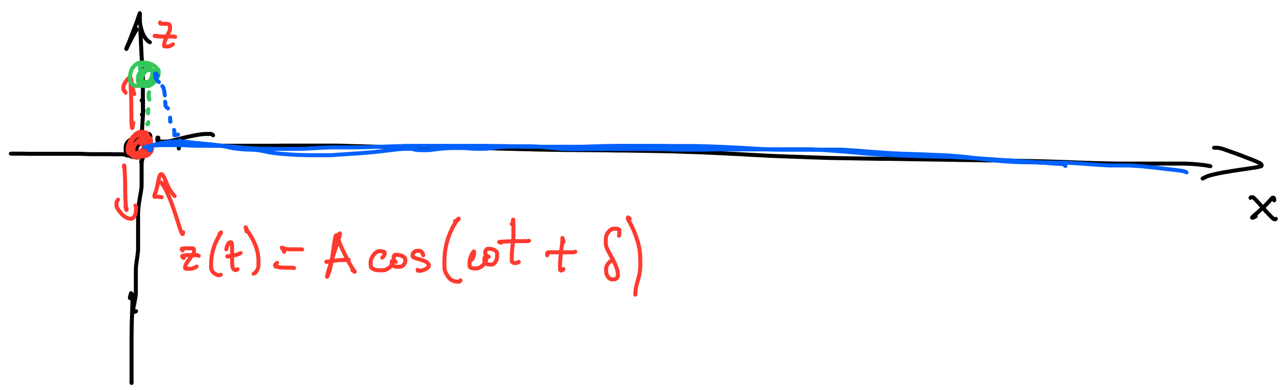
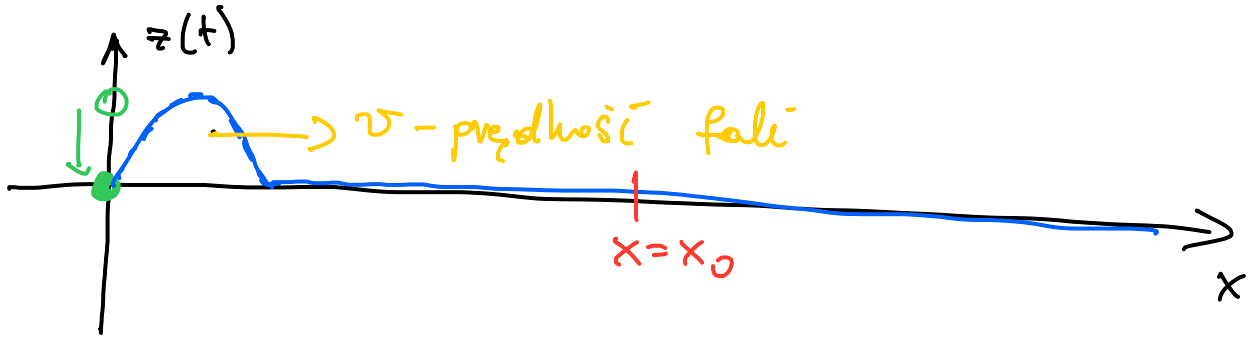
Przygotowanie przed olimpiadą 4.01.2021

9. Fale mechaniczne



$z(t) = A\cos(\omega t + \delta)$



$$z(x,t) = A\cos(kx - \omega t + \phi)$$

drgania w czasie

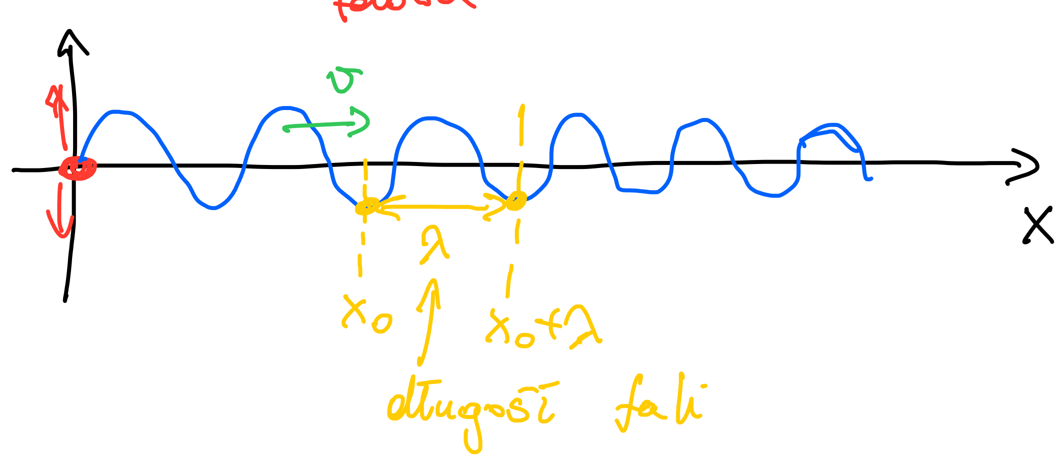
przesunięcie fazowe (odpowiada powtarzaniu przesunięcia w $x=0$)

