• Two-body operators and creation/annihilation operators:

First Pair number operator $\{|\alpha\rangle, |\beta\rangle\}$ $|\alpha_1 \dots \alpha_N\}$

$$\begin{cases} \text{if } |\alpha\rangle \neq |\beta\rangle & n_\alpha n_\beta \\ \text{if } |\alpha\rangle = |\beta\rangle & n_\alpha (n_\alpha - 1) \end{cases}$$

$$\hat{P}_{\alpha\beta} := \hat{n}_\alpha \hat{n}_\beta - \delta_{\alpha\beta} \hat{n}_\beta$$

Some rewriting: $\hat{P}_{\alpha\beta} = a_\alpha^+ (a_\alpha a_\beta^+) a_\beta - \delta_{\alpha\beta} a_\alpha^+ a_\alpha = a_\alpha^+ (\zeta a_\beta^+ a_\alpha + \cancel{\delta_{\alpha\beta}}) a_\beta - \cancel{\delta_{\alpha\beta} a_\alpha^+ a_\alpha}$

$$= \zeta a_\alpha^+ a_\beta^+ \underbrace{a_\alpha a_\beta}_{\zeta a_\beta a_\alpha} = a_\alpha^+ a_\beta^+ a_\beta a_\alpha$$

Now: take an arbitrary 2-body op. $\hat{V} = \frac{1}{2} \sum_{i \neq j} \hat{V}_{ij}$ and the corresponding eigenbasis.

$$\begin{cases} \hat{V} |\alpha\beta) = V_{\alpha\beta} |\alpha\beta) \\ V_{\alpha\beta} = (\alpha\beta| \hat{V} |\alpha\beta) \end{cases}$$

$$\hat{V} |\alpha_1 \dots \alpha_N\} = \hat{V} \sum_P \frac{1}{\sqrt{N!}} \zeta^P |\alpha_{P1} \dots \alpha_{PN}) = \frac{1}{2} \sum_{i \neq j} \hat{V}_{ij} \sum_P \frac{1}{\sqrt{N!}} \zeta^P |\alpha_{P1} \dots \alpha_{PN}) =$$

$$= \sum_P \frac{1}{\sqrt{N!}} \zeta^P \underbrace{\frac{1}{2} \sum_{i \neq j} V_{\alpha_{Pi} \alpha_{Pj}}}_{= \frac{1}{2} \sum_{i \neq j} V_{\alpha_i \alpha_j} = \frac{1}{2} \sum_{\alpha\beta} V_{\alpha\beta} P_{\alpha\beta}} |\alpha_{P1} \dots \alpha_{PN}) =$$

$$\hat{V} |\alpha_1 \dots \alpha_N\} = \frac{1}{2} \sum_{\alpha\beta} V_{\alpha\beta} \underbrace{P_{\alpha\beta} |\alpha_1 \dots \alpha_N\}}_{\hat{P}_{\alpha\beta} |\alpha_1 \dots \alpha_N\}} = \frac{1}{2} \sum_{\alpha\beta} V_{\alpha\beta} \hat{P}_{\alpha\beta} |\alpha_1 \dots \alpha_N\}$$

$$\boxed{\hat{V} = \frac{1}{2} \sum_{\alpha\beta} V_{\alpha\beta} \hat{P}_{\alpha\beta} = \frac{1}{2} \sum_{\alpha\beta} (\alpha\beta| \hat{V} |\alpha\beta) a_\alpha^+ a_\beta^+ a_\beta a_\alpha}$$

arbitrary basis

$$a^+_\alpha = \sum_{\tilde{\alpha}} \langle \tilde{\alpha} | \alpha \rangle a^+_{\tilde{\alpha}}$$

$$a_\alpha = \sum_{\tilde{\alpha}} \langle \alpha | \tilde{\alpha} \rangle a_{\tilde{\alpha}}$$

