega_T - \frac{1}{2}\omega_I\right) \qquad (*)$$

Full interior flow

T-P theorem $\qquad \frac{1}{2}\omega_I - \Omega_B = \Omega_T - \frac{1}{2}\omega_I$

That is $\qquad \omega_I = \Omega_T + \Omega_B$

In our case: $\qquad \Omega_B = 0 \qquad \Omega_T = \Omega\varepsilon$

we get: $\qquad \omega_I = \frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} = \Omega\varepsilon.$

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) = \Omega\varepsilon \qquad \text{in polar coordinates.}$$

$$\Rightarrow \quad v_\theta = \frac{1}{2}\Omega\varepsilon r$$

The vertical velocity:

$$v_z = \frac{1}{2}\sqrt{\nu\Omega}\,\varepsilon$$

From incompressibility also $v_r = 0$.