$$z(x_0,t) = A\cos(kx_0 - \omega t + \phi) = A\cos(\omega t + \delta_2)$$

liczba falowa

stała

$\delta_2 = -kx_0 - \phi$

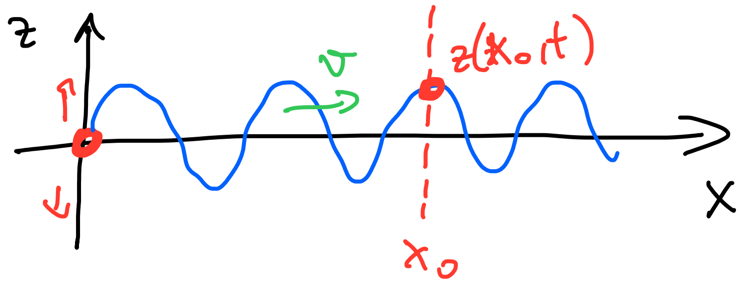


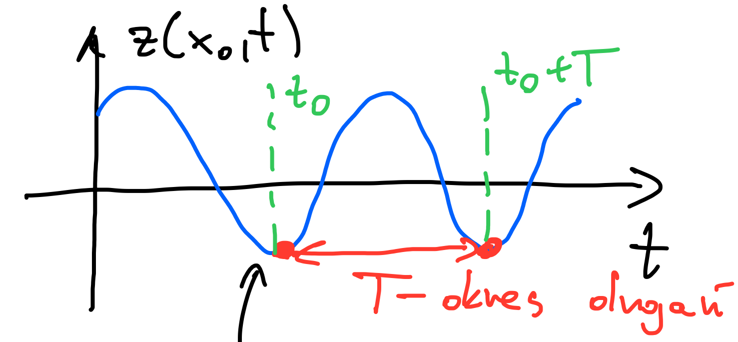
$$z(x_0+\lambda, t) = z(x_0, t)$$

$$A\cos(k(x_0+\lambda) - \omega t + \phi) = A\cos(kx_0 - \omega t + \phi)$$

$$A\cos(kx_0 + \underline{k\lambda} - \omega t + \phi) = A\cos(kx_0 - \omega t + \phi)$$

Aby to była prawda $\Rightarrow$ $k\lambda = 2\pi$ $\Rightarrow$ $k = \dfrac{2\pi}{\lambda}$





$$z(x_0,t) = A\cos(kx_0 - \omega t + \phi)$$

$$z(x_0,t_0) = A\cos(kx_0 - \omega t_0 + \phi) = A\cos(kx_0 - \underline{\underline{\omega}}(t_0 + \underline{\underline{T}}) + \phi) = z(x_0, t_0 + T)$$

$$\Rightarrow \quad \omega T = 2\pi \quad \Rightarrow \quad \omega = \frac{2\pi}{T}$$

częstość kołowa

Definicja prędkości fali:

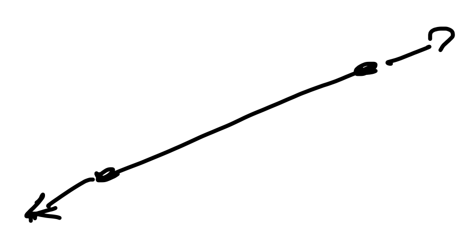
$$v = \frac{\lambda}{T} = \frac{\frac{2\pi}{k}}{\frac{2\pi}{\omega}} = \frac{\omega}{k} \quad \Rightarrow \quad \boxed{\omega = kv}$$

$$v = \frac{\lambda}{T} = \frac{\lambda}{\frac{2\pi}{\omega}} = \lambda\omega/2\pi = \lambda f$$

$\omega/2\pi = f = \nu$ − częstotliwość = ilość drgań w jednostce czasu

Proces drgań w strunie jest opisywany za pomocą równania falowego:

$$\boxed{\frac{1}{v^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}}$$



$$z(t,x) = e^{\lambda_1 t + \lambda_2 x}$$

$$\frac{\partial z(t,x)}{\partial t} = \lambda_1 e^{\lambda_1 t + \lambda_2 x} \qquad \frac{\partial z}{\partial x} = \lambda_2 e^{\lambda_1 t + \lambda_2 x}$$

$$\frac{\partial^2 z}{\partial t^2} = \lambda_1^2 e^{\lambda_1 t + \lambda_2 x} \qquad \frac{\partial^2 z}{\partial x^2} = \lambda_2^2 e^{\lambda_1 t + \lambda_2 x}$$

$$\frac{1}{v^2}\lambda_1^2 \cancel{e^{\lambda_1 t+\lambda_2 x}} = \lambda_2^2 \cancel{e^{\lambda_1 t+\lambda_2 x}}$$

$$\Rightarrow v^2 = \frac{\lambda_1^2}{\lambda_2^2} \Rightarrow v = \frac{\lambda_1}{\lambda_2}$$

