Sound Waves

Before : water waves
- ideal fluid
- potential flow $\quad \underline{v} = \nabla \phi$

implies $\underline{\nabla} \cdot \underline{v} = 0$

Now we take into account compressibility

$$\underline{\nabla} \cdot \underline{v} \neq 0$$

Euler's equation still holds:

$$\rho \left(\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla) \underline{u} \right) = - \nabla p \qquad (1)$$

Density $\rho = \rho(\underline{x}, t)$ is a variable.

Mass conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0 \qquad (2)$$

$$\hookrightarrow \underbrace{\frac{\partial \rho}{\partial t}} + \rho (\underline{\nabla} \cdot \underline{u}) + \underbrace{\underline{u} \cdot \nabla \rho} = 0$$

$$\frac{D\rho}{Dt} = - \rho (\nabla \cdot \underline{u})$$

We need the ρ vs p relationship
(thermodynamics)

Assume : · ideal gas
· slow heat conduction in the fluid

Then $p v^{\gamma} = \text{const}$ $\quad \gamma = c_p / c_v = 1.4$
for a fluid element at usual T, p

$$\frac{D}{Dt}\left(p \rho^{-\gamma}\right) = 0 \qquad (3)$$

Eqs (1) – (3) are the basis.

<u>Small-amplitude sound waves</u>

Undisturbed fluid (p_0, ρ_0) at rest $\underline{u} = 0$.
Trivial solution.

Consider a disturbance :

$$\underline{u} = \underline{u}_1 \,, \quad p = p_0 + p_1 \,, \quad \rho = \rho_0 + \rho_1$$

Perturbations are <u>small</u>, we aim to <u>linearise</u>.

$$p \rho^{-\gamma} = \text{const.} = p_0 \rho_0^{-\gamma}$$
(in the beginning

$$(p_0 + p_1)(\rho_0 + \rho_1)^{-\gamma} = p_0 \rho_0^{-\gamma}$$

$$\hookrightarrow \quad \left(1 + \frac{p_1}{p_0}\right)\left(1 + \frac{\rho_1}{\rho_0}\right)^{-\gamma} = 1$$

$$\left(1 + \frac{p_1}{p_0}\right)\left(1 - \gamma \frac{\rho_1}{\rho_0} + \underbrace{\ldots}_{\text{neglect.}}\right) = 1$$

we find:

$$\frac{p_1}{p_0} = \gamma \frac{\rho_1}{\rho_0}$$

We define $\quad c^2 = \dfrac{\gamma p_0}{\rho_0}$

really $\quad c^2 = \left(\dfrac{\partial p}{\partial \rho}\right)_s$

$$\boxed{p_1 = c^2 \rho_1}$$

Euler's equation

$$(\rho_0 + \cancel{\rho_1}^{\,0})\left(\frac{\partial \underline{u}_1}{\partial t} + \cancel{(\underline{u}_1 \cdot \nabla)\underline{u}_1}^{\,0}\right) = -\nabla(\cancel{p_0}^{\,0} + p_1)$$

$\sim u_1^2$

$$\rho_0 \frac{\partial \underline{u}_1}{\partial t} = -\nabla p_1 \qquad \text{(linearised Euler eq.)}$$

Mass conservation

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \underline{u}_1 = 0 \qquad \text{(linearised)}$$

Take $\nabla \cdot$(Euler eq.)

$$\nabla \cdot \left(\rho_0 \frac{\partial \underline{u}_1}{\partial t} \right) = \nabla \cdot \left(- \nabla p_1 \right)$$

$$\rho_0 \frac{\partial}{\partial t} \left(\nabla \cdot \underline{u}_1 \right) = - \nabla^2 p_1$$

$$\nabla \cdot \underline{u}_1 = \frac{1}{\rho_0} \frac{\partial \rho_1}{\partial t} \quad \text{(M.C.)}$$

$$p_1 = c^2 \rho_1$$

$$\boxed{\frac{\partial^2 p_1}{\partial t^2} = c^2 \nabla^2 p_1}$$

3D Wave equation for the pressure

For 1D waves

$$\frac{\partial^2 p_1}{\partial t^2} = c^2 \frac{\partial^2 p_1}{\partial x^2}$$

general solution:

$$p_1 = f(x - ct) + g(x + ct)$$

waves propagating without a change of shape.

