TOPICS IN QUANTUM MANY-BODY THEORY — Week II — 2025.03.06

(1)

Coherent states for fermions

Analogous construction for fermions? $\longrightarrow$ eigenvalues must anticommute!!

Algebra of anticommuting "numbers" – Grassmann algebra

vector space with additional multiplication operation (associative, anticommutative)

Grassmann algebra $\rightarrow$ set of generators $\{\xi_\alpha\}$. Consider $\alpha \in \{1,2,\dots,n\}$ now.

- generators anticommute $\xi_\alpha \xi_\beta + \xi_\beta \xi_\alpha = 0$ ($\xi_\alpha^2 = 0$ in particular)
- basis of the Grassmann algebra $\rightarrow$ distinct products of generators
- a "number" in the Grassmann algebra $\rightarrow$ linear combination (with complex coefficients) of the "numbers" $\{1, \xi_{\alpha_1}, \xi_{\alpha_1}\xi_{\alpha_2}, \dots \xi_{\alpha_1}\xi_{\alpha_2}\dots\xi_{\alpha_n}\}$ ($\alpha_1 < \alpha_2 < \dots < \alpha_n$ - convention)
- dimension of the Grassmann algebra $= 2^n$. (•)
- Addition defined as usual in vector spaces (commutative and associative)
- consider n even ($n = 2p$)
  - define a conjugation operation $\rightarrow$ pick p generators ξ_α, to each of them associate the conjugate generator ξ_α^*

$$
\left.
\begin{aligned}
(\xi_\alpha)^* &= \xi_\alpha^* \\
(\xi_\alpha^*)^* &= \xi_\alpha \\
(\lambda \xi_\alpha)^* &= \lambda^* \xi_\alpha^* \quad (\text{for } \lambda \in \mathbb{C}) \\
(\xi_{\alpha_1} \dots \xi_{\alpha_n})^* &= \xi_{\alpha_n}^* \xi_{\alpha_{n-1}}^* \dots \xi_{\alpha_1}^*
\end{aligned}
\right\}
\text{ Conjugation in Grassmann algebra.}
$$

Integration and differentiation

For simplicity – take the case with only 2 generators: ξ, ξ^* $\rightarrow$ algebra spanned by 4 elements: $\{1, \xi, \xi^*, \xi^* \xi\}$

(generalization straightforward)

Consider a Grassmann-valued function $A(\xi^*,\xi) = a_0 + a_1\xi + \bar{a}_1\xi^* + a_{12}\xi^*\xi$

Derivative – defined to be "identical" to the usual derivative, but in order to act with $\frac{\partial}{\partial\xi}$ on ξ, ξ must be adjacent to $\frac{\partial}{\partial\xi}$ ↳ e.g. $\frac{\partial}{\partial\xi}(\xi^*\xi) = \frac{\partial}{\partial\xi}(-\xi\xi^*) = -\xi^*$

"operational" definition (no sense of talking about "infinitesimals")

$$\frac{\partial}{\partial\xi}A(\xi^*,\xi) = a_1 - a_{12}\xi^*$$

$\frac{\partial}{\partial\xi}$ is nilpotent $\left(\left(\frac{\partial}{\partial\xi}\right)^2=0\right)$

$$\frac{\partial}{\partial\xi^*}A(\xi^*,\xi) = \bar{a}_1 + a_{12}\xi$$

$$\frac{\partial}{\partial\xi^*}\frac{\partial}{\partial\xi}A(\xi^*,\xi) = -a_{12} = -\frac{\partial}{\partial\xi}\frac{\partial}{\partial\xi^*}A(\xi^*,\xi)$$

($\frac{\partial}{\partial\xi}$, $\frac{\partial}{\partial\xi^*}$ anticommute)

(generalization to cases with more generators straightforward)

Integral – linear mapping with the property that the integral of an "exact differential form" is zero ↳

$$\int d\xi\, 1 = 0 \qquad \int d\xi^*\, 1 = 0$$

$$\int d\xi\,\xi = 1 \quad \text{(definition)} \qquad \int d\xi^*\,\xi^* = 1$$

"operational" definition (no analog to Riemann or Lebegue integral)

Mnemonic: Grassmann integration is identical to Grassmann differentiation...

$$\int d\xi\, A(\xi^*,\xi) = \int d\xi\,(a_0 + a_1\xi + \bar{a}_1\xi^* + a_{12}\xi^*\xi) = a_1 - a_{12}\xi^*$$

$$\int d\xi^* A(\xi^*,\xi) = \bar{a}_1 + a_{12}\xi$$

$$\int d\xi^* d\xi\, A(\xi^*,\xi) = -a_{12} = -\int d\xi\, d\xi^*\, A(\xi^*,\xi)$$

Two fundamental properties of ordinary integrals over functions vanishing at $\pm\infty$ are satisfied by this definition:

→ $\int d\psi_i \frac{\partial}{\partial\psi_i} f(\psi_1 \ldots \psi_n) = 0$

→ $\frac{\partial}{\partial\psi_i}\int d\psi_i\, f(\psi_1 \ldots \psi_n) = 0$

Why such definitions ??? → see later

Scalar product of Grassmann functions: $\langle f|g\rangle = \int d\xi^* \int d\xi\, e^{-\xi^*\xi} f^*(\xi)\, g(\xi^*)$

⟶ structure of a Hilbert space.