$$\lambda_1 \sim \omega, \quad \lambda_2 \sim k$$

Liczby zespolone – podstawy

$$x^2+1=0 \Rightarrow x^2=-1 \Rightarrow x_1=-i, \; x_2=+i$$

Wprowadzamy $i=\sqrt{-1}$, $i^2=-1$

jednostka urojona

$w, z \in \mathbb{C}$, $z = x+iy$, $w = a+ib$ $x, y, a, b \in \mathbb{R}$

1°) dodawanie

$$z+w = x+iy + a+ib = (x+a)+i(y+b)$$

2°) mnożenie

$$z\cdot w = (x+iy)(a+ib) = xa + ibx + iya + \underset{-1}{i^2}yb =$$

$$= (xa-yb)+i(bx+ya)$$

3°) sprzężenie zespolone

$$z = x+iy, \quad z^* = x-iy$$

sprzężenie

4°) moduł liczby zespolonej

$$|z|^2 = z\cdot z^* = (x+iy)(x-iy) = x^2+ixy-ixy-i^2y^2 = x^2+y^2$$

$$|z| = \sqrt{x^2+y^2}$$

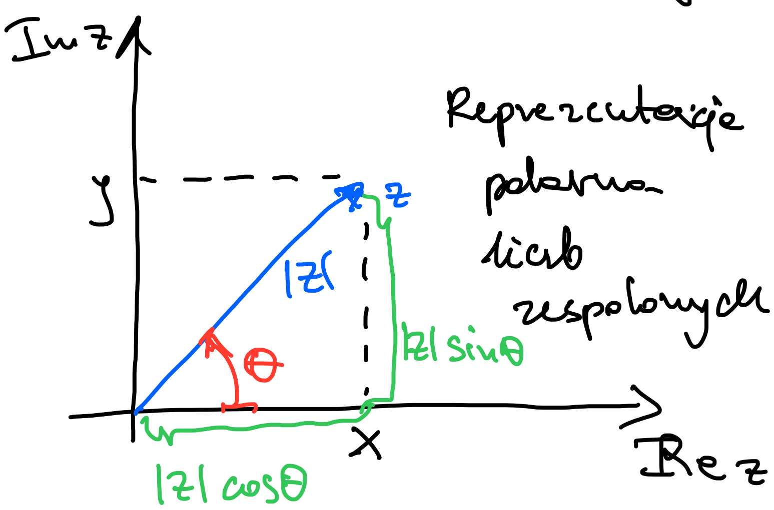
stąd mogę wyciągnąć interpretację geometryczną

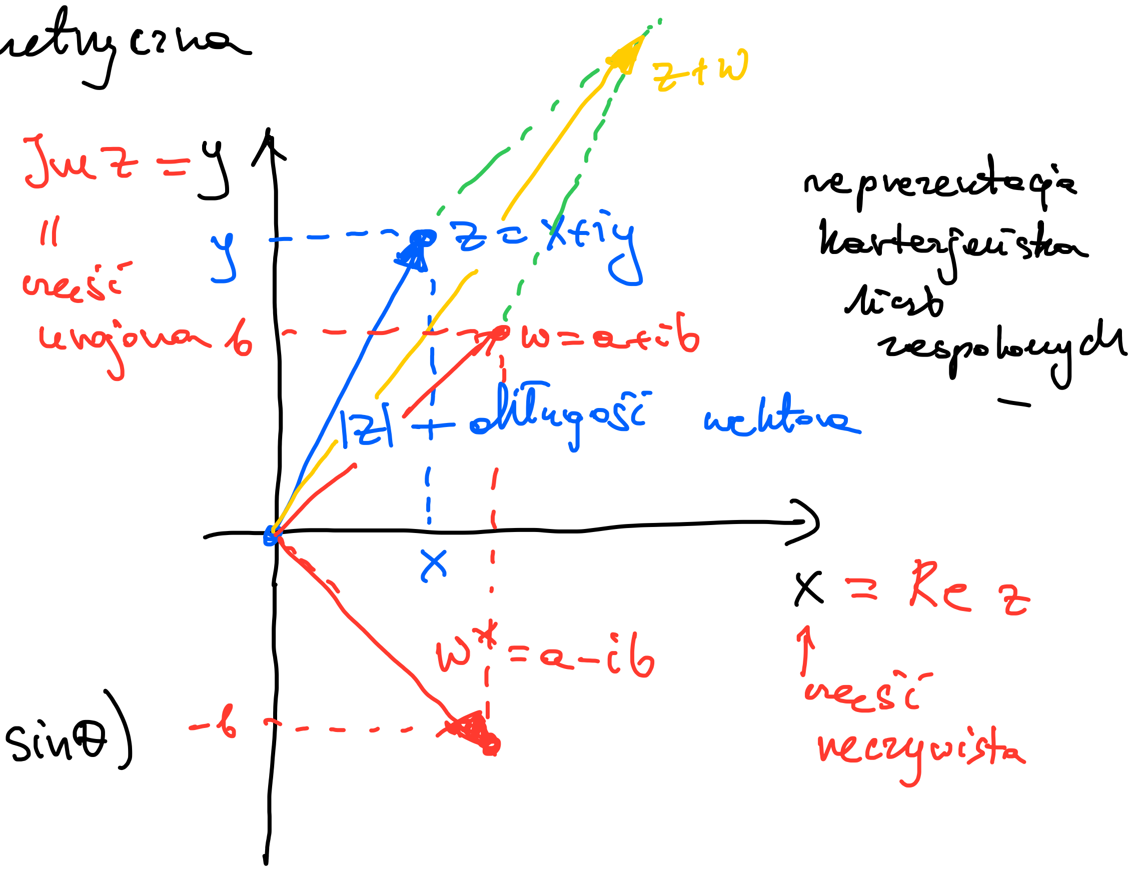
5°) dzielenie

$$z/w = \frac{z}{w}\,\frac{w^*}{w^*} = \frac{zw^*}{|w|^2} = \frac{(x+iy)(a-ib)}{a^2+b^2} =$$

$$= \frac{xa+iya-ixb-i^2yb}{a^2+b^2} = \frac{xa+yb}{a^2+b^2} + i\,\frac{ya-xb}{a^2+b^2}$$

6°) Interpretacja geometryczna





$$z = x+iy = |z|(\cos\theta + i\sin\theta)$$

$$e^x = e^0 + \frac{1}{1!}e^0(x-0) + \frac{1}{2!}\underset{=1}{e^0}(x-0)^2 + \frac{1}{3!}e^0(x-0)^3 + \dots =$$

rozwinięcie w szereg Taylora

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

definicja tej funkcji

$x = i\varphi$

$$e^{i\varphi} = 1 + i\varphi + \frac{(i\varphi)^2}{2!} + \frac{(i\varphi)^3}{3!} + \dots =$$

$$= \underbrace{\left(1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \frac{\varphi^6}{6!} + \dots\right)}_{\cos\varphi} + i\underbrace{\left(\varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \frac{\varphi^7}{7!} + \dots\right)}_{\sin\varphi}$$

$$\cos x = \underbrace{\cos 0}_{1} + \frac{1}{1!}(-\underbrace{\sin 0}_{0})(x-0) + \frac{1}{2!}(-\cos 0)(x-0)^2 + \frac{1}{3!}(\sin 0)(x-0)^3 + \dots$$

sz. Taylora

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\sin x = \underbrace{\sin 0}_{0} + \frac{1}{1!}(\underbrace{\cos 0}_{1})(x-0) + \frac{1}{2!}(-\sin 0)(x-0)^2 + \frac{1}{3!}(-\cos 0)(x-0)^3 + \dots =$$

sz. Taylora

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\boxed{e^{i\varphi} = \cos\varphi + i\sin\varphi} \qquad z = |z|\, e^{i\varphi}$$

