ELECTROMAGNETIC FIELD QUANTIZATION (FREE FIELD)

- Here – only elements
- Some affinity to phonons

Unlike the previous case (phonons) we must start from fields → Maxwell's equations

$$\begin{cases} \nabla \vec{E} = \frac{1}{\epsilon_0}\varrho \\ \nabla \times \bar{B} = \mu_0 \bar{J} + \mu_0\epsilon_0 \frac{\partial \bar{E}}{\partial t} \\ \nabla \bar{B} = 0 \\ \nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \end{cases} \quad \left.\begin{matrix} \\ \end{matrix}\right\} (\times)$$

$$\left.\begin{matrix} \varrho = \varrho(\bar{x},t) \\ \bar{J} = \bar{J}(\bar{x},t) \end{matrix}\right\} \text{sources}$$

Potentials: $\phi(\bar{x},t)$, $\vec{A}(\bar{x},t)$
$$\begin{cases} \bar{B} = \nabla\times\bar{A} \\ \bar{E} = -\nabla\phi - \frac{\partial \bar{A}}{\partial t} \end{cases}$$
(follows from $(\times)$)

$(\phi, \bar{A})$ not uniquely determined by $(\bar{E}, \bar{B})$

Gauge transformation:
$$\begin{cases} \phi \longrightarrow \phi + \frac{\partial f}{\partial t} \\ \bar{A} \longrightarrow \bar{A} - \nabla f \end{cases}$$
leaves $(\bar{E}, \bar{B})$ invariant

Plug $\bar{B} = \nabla\times\bar{A}$, $\bar{E} = -\nabla\phi - \frac{\partial \bar{A}}{\partial t}$ into Maxwell's equations.
$(\times)$ are then fulfilled automatically, while the first two equations give:

$$\begin{cases} -\Delta\phi - \partial_t(\nabla\bar{A}) = \frac{\varrho}{\epsilon_0} \\ \nabla(\nabla\bar{A}) - \Delta\bar{A} = \mu_0\bar{J} + \mu_0\epsilon_0\,\partial_t(-\nabla\phi - \partial_t\bar{A}) \end{cases}$$

$$\left\{ \begin{matrix} c^2 = \frac{1}{\mu_0\epsilon_0} \\ \nabla\times(\nabla\times\bar{A}) = \nabla(\nabla\bar{A}) - \Delta\bar{A} \end{matrix} \right\}$$

Now → Consider free field (no sources)

→ Choose Coulomb gauge ($\nabla \bar{A} = 0$)

Then: The first equation → $\Delta\phi = 0$ (and $\phi = 0$)

↳ the solution vanishing at ∞

{ $\phi = 0$ is the only (bounded) solution up to a constant. Recall: $\phi(\bar{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{d\bar{r}'\varrho(\bar{r}')}{|\bar{r}-\bar{r}'|}$ for charges in a bounded volume. }

The second equation: $\Delta \vec{A} = \frac{1}{c^2}\partial_t^2 \bar{A}$

$$\frac{1}{c^2}\frac{\partial^2 \bar{A}}{\partial t^2} - \Delta \bar{A} = 0 \qquad \otimes$$

Now: assume we are in a large box of volume $V = L^3$ and impose periodic b.c. for $\bar{A}$.

- $\otimes$ is fulfilled by $\bar{A}(\bar{x},t) = \bar{A}_0 e^{i(\bar{k}\bar{x} - \omega t)}$ if $-\frac{\omega^2}{c^2} + \bar{k}^2 = 0$

  → $\omega = \omega_{\bar{k}} = c|\bar{k}|$

  (compare phonons)

- The b.c require that $\bar{k} = \frac{2\pi}{L}(n_x, n_y, n_z)$, $n_x, n_y, n_z \in \mathbb{Z}$.