Back to fermions

To each a_α associate a generator ξ_α (Grassmann algebra $\mathcal{G}$)

$$a_\alpha^+ \longrightarrow \xi_\alpha^*$$

Generalized Fock space → set of linear combinations of states of the Fock space $\mathcal{F}$ with coefficients in $\mathcal{G}$.

Add a commutation rule between a's and ξ's : $[\tilde{\xi}, \tilde{a}]_+ = 0$ $\quad \tilde{\xi} \in \{\xi_\alpha, \xi_\alpha^*\}$

$$(\tilde{\xi}\tilde{a})^+ = \tilde{a}^+ \tilde{\xi}^* \qquad \tilde{a} \in \{a_\beta, a_\beta^+\}$$

<u>Define</u> a fermion coherent state $|\xi\rangle = e^{-\sum_\alpha \xi_\alpha a_\alpha^+}|0\rangle = \prod_\alpha (1-\xi_\alpha a_\alpha^+)|0\rangle$

(exact because any $\xi_\alpha a_\alpha^+$ commutes with any $\xi_\beta a_\beta^+$)

- Verify that this is an eigenstate of the annihilation operators:

For a single state α : $a_\alpha (1-\xi_\alpha a_\alpha^+)|0\rangle = -a_\alpha \xi_\alpha |1_\alpha\rangle = \xi_\alpha a_\alpha |1_\alpha\rangle =$
$= \xi_\alpha |0\rangle = \xi_\alpha (1-\xi_\alpha a_\alpha^+)|0\rangle$

{remains correct if $\xi_\alpha \longrightarrow \xi_\alpha' = \gamma \xi_\alpha$, $\gamma \in \mathbb{C}$}

$$a_\alpha |\xi\rangle = a_\alpha \prod_\beta (1-\xi_\beta a_\beta^+)|0\rangle = \prod_{\beta\neq\alpha}(1-\xi_\beta a_\beta^+)\, a_\alpha (1-\xi_\alpha a_\alpha^+)|0\rangle = \prod_{\beta\neq\alpha}(1-\xi_\beta a_\beta^+)\xi_\alpha \underbrace{(1-\xi_\alpha a_\alpha^+)}_{|0\rangle}$$
$$= \xi_\alpha \prod_\beta (1-\xi_\beta a_\beta^+)|0\rangle = \xi_\alpha |\xi\rangle$$

- The adjoint state : $\langle\xi| = \langle 0| e^{-\sum_\alpha a_\alpha \xi_\alpha^*} \longrightarrow \langle\xi| a_\alpha^+ = \langle\xi|\xi_\alpha^*$

- Action of a_α^+ on $|\xi\rangle$: $a_\alpha^+|\xi\rangle = a_\alpha^+(1-\xi_\alpha a_\alpha^+)\prod_{\beta\neq\alpha}(1-\xi_\beta a_\beta^+)|0\rangle =$
$$= a_\alpha^+ \prod_{\beta\neq\alpha}(1-\xi_\beta a_\beta^+)|0\rangle = -\frac{\partial}{\partial\xi_\alpha}(1-\xi_\alpha a_\alpha^+)\prod_{\beta\neq\alpha}(1-\xi_\beta a_\beta^+)|0\rangle$$
$$= -\frac{\partial}{\partial \xi_\alpha}|\xi\rangle$$

- Overlap of two coherent states :

$$\langle 0|\left(\prod_\beta (1-a_\beta \xi_\beta^*)\right)\prod_\alpha (1-\xi_\alpha' a_\alpha^+)|0\rangle \overset{(\bullet)}{=} \langle 0|\prod_\alpha (1+\xi_\alpha^* a_\alpha)(1-\xi_\alpha' a_\alpha^+)|0\rangle =$$
$$= \langle 0|\prod_\alpha (1+\xi_\alpha^* a_\alpha - \xi_\alpha' a_\alpha^+ + \xi_\alpha^* \xi_\alpha' a_\alpha a_\alpha^+)|0\rangle \overset{(\bullet)}{=} \langle 0|\prod_\alpha (1+\xi_\alpha^*\xi_\alpha'|0\rangle$$
$$= \prod_\alpha (1+\xi_\alpha^*\xi_\alpha') = e^{\sum_\alpha \xi_\alpha^* \xi_\alpha'}$$

• Closure relation: $\int \prod_\alpha d\xi^*_{\xi_\alpha} d\xi_{\xi_\alpha} e^{-\sum_\alpha \xi^*_\alpha \xi_\alpha} |\xi\rangle\langle\xi| = 1$ (unity in the physical fermionic Fock space!)