Wracamy do fal: $\varphi \longrightarrow kx - \omega t + \phi$

$|z| \longrightarrow A$

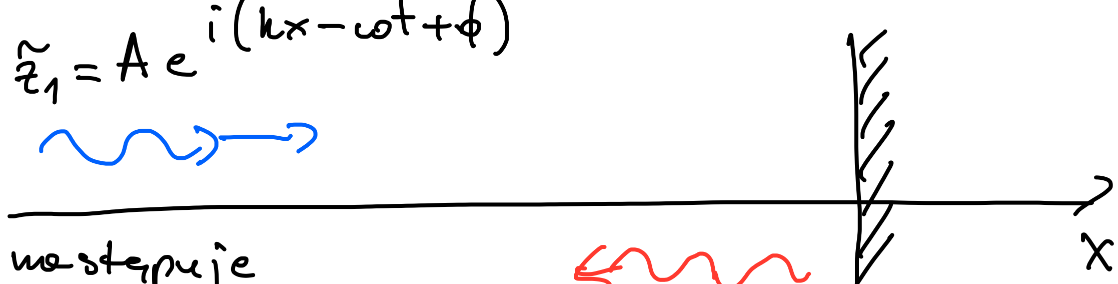
$$\tilde{z}(x,t) = A e^{i(kx - \omega t + \phi)}$$

$$z(x,t) = \mathrm{Re}\, \tilde{z}(x,t) = \mathrm{Re}\Big(A\big(\underline{\cos(kx-\omega t+\phi)} + i\sin(kx-\omega t+\phi)\big)\Big)$$

FALA STOJĄCA $= A\cos(kx - \omega t + \phi)$

$$\tilde{z}_1 = A e^{i(kx - \omega t + \phi)}$$

$\phi = 0$



następuje nakładanie fal

$$\tilde{z}_2 = A e^{i(kx + \omega t + \phi + \Delta)}$$

$$\tilde{z}_1 + \tilde{z}_2 = A e^{i(kx - \omega t)} + A e^{i(kx + \omega t + \Delta)} =$$

$$= A e^{i(kx - \omega t)} + A e^{i(kx + \omega t)} e^{i\Delta} = A e^{ikx}\left(e^{-i\omega t} + e^{i(\omega t + \Delta)}\right)$$

Dygresja:

$$z = |z|\left(\cos\varphi + i\sin\varphi\right) = |z|e^{i\varphi}$$

$$z^* = |z|\left(\cos\varphi - i\sin\varphi\right) = |z|e^{-i\varphi}$$

$$z + z^* = |z|\left(e^{i\varphi} + e^{-i\varphi}\right) = 2|z|\cos\varphi \Rightarrow \cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}$$

$$z - z^* = |z|\left(e^{+i\varphi} - e^{-i\varphi}\right) = 2i|z|\sin\varphi \Rightarrow \sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}$$

$$\tilde{z}_1 + \tilde{z}_2 = Ae^{ikx}\left(e^{-i\omega t} + e^{i\omega t + i\Delta}\right) = \left\{ \begin{array}{l} \text{mnożę przez } 1 = \\ = e^{i\Delta/2} e^{-i\Delta/2} \end{array} \right\}$$

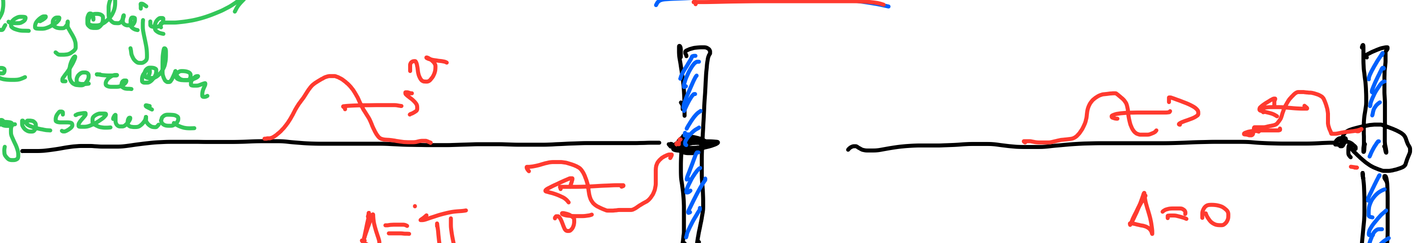
$$= Ae^{ikx} e^{i\Delta/2} \underbrace{\left(e^{-i\omega t - i\frac{\Delta}{2}} + e^{i\omega t + \underbrace{i\Delta - i\frac{\Delta}{2}}_{i\frac{\Delta}{2}}}\right)}_{2\cos\left(\omega t + \frac{\Delta}{2}\right)} =$$

$$= Ae^{ikx} e^{i\Delta/2}\, 2\cos\left(\omega t + \frac{\Delta}{2}\right)$$

$$z_1 + z_2 = \mathrm{Re}\left(\tilde{z}_1 + \tilde{z}_2\right) = \mathrm{Re}\left(A\underbrace{e^{ikx + \frac{i\Delta}{2}}}_{\cos\left(kx + \frac{\Delta}{2}\right) + i\sin\left(kx + \frac{\Delta}{2}\right)}\, 2\cos\left(\omega t + \frac{\Delta}{2}\right)\right) =$$