- The gauge condition is fulfilled if $\bar{A}_0 \bar{k} = 0$

For a given $\bar{k}$ we have an orthonormal basis:

$$\{\bar{\epsilon}_{\bar{k}1}, \bar{\epsilon}_{\bar{k}2}, \underbrace{\bar{\epsilon}_{\bar{k}3}}_{=\frac{\bar{k}}{|\bar{k}|}}\}$$

(freedom of choice of $\bar{\epsilon}_{\bar{k}1}, \bar{\epsilon}_{\bar{k}2}$)

Considering that $\vec{A}(\bar{x},t) \in \mathbb{R}$ the general solution of $\otimes$ may be written as:

$$\bar{A}(\bar{x},t) = \frac{1}{\sqrt{V}}\sum_{\bar{k}\lambda}\left[A_{\bar{k}\lambda} e^{i(\bar{k}\bar{x} - \omega_{\bar{k}}t)} + A^*_{\bar{k}\lambda} e^{-i(\bar{k}\bar{x} - \omega_{\bar{k}}t)}\right]\bar{\epsilon}_{\bar{k}\lambda}$$

$(\lambda \in \{1,2\})$

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Hamiltonian of the free E-M field: $\mathcal{H} = \frac{1}{2}\int d\bar{x} \left\{ \epsilon_0 \bar{E}^2 + \mu_0^{-1} \bar{B}^2 \right\}$

Now: Calculate $\bar{E} = -\frac{\partial \bar{A}}{\partial t}$, $\bar{B} = \nabla \times \bar{A}$, $\int d\bar{x}\, \bar{E}^2$, $\int d\bar{x}\, \bar{B}^2$

(somewhat boring) and plug into $\mathcal{H}$.

the result reads: $\mathcal{H} = 2\epsilon_0 \sum_{\bar{k}\lambda} \omega_{\bar{k}}^2 |A_{\bar{k}\lambda}|^2$

$$A_{\bar{k}\lambda} = A^R_{\bar{k}\lambda} + i A^I_{\bar{k}\lambda}$$

$$\mathcal{H} = 2\epsilon_0 \sum_{\bar{k}\lambda} \omega_{\bar{k}}^2 \left[(A^R_{\bar{k}\lambda})^2 + (A^I_{\bar{k}\lambda})^2 \right]$$

Consider time evolution

Rewrite $\bar{A}(\bar{x},t)$

$$\bar{A}(\bar{x},t) = \frac{1}{\sqrt{V}} \sum_{\bar{k}\lambda} \left[(A^R_{\bar{k}\lambda} + i A^I_{\bar{k}\lambda}) e^{i(\bar{k}\bar{x} - \omega_{\bar{k}} t)} + (A^R_{\bar{k}\lambda} - i A^I_{\bar{k}\lambda}) e^{-i(\bar{k}\bar{x} - \omega_{\bar{k}} t)} \right] \bar{\epsilon}_{\bar{k}\lambda} =$$

$$= \left\{ A_{\bar{k}\lambda}(t) := (A^R_{k\lambda} + i A^I_{k\lambda}) e^{-i\omega_{\bar{k}} t} \right\} =$$

$$= \frac{1}{\sqrt{V}} \sum_{\bar{k}\lambda} \left[A_{\bar{k}\lambda}(t) e^{i\bar{k}\bar{x}} + A^*_{\bar{k}\lambda}(t) e^{-i\bar{k}\bar{x}} \right]$$

use $e^{i\varphi} = \cos\varphi + i \sin\varphi$

$$A_{\bar{k}\lambda}(t) = \underbrace{A^R_{k\lambda} \cos(\omega_{\bar{k}} t) + A^I_{\bar{k}\lambda} (\sin \omega_{\bar{k}} t)}_{:= A^R_{\bar{k}\lambda}(t)} + i \underbrace{\left[A^I_{\bar{k}\lambda} \cos(\omega_{\bar{k}} t) - A^R_{\bar{k}\lambda} \sin(\omega_{\bar{k}} t) \right]}_{:= A^I_{\bar{k}\lambda}(t)}$$

