Lasers
lecture 3

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from lecture 2 we have

1. photon flux $F \equiv \frac{I}{\hbar \omega} \left[ \frac{1}{\text{cm}^2 \text{s}} \right]$ evolves according to:

$$\frac{\partial}{\partial z} F = \sigma_{12}(\omega) \cdot \Delta N \cdot F$$

2. the populations inversion $\Delta N \left[ \frac{1}{\text{cm}^3} \right]$ evolution is governed by:

$$\frac{\partial}{\partial t} \Delta N = -\frac{\Delta N}{T_1} - 2\sigma_{12} \cdot \Delta N \cdot F$$

**Note 1:** in more realistic models we will go beyond the two-level model and the second equation will be modified accordingly. In systems with a short lifetime of the lower level (the most common case) the factor 2 is missing.

**Note 2:** we switch from 0 to 12 when indexing the cross-section. From now on we will use $\sigma_{12}$ to signify a typical energy level system with 1 and 2 being the lower and upper level of laser transition.

2 variables and 2 first order differential equations. The problem is that the equations are nonlinear – there are no general analytic solutions.

**Options:**
- numerical integration
- approximate solutions
Note that, for the amplifier to work, we need some initial population inversion $\Delta N_0$. This modifies the second equation which now reads (time and space dependence are dropped for clarity):

$$\frac{\partial}{\partial t} \Delta N(t, z) = -\frac{\Delta N(t, z)}{T_1} - 2\sigma_{12} \Delta N(t, z) F(t, z)$$

We formally transform the two equations as follows: From the first one we calculate $\Delta N = \frac{\partial F}{\partial z} / (\sigma_{01} F)$ and insert it into the second equation:

$$\frac{\partial^2}{\partial t \partial z} \ln F + \sigma_{12} \frac{\partial}{\partial z} F + \frac{1}{T_1} \left( \frac{\partial}{\partial z} \ln F - \sigma_{12} \Delta N_0 \right) = 0$$

There are 2 characteristic time scales involved in this problem: (1) the population decay time $T_1$ and (2) light pulse duration ($t_p$). We will attempt to solve those equations in two limiting cases.
„short” pulse laser amplifier

„short” pulses ($t_p \ll T_1$). They actually cannot be too short – we have previously neglected the transverse relaxation time so we need $t_p \gg T_2$

$$\frac{\partial^2}{\partial t \partial z} \ln F + 2\sigma_{12} \frac{\partial}{\partial z} F + \frac{1}{T_1} \left( \frac{\partial}{\partial z} \ln F - \sigma_{01} \Delta N_0 \right) = 0$$

this term is small; we will drop it

solutions

$$F(z, t) = \frac{F(0, t)}{1 - e^{-S(t)} \left( 1 - e^{-G(z)} \right)}$$

$$\Delta N(z, t) = \frac{\Delta N(z, 0) e^{-G(z)}}{e^{S(t)} + e^{-G(z)} - 1}$$

$$S(t) = 2\sigma_{12} \int_{-\infty}^{t} F(0, t') dt'$$

$$G(z) = \sigma_{12} \int_{0}^{z} \Delta N(z', 0) dz'$$

"short" pulse laser amplifier, 2

Gain saturation leads to pulse shaping.

\[
F(z, t) = \frac{F(0, t)}{1 - e^{-S(t)}(1 - e^{-G(z)})}
\]

\[
\Delta N(z, t) = \frac{\Delta N(z, 0)e^{-G(z)}}{e^{S(t)} + e^{-G(z)} - 1}
\]

\[
S(t) = 2\sigma_{12} \int_{-\infty}^{t} F(0, t') dt'
\]

\[
G(z) = \sigma_{12} \int_{0}^{z} \Delta N(z', 0) dz'
\]
„long” amplified pulses ($\tau_p \gg T_1, T_2$).

\[
\frac{\partial^2}{\partial t \partial z} \ln F + \sigma_{12} \frac{\partial}{\partial z} F + \frac{1}{T_1} \left( \frac{\partial}{\partial z} \ln F - \sigma_{12} \Delta N_0 \right) = 0
\]

small – we drop it

\[
\frac{\partial}{\partial z} F = \frac{\gamma_0 F}{1 + F/F_s}
\]

the formal solution is

\[
\ln \frac{F(z, t)}{F(0, t)} + \frac{F(z, t) - F(0, t)}{F_s} = \gamma_0 z
\]

two limits:

\[
I \ll I_s \quad \Rightarrow \quad F(z, t) = e^{\gamma_0 z} F(0, t)
\]
unsaturated laser amplifier

\[
I \gg I_s \quad \Rightarrow \quad F(z, t) = F(0, t)
\]
completely saturated laser amplifier

\[
\gamma_0 = \sigma_{12} \Delta N_0 \quad - \text{unsaturated gain coefficient}
\]
\[
F_s \equiv \frac{1}{T_1 \sigma_{12}} \quad - \text{saturating photon flux} \left[ \frac{1}{\text{s} \cdot \text{cm}^2} \right]
\]
\[
I_s = \hbar \omega F_s \quad - \text{saturating intensity} \left[ \frac{\text{W}}{\text{cm}^2} \right]
\]