$$= 2A\cos\left(kx + \frac{\Delta}{2}\right)\cos\left(\omega t + \frac{\Delta}{2}\right)$$

on decyduje
gdzie będą
wygaszenia

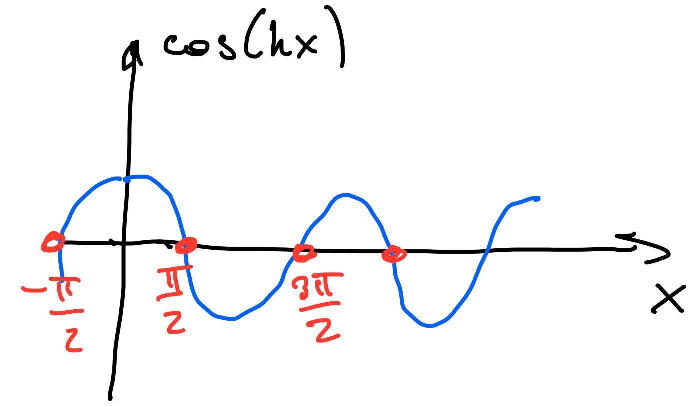


Gdy $\cos\left(kx+\frac{\Delta}{2}\right)=0$ nie ma drgań

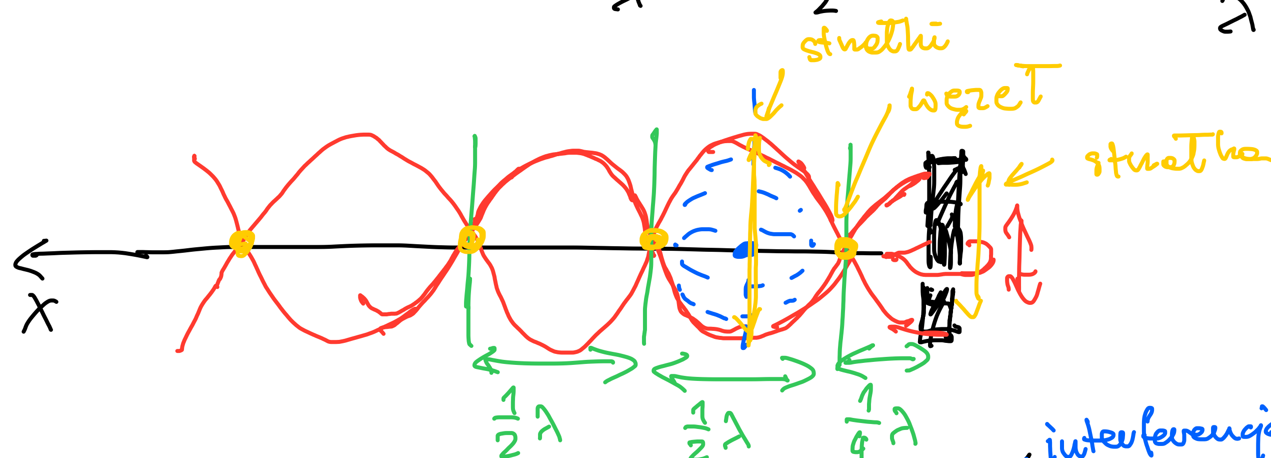
$\Delta=0$: $\cos(kx)=0 \Rightarrow$

$kx=\frac{\pi}{2}+n\pi$, $n\in\mathbb{Z}$

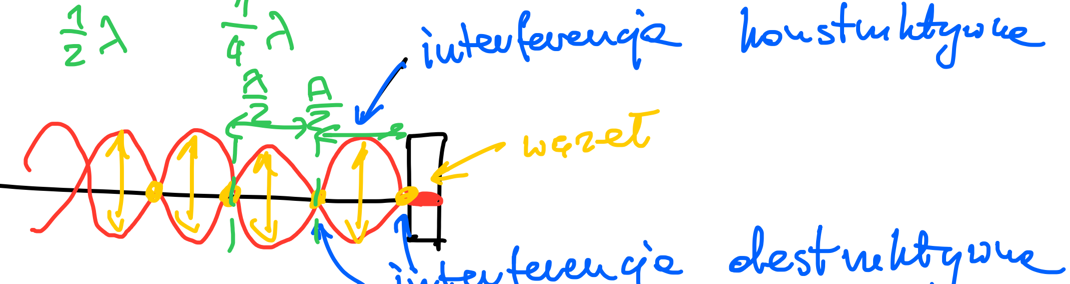
następuje wygaszanie fal



$$k=\frac{2\pi}{\lambda} \Rightarrow x\frac{2\pi}{\lambda}=\frac{\pi}{2}+n\pi \Rightarrow \frac{x}{\lambda}=\frac{1}{4}+\frac{n}{2}$$