Proof:

$$A := \int \prod_\alpha d\xi^*_{\xi_\alpha} d\xi_{\xi_\alpha} e^{-\sum_{\alpha'} \xi^*_{\alpha'}\xi_{\alpha'}} |\xi\rangle\langle\xi|$$

$|\alpha_1 \dots \alpha_n\rangle$, $|\beta_1 \dots \beta_m\rangle$ } basis states from the Fock space

$= (-1)^P$ if $(\alpha_1 \dots \alpha_n)$ is a permutation of $(\beta_1 \dots \beta_m)$ and 0 otherwise

$$\langle \alpha_1 \dots \alpha_n | \beta_1 \dots \beta_m\rangle = \langle \alpha_1 \dots \alpha_n | A | \beta_1 \dots \beta_m\rangle \quad (*)$$

(This we show.)

$$\langle \alpha_1 \dots \alpha_n | \xi\rangle = \langle 0 | \left(a_{\alpha_n} \dots a_{\alpha_1} |\xi\rangle\right) = \xi_{\alpha_n} \dots \xi_{\alpha_1} \qquad (\langle 0|\xi\rangle = 1)$$

$$\langle \alpha_1 \dots \alpha_n | A | \beta_1 \dots \beta_m\rangle = \int \prod_\alpha d\xi^*_{\xi_\alpha} d\xi_{\xi_\alpha} e^{-\sum_\beta \xi^*_\beta \xi_\beta} \langle \alpha_1 \dots \alpha_n|\xi\rangle \langle\xi|\beta_1 \dots \beta_m\rangle$$

$$= \int \prod_\alpha d\xi^*_{\xi_\alpha} d\xi_{\xi_\alpha} \prod_\beta (1 - \xi^*_\beta \xi_\beta)\, \xi_{\alpha_n} \dots \xi_{\alpha_1} \xi^*_{\beta_1} \dots \xi^*_{\beta_m}$$

Consider the integral arising for a particular state γ:

$$\int d\xi^*_{\xi_\gamma} d\xi_{\xi_\gamma} (1 - \xi^*_\gamma \xi_\gamma) \begin{Bmatrix} \xi_\gamma \xi^*_\gamma \\ \xi^*_\gamma \\ \xi_\gamma \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{Bmatrix}$$

Integral non-zero only if each γ is either occupied in both or not occupied in both

$|\alpha_1 \dots \alpha_n\rangle$, $|\beta_1 \dots \beta_m\rangle \to (\beta_1 \dots \beta_m)$ is a permutation of $(\alpha_1 \dots \alpha_n) \to n = m$, $\beta_i = \alpha_{P(i)}$ (PERMUTATION)

(•) evaluate the integral

Hint: $\xi_{\alpha_n} \dots \xi_{\alpha_1} \xi^*_{\beta_1} \dots \xi^*_{\beta_m} = (-1)^P \xi_{\alpha_n} \dots \xi_{\alpha_1} \xi^*_{\alpha_1} \dots \xi^*_{\alpha_n}$.

$$\int \prod_\alpha d\xi^*_{\xi_\alpha} d\xi_{\xi_\alpha} \prod_\beta (1 - \xi^*_\beta \xi_\beta)\, \xi_{\alpha_n} \dots \xi_{\alpha_1} \xi^*_{\alpha_{P(1)}} \dots \xi^*_{\alpha_{P(n)}} = \int \prod_\alpha d\xi^*_{\xi_\alpha} d\xi_{\xi_\alpha} \prod_\beta (1 - \xi^*_\beta \xi_\beta) (-1)^P \xi_{\alpha_n} \dots \underbrace{\xi_{\alpha_1} \xi^*_{\alpha_1}} \dots \xi^*_{\alpha_n} =$$

commutes with ξ_{α_i} ($i \neq 1$)
→ move to the left
→ integrate
→ do the same with $\xi^*_{\alpha_2} \xi_{\alpha_2}$ and so on.

$$= (-1)^P$$

This demonstrates (*)

• Trace of an operator

$$\mathrm{Tr}\, A = \sum_n \langle n | A | n \rangle \qquad \text{e.g. } A = e^{-(\hat{H} - \mu \hat{N})}$$

$$\cdot\ \mathrm{Tr}\, A = \int \prod_\alpha d\xi_\alpha^* d\xi_\alpha e^{-\sum_\alpha \xi_\alpha^* \xi_\alpha} \sum_n \langle n | \xi \rangle \langle \xi | A | n \rangle = (*)$$