Observe

$$\dot{A}^R_{\bar{k}\lambda} = \omega_{\bar{k}} A^I_{k\lambda}$$
$$\dot{A}^I_{\bar{k}\lambda} = -\omega_{\bar{k}} A^R_{\bar{k}\lambda}$$

On the other hand:

$$\frac{\partial \mathcal{H}}{\partial A^R_{\bar{k}\lambda}} = 4\epsilon_0 \omega_{\bar{k}}^2 A^R_{\bar{k}\lambda}$$

$$\frac{\partial \mathcal{H}}{\partial A^I_{\bar{k}\lambda}} = 4\epsilon_0 \omega_{\bar{k}}^2 A^I_{k\lambda}$$

$$\begin{cases} \dot{A}^R_{\bar{k}\lambda} = \dfrac{1}{4\epsilon_0 \omega_k} \dfrac{\partial H}{\partial A^I_{\bar{k}\lambda}} \\ \dot{A}^I_{\bar{k}\lambda} = -\dfrac{1}{4\epsilon_0 \omega_{\bar{k}}} \dfrac{\partial H}{\partial A^R_{\bar{k}\lambda}} \end{cases}$$

Scale away the unwanted factors.

$$Q_{\bar{k}\lambda} := 2\sqrt{\epsilon_0}\, A^R_{\bar{k}\lambda}$$
$$P_{\bar{k}\lambda} := 2\sqrt{\epsilon_0}\, \omega_{\bar{k}}\, A^I_{\bar{k}\lambda}$$

- $$H = \sum_{\bar{k}\lambda} \frac{1}{2}\left(P^2_{\bar{k}\lambda} + \omega_{\bar{k}}^2 Q^2_{\bar{k}\lambda}\right)$$

- $$\begin{cases} \dot{Q}_{\bar{k}\lambda} = \dfrac{\partial H}{\partial P_{\bar{k}\lambda}} \\ \dot{P}_{\bar{k}\lambda} = -\dfrac{\partial H}{\partial Q_{\bar{k}\lambda}} \end{cases}$$

We know how to quantize oscillators!

$$Q_{\bar{k}\lambda} \to \hat{Q}_{\bar{k}\lambda} = \sqrt{\frac{\hbar}{2\omega_{\bar{k}}}}\left(a^\dagger_{\bar{k}\lambda} + a_{\bar{k}\lambda}\right)$$
$$P_{\bar{k}\lambda} \to \hat{P}_{k\lambda} = \sqrt{\frac{\hbar\omega_{\bar{k}}}{2}}\, i\left(a^\dagger_{\bar{k}\lambda} - a_{\bar{k}\lambda}\right)$$

$$H \to \hat{H} = \sum_{\bar{k}\lambda} \hbar\omega_{\bar{k}\lambda}\left(a^\dagger_{\bar{k}\lambda} a_{\bar{k}\lambda} + \frac{1}{2}\right)$$ $a^\dagger_{\bar{k}\lambda}$ (Photon creation op)

$$\left[a_{\bar{k}\lambda}, a^\dagger_{\bar{k}'\lambda'}\right] = \delta_{\bar{k}\bar{k}'}\delta_{\lambda\lambda'} \quad \text{(Bosons)}.$$

Summary / comparison

- Lattice vibrations (phonons)
- Electromagnetic field (photons)

$\Big\} \to H = \sum_{\bar{k}\lambda} \hbar\omega_{\bar{k}\lambda}\left(a^+_{\bar{k}\lambda} a_{\bar{k}\lambda} + \frac{1}{2}\right)$

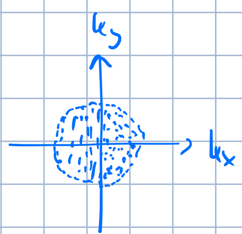
$$\left[a_{\bar{k}\lambda}, a^+_{\bar{k}'\lambda'}\right] = \delta_{\bar{k}\bar{k}'}\delta_{\lambda\lambda'}$$

(Bosons)