- Sound waves are <u>non-dispersive</u>. $c = \text{const.}$

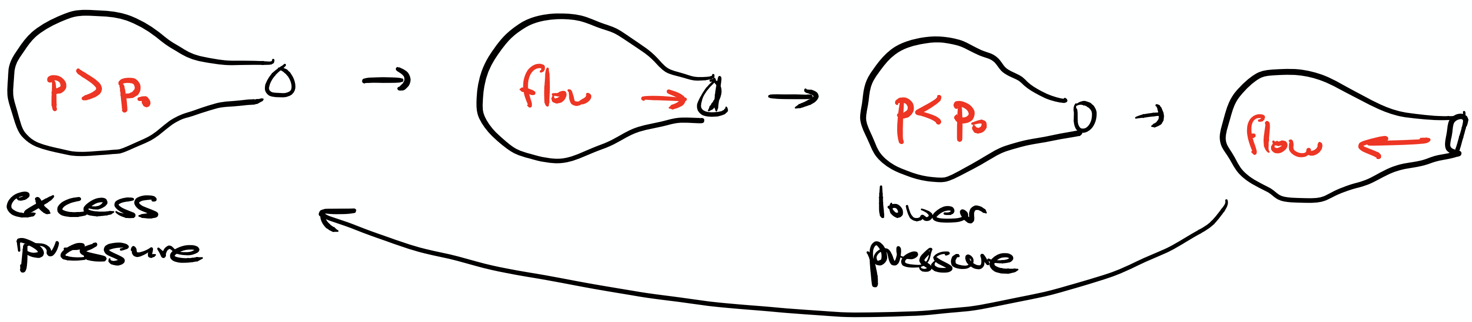
$$c = \sqrt{\frac{\gamma p_0}{\rho_0}} \simeq 340 \frac{m}{s} \quad \text{at } p_{atm},\ T = 20°C$$

- In spherical geometry $p_1 = \frac{1}{r} F(r - ct)$.

Resonance and cavities

(Helmholtz resonator)

Consider a bottle or a cavity with a thin neck

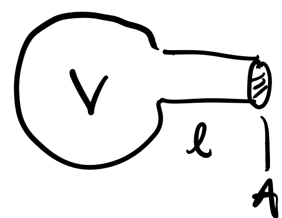


Fully reversible process (lack of dissipation & inertia) with a period T.

→ Viscous attenuation of sound waves.

How long does it take?

Parameters:



* Δp excess pressure
* v peak air velocity in the tube
* p_0 atmospheric pressure
* ρ air density

* A area of the neck
* l length of the neck
* V volume of the cavity

Let's estimate the period.

(1) How long does it take for the air to speed up?

The force on the air in the neck

$$F = A\,\Delta p = A(p - p_0)$$

so the acceleration

$$a = \frac{F}{m} = \frac{\Delta p\, A}{\rho\, l A} = \frac{\Delta p}{\rho l}$$

so the velocity after a time t

$$v = \frac{\Delta p}{\rho l}\, t$$

so the volume change

$$\Delta V = A v t = \frac{A\,\Delta p}{\rho l}\, t^2$$

How much time does the air take for the pressure to fall to atmospheric p?

$$\frac{\Delta p}{p_0} = \frac{\Delta V}{V}$$

$$t^2 = \frac{\rho l\,\Delta V}{A\,\Delta p} = \frac{\rho l\;\Delta V}{A p_0\,\Delta V}\, V = \frac{\rho l}{A p_0}\, V$$

but $c^2 = \gamma \frac{p_0}{\rho}$; $c^2 \approx \frac{p_0}{\rho}$

$t^2 = \frac{l V}{A c^2}$ is $\frac{1}{4}$ of the resonance time

Therefore

$$T = 4\sqrt{\frac{\ell V}{A c^2}}$$

is under-estimated

(we assumed Δp = const. α = const.)

The actual answer is

$$T = 2\pi\sqrt{\frac{\ell V}{A c^2}}$$

A : wider neck → faster flow → lower T

ℓ : longer neck → longer to get the air moving

V : larger volume : more air has to leave before p falls

c : faster speed → quicker response

The general procedure for finding resonance frequencies in a cavity:

- assume sinusoidal changes of pressure

Basic linearised eqs.

$$\frac{\partial^2 p}{\partial t^2} = c^2 \nabla^2 p \quad \leftarrow \quad \begin{cases} \dfrac{D\underline{u}}{Dt} = \dfrac{1}{\rho}\nabla p \\[2ex] \dfrac{Dp}{Dt} = \gamma p_0 (\underline{\nabla}\cdot\underline{u}) \end{cases}$$

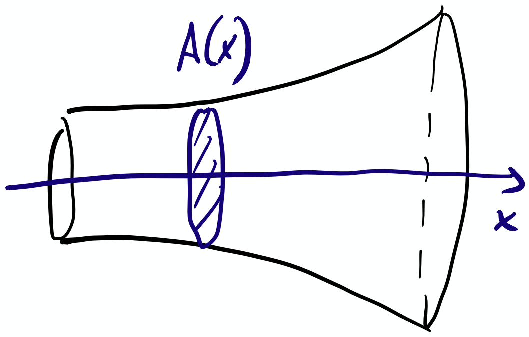
look for solutions that satisfy

$$\boxed{\nabla^2 p = -\frac{\omega^2}{c^2} p}$$

with a b.c. of $p=0$ at the openings.