Be careful about interchanging the order of $\langle n | \xi \rangle$ and $\langle \xi | A | n \rangle$

Assume A does not change the particle number.

$$A | n \rangle = \sum_m a_{nm} | m \rangle \qquad p - \text{number of particles}$$

$$| n \rangle = a_{\alpha_1}^+ \dots a_{\alpha_p}^+ | 0 \rangle \qquad \langle n | = \langle 0 | a_{\alpha_p} \dots a_{\alpha_1}$$

$$| m \rangle = a_{\beta_1}^+ \dots a_{\beta_p}^+ | 0 \rangle \qquad \langle n | \xi \rangle = \xi_{\alpha_p} \dots \xi_{\alpha_1}$$

$$\langle n | \xi \rangle \langle \xi | m \rangle = (\xi_{\alpha_p} \dots \xi_{\alpha_1})(\xi_{\beta_1}^* \dots \xi_{\beta_p}^*) = ((-1)^p)^p (\xi_{\beta_1}^* \dots \xi_{\beta_p}^*)(\xi_{\alpha_p} \dots \xi_{\alpha_1})$$

$$= (-1)^{p^2} \langle \xi | m \rangle \langle n | \xi \rangle \qquad (-1)^{p^2} = (-1)^p$$

$$\langle n | \xi \rangle \langle \xi | m \rangle = (-1)^{\overset{\nearrow \text{PARTICLE NUMBER}}{p}} \langle \xi | m \rangle \langle n | \xi \rangle$$

$$\langle n | \xi \rangle \langle \xi | m \rangle = (-1)^p \langle \xi | m \rangle \langle n | \xi \rangle = \langle -\xi | m \rangle \langle n | \xi \rangle$$

E.g. $| \xi \rangle = (1 - \xi_1 a_1^+)(1 - \xi_2 a_2^+) | 0 \rangle \qquad (n = 2)$

$$= | 0 \rangle - \xi_1 | 10 \rangle - \xi_2 | 01 \rangle - \xi_1 \xi_2 | 11 \rangle$$

$$| -\xi \rangle = | 0 \rangle + \xi_1 | 10 \rangle + \xi_2 | 01 \rangle - \xi_1 \xi_2 | 11 \rangle$$

(added a minus to the components with odd particle number)

- $\langle 0 | \xi \rangle \langle \xi | 0 \rangle = 1 \cdot 1 = \langle -\xi | 0 \rangle \langle 0 | \xi \rangle$
- $\langle 10 | \xi \rangle \langle \xi | 10 \rangle = (-\xi_1)(-\xi_1^*) = -\xi_1^* \xi_1 = -\langle \xi | 10 \rangle \langle 10 | \xi \rangle = \langle -\xi | 10 \rangle \langle 10 | \xi \rangle$
- $\langle 11 | \xi \rangle \langle \xi | 11 \rangle = (-\xi_1 \xi_2)(-\xi_2^* \xi_1^*) = (\xi_2^* \xi_1^*)(\xi_1 \xi_2) = \langle \xi | 11 \rangle \langle 11 | \xi \rangle$

$\langle -\xi | 11 \rangle \langle 11 | \xi \rangle =$

Going back to ⊗:

$$\circledast = \int \prod_\alpha d\xi_\alpha^* d\xi_\alpha \, e^{-\sum_\alpha \xi_\alpha^* \xi_\alpha} \sum_n \sum_m a_{nm} \langle -\xi | m \rangle \langle n | \xi \rangle$$

$$= \Big\{ \sum_m a_{nm} |m\rangle = A|n\rangle \Big\} = \int \prod_\alpha d\xi_\alpha^* d\xi_\alpha \, e^{-\sum_\alpha \xi_\alpha^* \xi_\alpha} \langle -\xi | A | \sum_n |n\rangle\langle n | \xi \rangle$$

$$= \int \prod_\alpha d\xi_\alpha^* d\xi_\alpha \, e^{-\sum_\alpha \xi_\alpha^* \xi_\alpha} \langle -\xi | A | \xi \rangle$$

$$\mathrm{Tr}A = \int \prod_\alpha d\xi_\alpha^* d\xi_\alpha \, e^{-\sum_\alpha \xi_\alpha^* \xi_\alpha} \langle -\xi | A | \xi \rangle$$

Note: Fermionic coherent states are not contained in the Fock space.

In particular: $\langle \hat{N} \rangle = \dfrac{\langle \xi | \hat{N} | \xi \rangle}{\langle \xi | \xi \rangle} = \sum_\alpha \xi_\alpha^* \xi_\alpha \notin \mathbb{R}$.