Differences:
- $\lambda \in \{1,2,3\}$ vs $\lambda \in \{1,2\}$
- Different $\omega_{\bar{k}\lambda}$ (but linear for $|\bar{k}|$ small in each case)
- Upper bound for $|\bar{k}|$ in case of phonons (implies the classical limit for T large)
- Completely different starting points (fields vs vibrating ions)

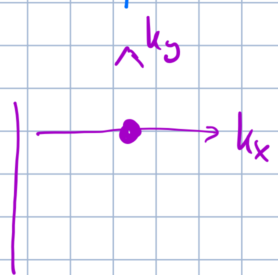
FREE QUANTUM GASES

Exercise class — BOSONS



GS $\to$ Fermi sea

LECTURE — FERMIONS



GS $\to$ many particles "close to $\bar{k}=0$".

Simplest case: system of non-interacting bosonic particles:

(neglect spin for simplicity)

$$H = \sum_{\bar{k}} \frac{\hbar^2\bar{k}^2}{2m} a^+_{\bar{k}} a_{\bar{k}} = H_{kin}$$

- analyzed first in 1920s (Bose, Einstein)

{see Stat. phys course for detailed solution}

- phenomenon of Bose-Einstein condensation:

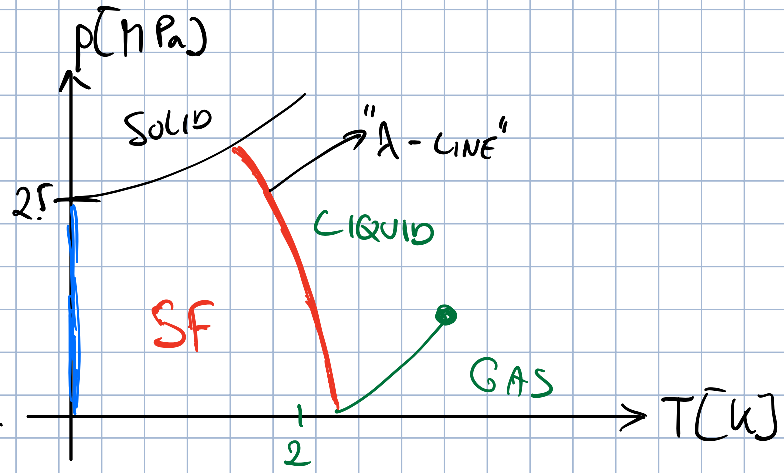
Consider $N_0 = \langle N_{\vec{k}=0} \rangle$ $\quad \frac{N_0}{V} \xrightarrow[\frac{N}{V}=\text{const}]{N,V\to\infty} \begin{cases} n_0 > 0 & T < T_c(n) \\ 0 & T \geq T_c(n) \end{cases}$

$$T_c(n) = \frac{2\pi\hbar^2}{mk_B}\left(\frac{n}{\zeta(\frac{3}{2})}\right)^{\frac{2}{3}} = \text{const}\cdot n^{\frac{2}{3}}$$

$T = T_c$ – phase transition

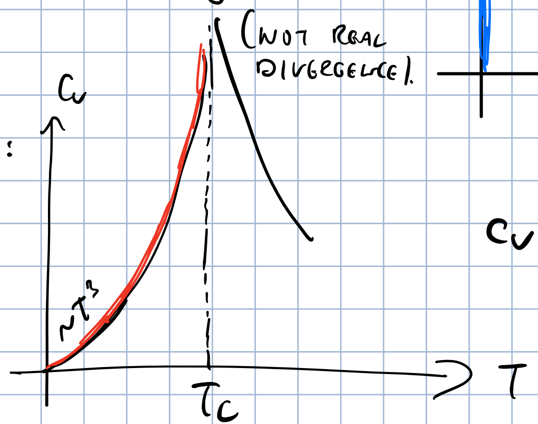
1930s – experiments on liquid helium (^4He) reveal a "superfluid" state at $T < T_c \approx 2K$.