Note: no factor 2 in this eq. - explanation will be given later.
\[ \ln \frac{F(z, t)}{F(0, t)} + \frac{F(z, t) - F(0, t)}{F_s} = \gamma_0 z \]

\[ F_s = (\sigma_0 T_1)^{-1} \]

\[ F(z, t) = e^{\gamma_0 z} F(0, t) \]
„intermediate” pulses ($t_p \cong T_1$). The equation has to be integrated numerically in its full splendor

$$\frac{\partial^2}{\partial t \partial z} \ln F + 2\sigma_{01} \frac{\partial}{\partial z} F + \frac{1}{T_1} \left( \frac{\partial}{\partial z} \ln F - \sigma_{01} \Delta N_0 \right) = 0$$
a simpler picture for light-matter interaction; Einstein coefficients

a two-level atom/ion. There are 3 radiative transitions:

a. spontaneous emission
b. absorption
c. stimulated emission

populations: \( N_1 + N_2 = N \)

Simple properties of the radiative transitions:

- spontaneous emission

\[
\frac{dN_2}{dt} = -A_{21} N_2, \quad A_{21} \text{ is a constant (coefficient)}
\]
\[ \frac{dN_2}{dt} = -\frac{dN_1}{dt} = B_{12} \varrho(\omega_{12})N_1 \quad B_{12} \quad \text{coefficient,} \quad \varrho(\omega_{12}) \quad \text{power density of em filed} \]

\[ \frac{dN_1}{dt} = -\frac{dN_2}{dt} = B_{21} \varrho(\omega_{12})N_2 \]

**Absorption**

**Stimulated Emission**
relations between Einstein coefficients

\[ B_{21} = B_{12} \]

\[ A_{21} = \frac{\hbar \omega_{21}^3}{\pi^3 c^3} B_{21} \]

population evolution:

\[ \frac{dN_2}{dt} = -A_{21} N_2 + B_{21} q(\omega_{21})(N_1 - N_2) \]

\[ N_1 + N_2 = N \]

consequences:

- the same speed of stimulated transitions
- stimulated transitions dominate at low frequencies
- at high frequencies the spontaneous emission dominates

\[ q_{cr} \left[ \frac{1}{m^3 \text{Hz}} \right] \] - critical spectral density and critical intensity of the em field:

\[ A_{21} = B_{21} q_{cr}(\omega) = \frac{\hbar \omega_{21}^3}{\pi^3 c^3} B_{21} \Rightarrow q_{cr}(\omega) = \frac{\hbar \omega_{21}^3}{\pi^3 c^3} \]

\[ I(\omega) = c \, q_\omega(\omega) d\omega \Rightarrow I_{cr}(\omega) = \frac{\hbar \omega_{21}^3}{\pi^3 c^2} d\omega \]

in vacuum
energy transport equation

\[ I(z, t) = q_{em}(z, t) d\omega \cdot v_g \]

spectral density of em field \[ \text{[J/m}^3\text{Hz]} \]  
group velocity \[ v_g \equiv \frac{d\omega}{dk} \]

em field propagates in the \( z \) direction. Consider a slice with area \( S \) and thickness \( dz \). Em field energy change within the slice:

\[
\frac{d\rho}{dt} d\omega S dz = \left[ I(z, t) - I(z + dz, t) \right] S + \hbar \omega B_{21} q d\omega \Delta N(z, t) S dz /
\]

\[ \text{(Sdz)} \]

\[ \frac{dI}{dt} \frac{1}{v_g} \]

\[ \frac{\partial I}{\partial z} + \frac{1}{v_g} \frac{\partial I}{\partial t} = \sigma \cdot \Delta N \cdot I \]

\[ \frac{\partial F}{\partial z} + \frac{1}{v_g} \frac{\partial F}{\partial t} = \sigma \cdot \Delta N \cdot F \]