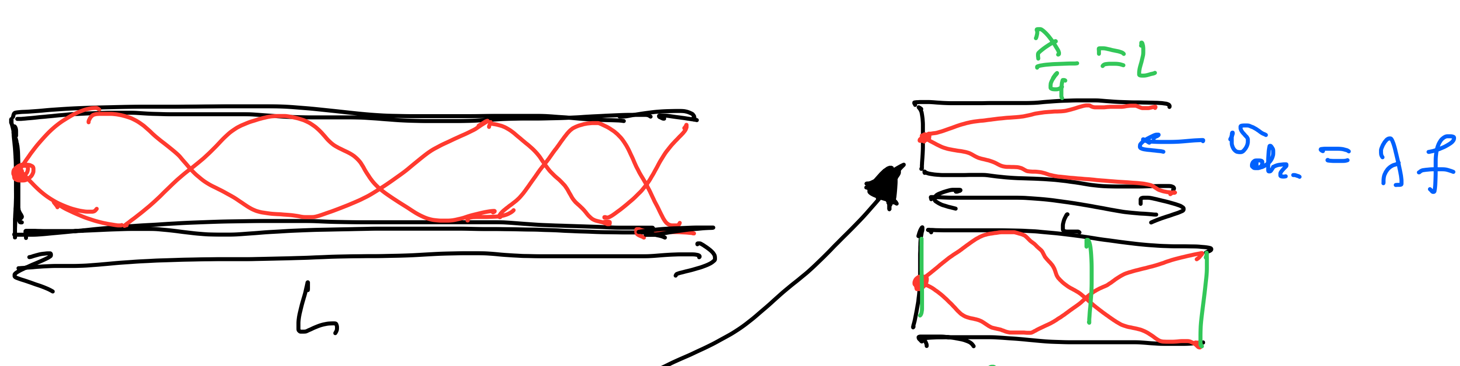


$\Delta=\pi$



$$\cos\left(kx+\frac{\pi}{2}\right)=0 \Rightarrow kx+\frac{\pi}{2}=\frac{\pi}{2}+n\pi, \quad n\in\mathbb{Z}$$

$$k=\frac{2\pi}{\lambda} \Rightarrow \frac{2\pi}{\lambda}x=n\pi \Rightarrow \frac{x}{\lambda}=\frac{n}{2}$$



$$\frac{\lambda}{2}+\frac{\lambda}{4}=\frac{3}{4}\lambda=L$$

$$\frac{4}{3}L=\lambda$$

$$v_{dź.}=4Lf_1 \Rightarrow f_1=\frac{v_{dź}}{4L}$$

$$v_{dź.}=\frac{4L}{3}f_2 \Rightarrow f_2=\frac{3v_{dź}}{4L}$$

DUDNIENIA

Mamy dwa źródła: $\begin{cases} f_1 = f_0 + \delta \\ f_2 = f_0 - \delta \end{cases}$ ← powinna być mała

częstotliwości

$$z_1 = A\cos(kx - \omega_1 t + \phi) = A\cos(kx - 2\pi(f_0+\delta)t + \phi)$$

$$z_2 = A\cos(kx - \omega_2 t + \phi) = A\cos(kx - 2\pi(f_0-\delta)t + \phi)$$

$$\tilde{z}_1 = A e^{i(kx - 2\pi(f_0+\delta)t + \phi)}$$

$$\tilde{z}_2 = A e^{i(kx - 2\pi(f_0-\delta)t + \phi)}$$

$$\tilde{z}_1 + \tilde{z}_2 = A e^{i(kx+\phi)} e^{-i2\pi f_0 t} \underbrace{\left(e^{-i\delta t} + e^{i\delta t}\right)}_{2\cos\delta t} =$$

$$= 2A e^{i(kx - 2\pi f_0 t + \phi)} \cos(\delta t)$$

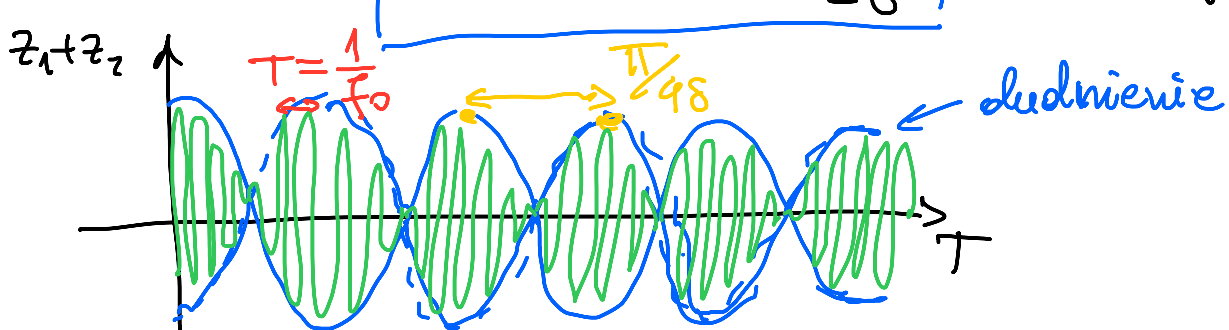
Superpozycja fal

$$z_1 + z_2 = \mathrm{Re}(\tilde{z}_1 + \tilde{z}_2) = 2A\cos(kx - 2\pi f_0 t + \phi)\underline{\cos(\delta t)}$$

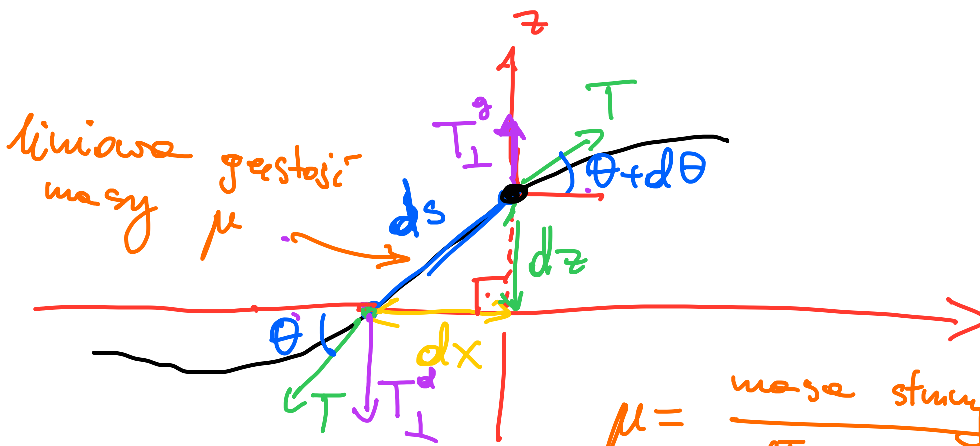
$$\cos(\delta t) = 0 \Rightarrow \delta t = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}$$