Phase diagram (^4He)



SF – "frictionless" flow

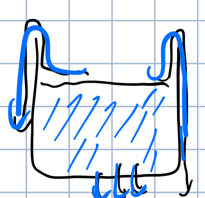
λ-line:



$$c_v = \begin{cases} c(T) + A_+ |T - T_c|^{-\alpha} & T > T_c \\ c(T) + A_- |T - T_c|^{-\alpha} & T < T_c \end{cases}$$

$c(T)$ – smooth

$\alpha \simeq -0.009$



See YouTube movies

Connection between B-E condensation and superfluidity of ^4He?

{take m, n pertinent to $^4He \to T_c \approx 3.1K$}

Strongly interacting particles.

Landau: No superfluidity for non-interacting bosons with $\epsilon \sim \bar{p}^2$.

Bogolyubov - (approximate) theory of (weakly) interacting bosons

↳ more adequate to BECs in ultracold atomic systems than to ^4He.

Consider weakly interacting bosons (spinless for simplicity)

Focus on $T=0 \to |GS\rangle$

Recall: $H = H_{kin} + H_{int} = \sum_{\bar{k}} \epsilon_{\bar{k}} a^+_{\bar{k}} a_{\bar{k}} + \frac{1}{2V} \sum_{\bar{k}\bar{k}'\bar{q}} V_{\bar{q}}\, a^+_{\bar{k}-\bar{q}}\, a^+_{\bar{k}'+\bar{q}}\, a_{\bar{k}'} a_{\bar{k}}$

$\left\{ V_{\bar{q}} = \int d\bar{r}\, V(\bar{r}) e^{i\bar{q}\bar{r}} \right\}$

Non-interacting situation

→ all particles in state $\bar{k}=0$

assumption: after switching (very weak) interactions, a "very large" fraction of particles will still be in the $\bar{k}=0$ state.

$$N - N_0 \ll N_0$$

$N \approx 10^{23}$
$N_0 \approx 10^{23}$

$$a_0 |GS\rangle \simeq \sqrt{N_0}\, |GS\rangle$$
$$a_0^+ |GS\rangle \simeq \sqrt{N_0}\, |GS\rangle$$

→ Bogolyubov approximation

$$\left\{ [a_0, a_0^+] = 1 \qquad \underbrace{[a_0, a_0^+]}_{\mathcal{O}(1)} |GS\rangle = |GS\rangle \qquad 1 \ll \sqrt{N_0}. \right\}$$

$$H_{int} = \frac{1}{2V} \sum_{\bar{k}\bar{k}'\bar{q}} V_{\bar{q}}\, a^{\dagger}_{\bar{k}-\bar{q}}\, a^{\dagger}_{\bar{k}'+\bar{q}}\, a_{\bar{k}'}\, a_{\bar{k}} = \frac{1}{2V}\Big[V_0\, a^{\dagger}_0 a^{\dagger}_0 a_0 a_0 + \sum_{\bar{q}\neq 0} V_{\bar{q}}\, a^{\dagger}_{-\bar{q}} a^{\dagger}_{\bar{q}} a_0 a_0 +$$

$$+ \sum_{\bar{q}\neq 0} V_{\bar{q}}\, a^{\dagger}_0 a^{\dagger}_0 a_{-\bar{q}} a_{\bar{q}} + \sum_{\bar{q}\neq 0} V_{\bar{q}}\, a^{\dagger}_{-q} a^{\dagger}_0 a_{-\bar{q}} a_0 + V_0 \sum_{\bar{k}'\neq 0} a^{\dagger}_0 a^{\dagger}_{\bar{k}'} a_{\bar{k}'} a_0 +$$

$$+ V_0 \sum_{\bar{k}\neq 0} a^{\dagger}_{\bar{k}} a^{\dagger}_0 a_0 a_{\bar{k}} + \sum_{\bar{q}\neq 0} V_{\bar{q}}\, a^{\dagger}_0 a^{\dagger}_{\bar{q}} a_0 a_{\bar{q}} \Big] + R \simeq$$