- the same as in lecture 2
from Einstein’s eqs.:

\[
\frac{dN_2}{dt} = -A_{21}N_2 + B_{21}q(\omega_{21})(N_1 - N_2) \\
N_1 + N_2 = N
\]

which gives

\[
\Delta N = N_2 - N_1 = 2N_2 - N
\]

\[
\frac{d}{dt}\Delta N = 2 \frac{dN_2}{dt} = -A_{21}(\Delta N + N) - 2B_{21}q(\omega_{21})\Delta N
\]

something is wrong; we know from lecture 2 that

\[
\frac{d}{dt}\Delta N = -\frac{1}{T_1}(\Delta N - \Delta N_0) - 2\sigma\Delta NF
\]

- the rate of stimulated emission is OK
- In lecture 2 we have ignored spontaneous emission!
spontaneous emission

Again, consider a thin slice of the amplifying medium. The thing we measure is intensity and the detector cannot distinguish between photons from stimulated and spontaneous emission.

\[
\Delta I(v) = \sigma(v) \Delta N I(v) + h\nu \times A_{21} \times g(v) \Delta \nu \times \frac{1}{2} \times \frac{d\Omega}{4\pi} \times N_2 \Delta z
\]
Amplified Spontaneous Emission (ASE)

when the medium is long and/or amplification coefficient large $\gamma_0 l \gg 1$ the spontaneous emission can be amplified to macroscopic intensities

No simple and convenient formulas for accounting for ASE. The spontaneous emission rate is given by $\frac{1}{2} h\nu$ for every spatio-temporal mode of the amplifier.

Consequences of ASE:
- Noise at the output of the amplifier
- In extreme cases ASE can saturate amplifier

Simple rule: to avoid problems with ASE the input intensity has to be much larger than the spontaneous emission intensity
**homogenous line-broadening** (we cannot address atoms by spectral methods)

- Natural broadening (always present). At least one of the two energy levels involved in light amplification corresponds to an excited state which has a finite life-time because atoms spontaneously drop to lower energy levels while emitting photons. In addition, in condensed phase, the life-time can shortened by non-radiative transition which increase the total transition rate.

$$\frac{\Delta \nu}{2\pi} = \frac{A_{21}}{2\pi}$$

spontaneous emission leads to the Lorentzian line – shape

$$g(\nu) = \frac{\Delta \nu}{2\pi[(\nu-\nu_0)^2+(\Delta \nu/2)^2]} , \quad g(\omega) = 2\pi g(\nu)$$

with FWHM line-width

$$\Delta \nu = \frac{A_{21}}{2\pi}$$

**FWHM** – Full Width at Half-Maximum
Pressure broadening

probability density for atomic collisions in gas phase

\[ p(\tau) = \frac{e^{-\tau/\tau_c}}{\tau_c} \]

\[ p(\tau) \, d\tau \] - the probability for that the atom to undergo a collision in the time interval \( \tau, \tau + d\tau \)

calculations.....

\[ g(\nu) = \frac{\Delta \nu}{2\pi[(\nu - \nu_0)^2 + (\Delta \nu / 2)^2]} \]

with \( \Delta \nu = \frac{\pi}{\tau_c} \).

in glasses and crystals the interrupting events are phonons

for gases \( \cong \text{MHz/mbar} \)
inhomogenous line-broadening (we can address atoms by spectra methods)

- Doppler broadening
  Doppler shift; if the atoms moves slowly compared to the speed of light in vacuum ($v \ll c$) the largest shift comes form linear Doppler effect, which depends on the velocity component parallel to the direction of observation (we assume $v_z$):

$$v' = \left(1 + \frac{v_z}{c}\right)v \Rightarrow v_z = \frac{c}{v_z}(v' - v)$$

the convention is that $v_z > 0$ for atom moving towards the light source (absorption) or the observer (emission).

For gas at the temperature $T$ the velocity distribution is given by Maxwell function

$$p(v) = \left(\frac{M}{2\pi kT}\right)^{1/2} \exp[-(Mv_z^2/T)]$$

Let’s mark the resonant frequency in atom is by $\nu_0$ and let’s assume that homogenous line-broadening is small. Then the line-shape function id given by Gaussian function:

$$g(\nu) = \frac{1}{\nu_0 \left(\frac{M}{2\pi kT}\right)^{1/2}} \exp\left[-\frac{Mc^2}{2kT} \left(\frac{\nu - \nu_0}{\nu_0}\right)^2\right]$$

with FWHM:

$$\Delta \nu = 2\nu_0 \left(\frac{2kT \ln 2}{Mc^2}\right)^{1/2}$$
Gauss vs Lorentz
**mixed line-broadening**

**example:** Doppler broadening + collisional broadening

Atoms with a given value of $v_z$ are characterized by homogenously broadened

$$g_h(v') = \frac{\Delta \nu}{2\pi[(v' - v_0')^2 + (\Delta \nu/2)^2]}$$

Index $h$ signifies homogenous (in this example collisional broadening), $\Delta \nu$ is the linewidth of homogenous broadening, and $v_0' = (1 + \frac{v_z}{c})v_0$ the Doppler shifted resonance frequency.