$$t_{\text{dudnienia}} = \frac{\pi}{2\delta} + \frac{n\pi}{\delta}, \quad n = 0$$

$$\boxed{t_{\text{dudnienie}} = \frac{\pi}{2\delta}} \qquad f_0 \gg \delta$$



DRGANIA STRUNY – JAKI JEST SENS RÓWNANIA FALOWEGO



$$\mu = \frac{\text{masa struny}}{\text{długość struny}}$$

Tw. Pitagorasa

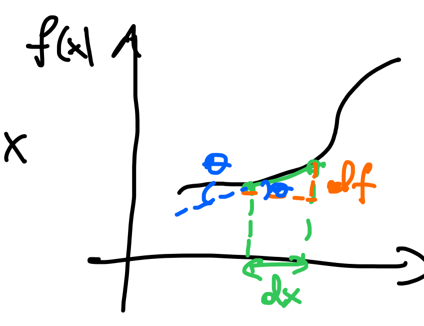
$$(ds)^2 = (dz)^2 + (dx)^2$$

$$(ds)^2 = \left(\left(\frac{dz}{dx}\right)^2 + 1\right)(dx)^2$$

$$ds = \sqrt{1 + \underbrace{\left(\frac{dz}{dx}\right)^2}_{\text{gdy to małe}}}\; dx \simeq dx$$

$$T_\perp^g - T_\perp^d = T\sin(\theta + d\theta) - T\sin\theta = \mu\, dx\, \frac{\partial^2 z}{\partial t^2}$$

Gdy nachylenie struny jest małe (lub mówiąc inaczej dx jest małe), wtedy



$dx \to 0$

$$\frac{df}{dx} = \mathrm{tg}\,\theta$$

$$\sin\theta \approx \mathrm{tg}\,\theta = \frac{\sin\theta}{\cos\theta} = \frac{\partial z}{\partial x}$$

θ – małe, czyli $\cos\theta \simeq 1$

$$T\left(\left.\frac{\partial z}{\partial x}\right|_{x+dx} \quad \left.\frac{\partial z}{\partial x}\right|_{x}\right) = \mu\, dx\, \frac{\partial^2 z}{\partial t^2}$$

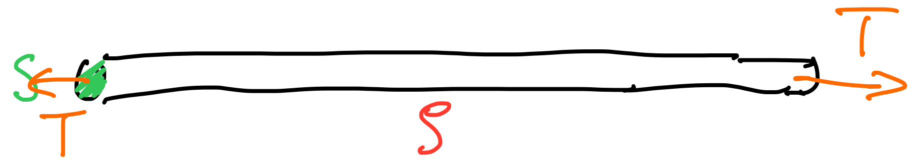
$$\frac{\left.\frac{\partial z}{\partial x}\right|_{x+dx} - \left.\frac{\partial z}{\partial x}\right|_{x}}{dx} \xrightarrow[dx\to 0]{} \frac{\partial^2 z}{\partial x^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \underbrace{\left(\frac{\mu}{T}\right)}_{\frac{1}{v^2}} \frac{\partial^2 z}{\partial t^2} \implies v_{fali} = \sqrt{\frac{T}{\mu}}$$

naprężenie struny

Prędkość fali:

$$\boxed{v_{fali} = \sqrt{\frac{\text{„sprężystość"}}{\text{„bezwładność"}}}}$$



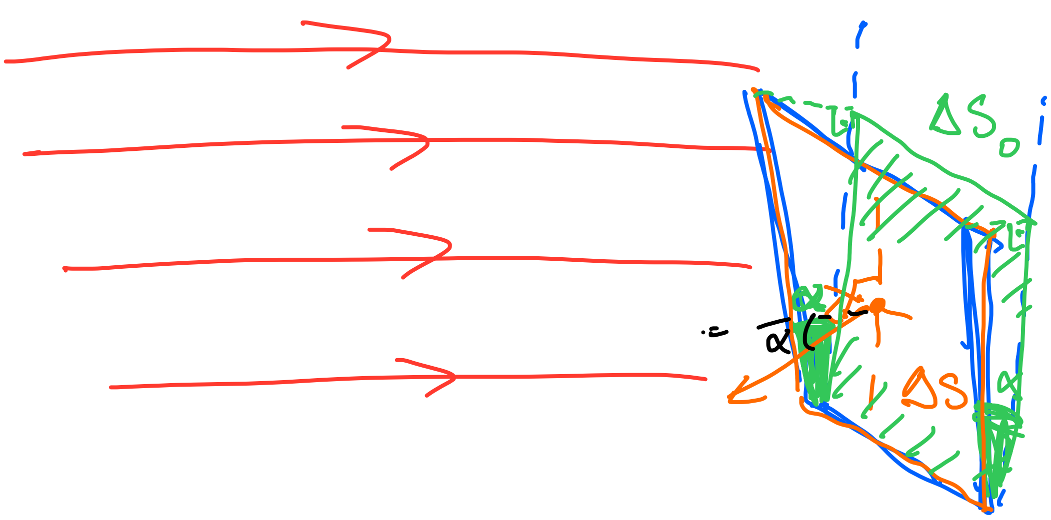
$$v_{fali} = \sqrt{\frac{T}{\varrho S}}$$

Opis energetyczny fal

Natężenie fali:

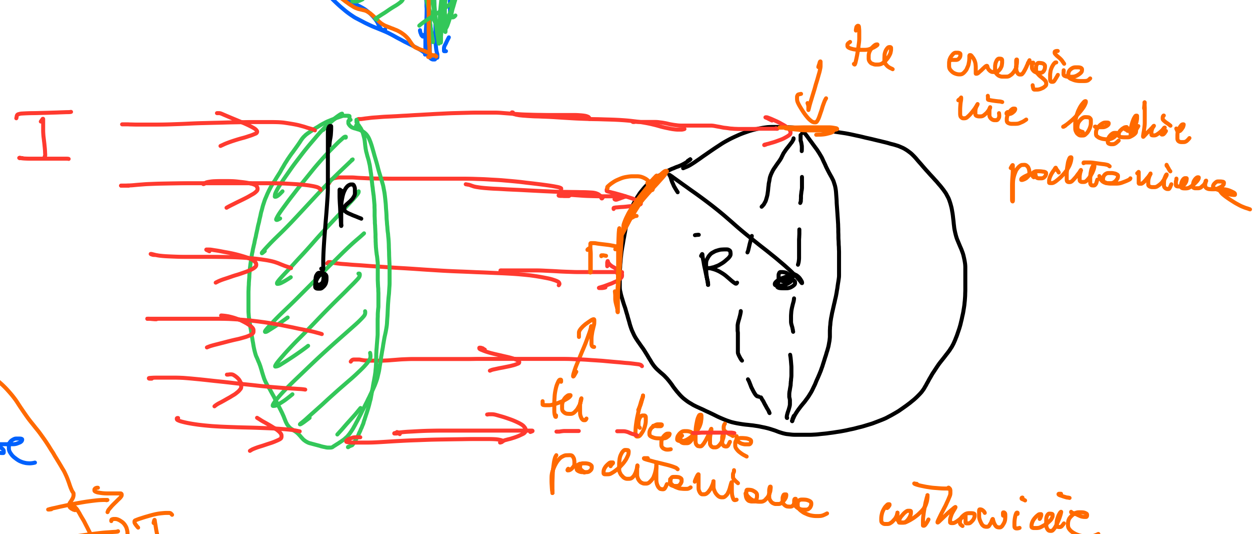
$$I = \frac{\Delta E}{\Delta S\, \Delta t} \propto \left(z(x,t)\right)^2$$

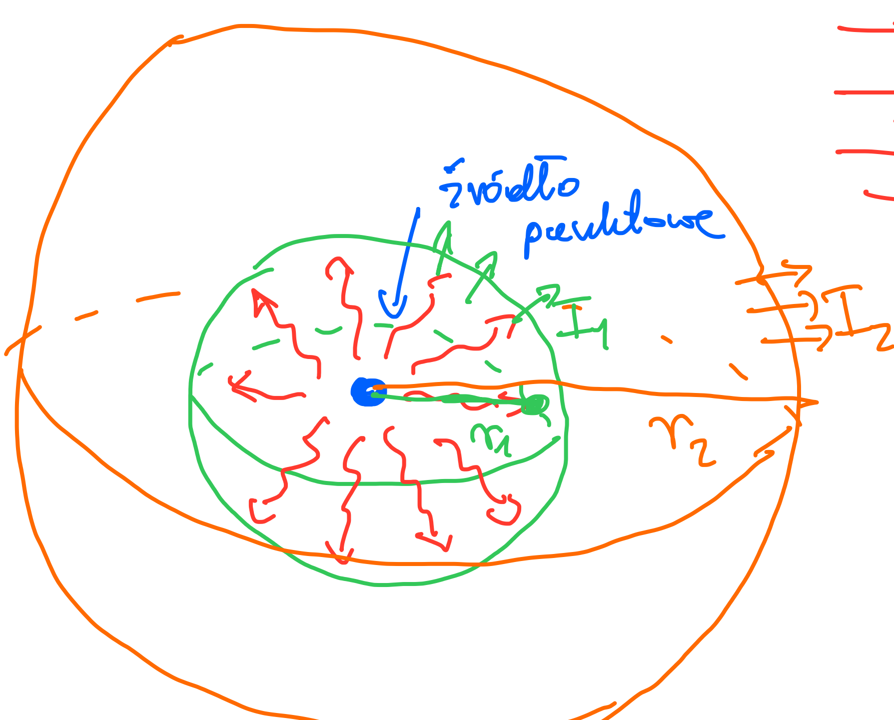
$$\Phi = \frac{\Delta E}{\Delta t} \text{ – strumień energii}$$



$$\Phi = I \Delta S_0 = $$
$$= I \Delta S \cos\alpha$$

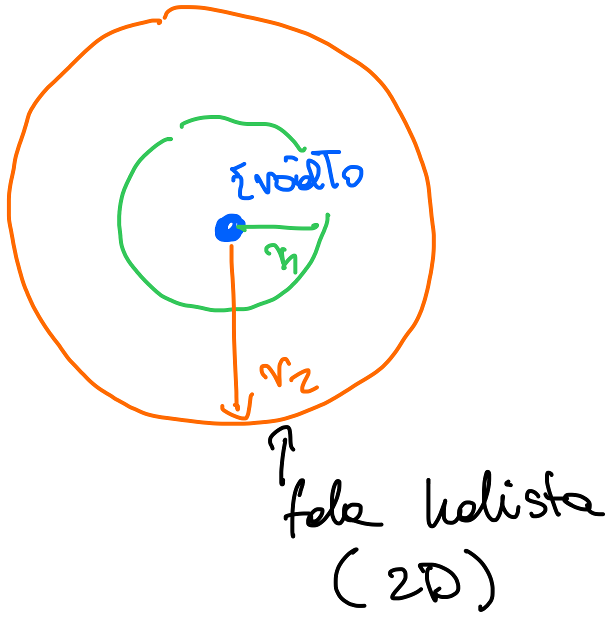
$$\Phi = I \pi R^2$$





$$\Phi = \text{const} = I_1\, 4\pi r_1^2 = I_2\, 4\pi r_2^2$$

$$I_1 = I_2 \frac{r_2^2}{r_1^2}$$



$$I_1\, 2\pi r_1 = I_2\, 2\pi r_2$$

$$I_1 = I_2\, \frac{r_2}{r_1}$$