R ↳ Neglect

$$= \frac{1}{2V}\Big[V_0 N_0^2 + 2V_0 N_0 \sum_{\bar{k}\neq 0} a^{\dagger}_{\bar{k}} a_{\bar{k}} + \underbrace{2 N_0 \sum_{\bar{q}\neq 0} V_{\bar{q}}\, a^{\dagger}_{\bar{q}} a_{\bar{q}}}_{\text{4th and 7th terms, } V_{\bar{q}} = V_{-\bar{q}}} +$$

$$+ N_0 \sum_{\bar{q}\neq 0} V_{\bar{q}} \left(a^{\dagger}_{-\bar{q}} a^{\dagger}_{\bar{q}} + a_{-\bar{q}} a_{\bar{q}} \right) \Big]$$

$$\Big\{ V_0 N_0^2 + 2V_0 N_0 \underbrace{\sum_{\bar{k}\neq 0} a^{\dagger}_{\bar{k}} a_{\bar{k}}}_{\delta N} = V_0 N^2 + \mathcal{O}(a_k^4) \Big\}$$

$$N^2 = (N_0 + \delta N)^2 = N_0^2 + 2N_0 \delta N + \delta N^2$$

• Add the kinetic term: $\displaystyle \sum_{\bar{k}} \epsilon_{\bar{k}}\, a^{\dagger}_{\bar{k}} a_{\bar{k}} \underset{\epsilon_0 = 0}{=} \sum_{\bar{k}\neq 0} \epsilon_{\bar{k}}\, a^{\dagger}_{\bar{k}} a_{\bar{k}}$

$$\eta_{\bar{k}} := \frac{N_0}{V} V_{\bar{k}}$$

$$H \simeq \frac{1}{2}\frac{V_0 N^2}{V} + \underbrace{\sum_{\bar{k}\neq 0} \left(\epsilon_{\bar{k}} + \eta_{\bar{k}}\right) a^+_{\bar{k}} a_{\bar{k}}}_{\text{(Familiar)}\ \sum_{\bar{k}} \hbar \Omega_{\bar{k}} a^+_{\bar{k}} a_{\bar{k}}} + \underbrace{\frac{1}{2}\sum_{\bar{k}} \eta_{\bar{k}} \left(a^+_{\bar{k}} a^+_{-\bar{k}} + a_{\bar{k}} a_{-\bar{k}}\right)}_{\text{(Particle number not conserved)}}$$

Diagonalization of H – Bogolyubov transformation.

$$\alpha_{\bar{k}} := \mathrm{ch}\theta_{\bar{k}}\, a_{\bar{k}} - \mathrm{sh}\theta_{\bar{k}}\, a^+_{-\bar{k}}$$

{Homework – show that $\{\alpha_{\bar{k}}\}$ preserve the commutation relations}

Consider
$$\alpha^+_{k}\alpha_{\bar{k}} = \mathrm{ch}^2\theta_{\bar{k}}\, a^+_{\bar{k}} a_{\bar{k}} - \mathrm{ch}\theta_{\bar{k}}\, a^+_{\bar{k}}\, \mathrm{sh}\theta_{\bar{k}}\, a^+_{-\bar{k}} + $$
$$- \mathrm{sh}\theta_{\bar{k}}\, a_{-\bar{k}}\, \mathrm{ch}\theta_{\bar{k}}\, a_{\bar{k}} + \mathrm{sh}^2\theta_{\bar{k}} \underbrace{a_{-\bar{k}} a^+_{-\bar{k}}}_{1 + a^+_{-\bar{k}} a_{-\bar{k}}}$$

Take: $\sum_{\bar{k}} \hbar \omega_{\bar{k}} \alpha^+_{\bar{k}} \alpha_{\bar{k}}$ with $\omega_{-\bar{k}} = \omega_{\bar{k}}$ $\theta_{-\bar{k}} = \theta_{\bar{k}}$.