The probability of atom having a given value of $v_z$ is given by Maxwell’s distribution, we integrate over the possible values of $v_0'$

$$g_V(v) = \left(\frac{M}{2\pi kT}\right)^{1/2} \int_{-\infty}^{\infty} d\nu_0' \frac{\Delta \nu}{2\pi[(v - \nu_0')^2 + (\Delta \nu/2)^2]} \exp\left[-\frac{M c^2}{2kT} \frac{(v - \nu_0')^2}{\nu_0'^2}\right]$$

$g_V(v)$ is Voigt’s profile.
mixed line-broadening, 1

Voigt’s profile is a convolution of Lorentz and Gauss functions

\[ g_V(x) = \int_{-\infty}^{\infty} dx' G(x'; \sigma)L(x - x'; \gamma) \]

\[ G(x; \sigma) \equiv \frac{e^{-\frac{x^2}{(2\sigma^2)}}}{\sigma\sqrt{\pi}}, \quad L(x; \gamma) \equiv \frac{\gamma}{\pi(x^2 + \gamma^2)} \]
absorption coefficient $\alpha$  

$\alpha \propto p \times \frac{1}{\Delta \nu}$

Doppler broadening:  $\alpha \propto p$

pressure broadening:  $\alpha \propto p \times \frac{1}{p} = \text{const}$

FIGURE 7.9. Absorption coefficient in CO$_2$ at 10.6 $\mu$m as a function of CO$_2$ pressure.  
### "typical" linewidths

<table>
<thead>
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<th></th>
<th>effect</th>
<th>gas</th>
<th>liquid</th>
<th>condensed matter</th>
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<td>natural</td>
<td>0.001Hz-10MHz</td>
<td>n *</td>
<td>n</td>
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<tr>
<td>atomnc collision</td>
<td>5-10MHz/mbar</td>
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<td>≈ 300 cm⁻¹</td>
<td>----</td>
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<td>phonons</td>
<td>---</td>
<td></td>
<td>---</td>
<td>≈ 10 cm⁻¹</td>
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<td>inhomogeneous</td>
<td>Doppler</td>
<td>50MHz-1GHz</td>
<td>n</td>
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<td>Local fields</td>
<td>---</td>
<td></td>
<td>≈ 500 cm⁻¹</td>
<td>1-500 cm⁻¹</td>
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</tbody>
</table>

cm⁻¹ units are often used in spectroscopy

\[ \tilde{\nu} \equiv \frac{1}{\lambda [\text{cm}]} \]

\[ \tilde{\nu} \equiv \frac{1}{\lambda [\text{cm}^{-1}]} = \frac{\nu}{c \left[ \frac{\text{cm}}{\text{s}} \right]} = 10^{-2} \frac{\nu}{c} \]

numbers: \( \lambda = 1 \mu m \Leftrightarrow 10000 \text{ cm}^{-1} \) for \( \lambda = 1 \mu m \): \( 1 \text{ cm}^{-1} = 30 \text{GHz} \)

* n - negligible
gain saturation in media with different line broadening

We will concentrate on the case $\tau_p \gg T_1$. Similar reasoning can be extended to other cases as well.

$$\gamma(F) = \frac{\gamma_0}{1 + F/F_s}$$

Homogenous broadening dominates. As the population inversion decreases the gain drops for all frequencies because all atoms interact with the em wave in the same way. – Saturation requires higher intensities for frequencies far away for the resonance.

$$\gamma_0(\nu) \propto \frac{\Delta \nu}{2\pi[(\nu - \nu_0)^2 + (\Delta \nu/2)^2]}$$

$$\gamma(F, \nu) = \frac{\gamma_0(\nu)}{1 + F/F_s}$$
inhomogeneous broadening dominates. A monochromatic em wave of frequency $\nu$ interacts only with atoms that have their resonant frequencies close to $\nu$ (closer than homogenous linewidth). The saturation affects only this selected group of atoms – other groups „do not see“ the em field.

$$\gamma_0(\nu) \propto \int_{-\infty}^{\infty} d\nu_0' \frac{\Delta \nu}{2\pi[(\nu - \nu_0')^2 + (\Delta \nu/2)^2]} g(\nu_0')$$

$g_j(\nu, \nu_0')$ homogenously broaden line centered at $\nu_0'$.

Saturation „burns a hole“ in the gain profile. Its width corresponds to the homogenous linewidth. The depth of the hole scales with saturation (em field intensity).