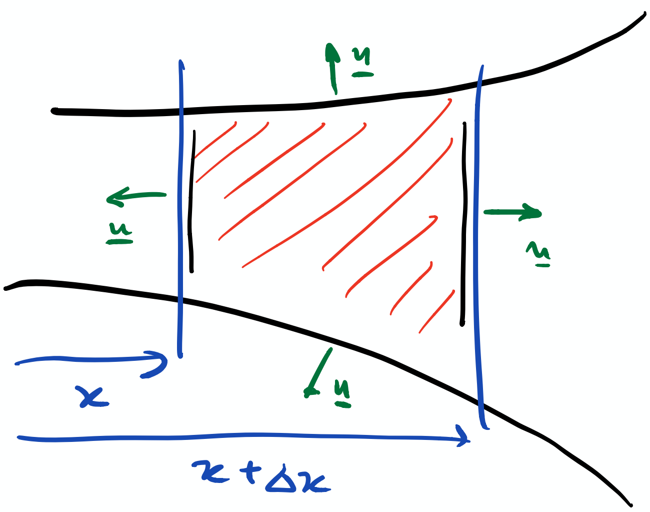
$\rightarrow$ discrete set of resonance frequencies ω.

Webster Horn Equation



Quasi-1D model of sound propagation in a rigid-walled duct of variable cross-sectional area $A(x)$

Quasi 1D $\quad p = p(x,t)$



Wave equation for the acoustic pressure over the volume

$$\int \frac{\partial^2}{\partial t^2} p \, dV = \int c^2 \frac{\partial^2}{\partial x^2} p \, dV$$

Gauss' theorem $\quad \int dV \, (\nabla \cdot \underline{f}) = \int dS \, (\underline{f} \cdot \underline{n})$

$$\frac{\partial^2}{\partial t^2} \int p \, dV - c^2 \int \frac{\partial}{\partial x} \cdot \frac{\partial p}{\partial x} \, dV = \frac{\partial^2}{\partial t^2} \int dV \, p - c^2 \int dS \, (\nabla p \cdot \underline{n})$$

on the walls: $\quad \underline{\nabla} p \cdot \underline{n} = 0$

$$c^2 \left(\int ds \, \frac{\partial p}{\partial x} \Big|_{x+\Delta x} - \int ds \, \frac{\partial p}{\partial x} \Big|_{x} \right) = \frac{\partial^2}{\partial t^2} \int ds \int dx \, p$$

$$c^2 \Delta x \frac{\partial}{\partial x} \int ds \frac{\partial p}{\partial x} = \frac{\partial^2}{\partial t^2} \int ds \; p \Delta x$$

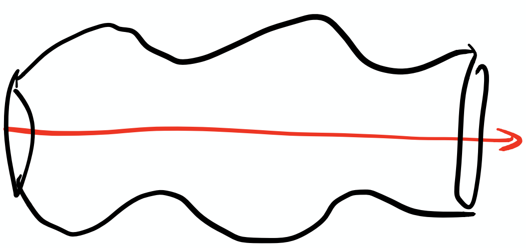
In the limit of $\Delta x \to 0$, we get

$$\frac{\partial}{\partial x} \int ds \frac{\partial p}{\partial x} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \int ds \, p = 0$$

If p is uniform over a cross-section, we get the Webster horn equation:

$$\frac{1}{A(x)} \frac{\partial}{\partial x} \left(A(x) \frac{\partial p}{\partial x} \right) - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0$$

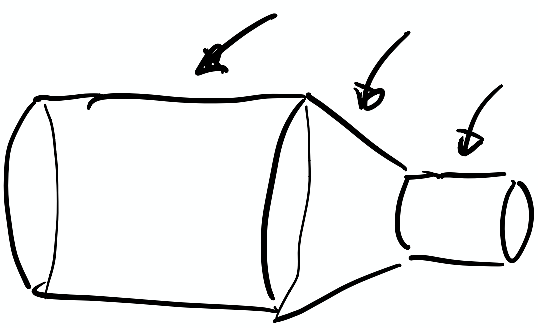
$$\left\{ \frac{\partial^2}{\partial x^2} + \frac{1}{A} \frac{\partial A}{\partial x} \frac{\partial}{\partial x} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right\} p = 0$$



$\to p$

If the velocity is needed, we use the Euler's eq.

$$\rho \frac{\partial v_x}{\partial t} = - \frac{\partial p}{\partial x}$$



$\longrightarrow \omega \, ?$

$$\frac{1}{A}\frac{\partial A}{\partial x} = \alpha = \text{const.}$$

$$\frac{\partial A}{\partial x} = \alpha A$$

$$\left\{ \partial_x^2 + \partial_x - \frac{1}{c^2}\partial_t^2 \right\} p = 0$$

$$\underline{A = B e^{\alpha x}}$$

(Exponential horns)