$$\sum_{\bar{k}\neq 0} \hbar \omega_{\bar{k}} \alpha^+_{\bar{k}} \alpha_{\bar{k}} = \sum_{\bar{k}\neq 0} \hbar \omega_{k} \overbrace{\left(\mathrm{ch}^2\theta_{\bar{k}} + \mathrm{sh}^2\theta_{\bar{k}}\right)}^{\mathrm{ch}2\theta_{\bar{k}}} a^+_{\bar{k}} a_{\bar{k}} + \sum_{\bar{k}\neq 0} \hbar \omega_{\bar{k}}\, \mathrm{sh}^2\theta_{\bar{k}}$$
$$- \frac{1}{2}\sum_{\bar{k}\neq 0} \hbar \omega_{\bar{k}}\, \mathrm{sh}2\theta_{\bar{k}} \left(a^+_{\bar{k}} a^+_{-\bar{k}} + a_{\bar{k}} a_{-\bar{k}}\right)$$

Structurally this is equivalent to H after identifying:

$$\begin{cases} \hbar\omega_{\bar{k}}\,\mathrm{ch}\,2\theta_{\bar{k}} = \hbar\Omega_{\bar{k}} \\ \hbar\omega_{\bar{k}}\,\mathrm{sh}\,2\theta_{\bar{k}} = -\eta_{\bar{k}} \end{cases}$$

Take squares of these equations and subtract.

$$\longrightarrow \quad \hbar\omega_{\bar{k}}^2 = (\hbar\Omega_{\bar{k}})^2 - \eta_{\bar{k}}^2$$

$$\hbar\omega_{\bar{k}} = \sqrt{\left(\epsilon_{\bar{k}} + \frac{N_0}{V}V_{\bar{k}}\right)^2 - \left(\frac{N_0 V_{\bar{k}}}{V}\right)^2}$$

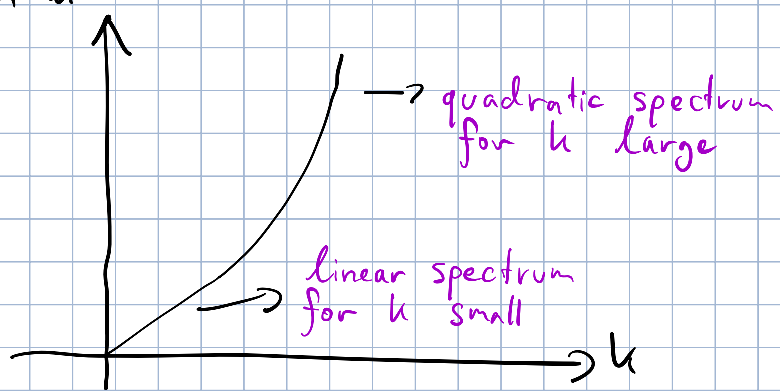
Short-ranged interactions:

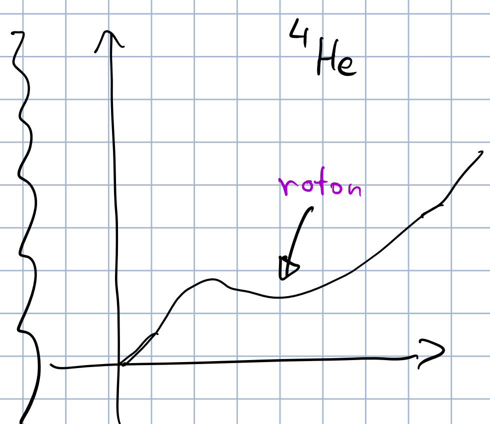
$$V_{\bar{k}} \xrightarrow{k\to 0} \text{const} = V_0$$

$$k \text{ small} \longrightarrow \hbar\omega_{\bar{k}} \sim \sqrt{\epsilon_{\bar{k}}} \sim k$$

$$k \text{ large} \longrightarrow \hbar\omega_{\bar{k}} \sim \epsilon_{\bar{k}} \sim k^2$$

$\hbar\omega_{\bar{k}}$





Why is this important?

Landau: No superfluidity for bosons with quadratic dispersion.

(heuristic arguments based on Galilean transformation)