$$\hat{V} = \frac{1}{2} \sum_{\alpha\beta} V_{\alpha\beta} \sum_{\tilde{\alpha}_1} \sum_{\tilde{\alpha}_2} \sum_{\tilde{\beta}_1} \sum_{\tilde{\beta}_2} \langle \tilde{\alpha}_1 | \alpha \rangle a^+_{\tilde{\alpha}_1} \langle \tilde{\beta}_1 | \beta \rangle a^+_{\tilde{\beta}_1} \langle \beta | \tilde{\beta}_2 \rangle a_{\tilde{\beta}_2} \langle \alpha | \tilde{\alpha}_2 \rangle a_{\tilde{\alpha}_2} =$$

$$= \frac{1}{2} \sum_{\tilde{\alpha}_1\tilde{\alpha}_2} \sum_{\tilde{\beta}_1\tilde{\beta}_2} \left(\langle \tilde{\alpha}_1 | \otimes \langle \tilde{\beta}_1 | \right) \left[\sum_{\alpha\beta} V_{\alpha\beta} | \alpha \rangle \otimes | \beta \rangle \langle \alpha | \otimes \langle \beta | \right] \left(| \tilde{\alpha}_2 \rangle \times | \tilde{\beta}_2 \rangle \right) \cdot$$

$$\cdot\, a^+_{\tilde{\alpha}_1} a^+_{\tilde{\beta}_1} a_{\tilde{\beta}_2} a_{\tilde{\alpha}_2}$$

$$\hat{V} = \frac{1}{2} \sum_{\tilde{\alpha}_1\tilde{\alpha}_2\tilde{\beta}_1\tilde{\beta}_2} (\tilde{\alpha}_1 \tilde{\beta}_1 | \hat{V} | \tilde{\alpha}_2 \tilde{\beta}_2) a^+_{\tilde{\alpha}_1} a^+_{\tilde{\beta}_1} a_{\tilde{\beta}_2} a_{\tilde{\alpha}_2}$$

$$\boxed{\hat{V} = \frac{1}{2} \sum_{\beta_1\beta_2\beta_3\beta_4} (\beta_1 \beta_2 | \hat{V} | \beta_4 \beta_3 \rangle a^+_{\beta_1} a^+_{\beta_2} a_{\beta_3} a_{\beta_4}}$$

<u>Important example</u>

2-body interaction operator (such as Coulomb)

$\hat{V}$ - diagonal in position basis $|\bar{r}, \sigma\rangle$

$$(\bar{r}_1 \sigma_1 \bar{r}_2 \sigma_2 | \hat{V} | \bar{r}_3 \sigma_3 \bar{r}_4 \sigma_4) = \underbrace{V(\bar{r}_3 - \bar{r}_4)}_{\text{for Coulomb}} \delta(\bar{r}_1 - \bar{r}_3) \delta(\bar{r}_2 - \bar{r}_4) \delta_{\sigma_1\sigma_3} \delta_{\sigma_2\sigma_4}$$

this is $\dfrac{\tilde{e}^2}{|\bar{r}_3 - \bar{r}_4|} = \dfrac{e^2}{4\pi\epsilon_0 |\bar{r}_3 - \bar{r}_4|}$

$$\hat{V} = \frac{1}{2} \sum_{\sigma_1\sigma_2} \int d\bar{r}_1 \int d\bar{r}_2\, V(\bar{r}_1 - \bar{r}_2)\, \Psi^+_{\sigma_1}(\bar{r}_1)\, \Psi^+_{\sigma_2}(\bar{r}_2)\, \Psi_{\sigma_2}(\bar{r}_2)\, \Psi_{\sigma_1}(\bar{r}_1)$$

$\hat{V}$ in position basis.

Transformation to momentum basis

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$$\hat{V} = \frac{1}{2} \sum_{\bar{k}_1\sigma_1} \sum_{\bar{k}_2\sigma_2} \sum_{\bar{k}_3\sigma_3} \sum_{\bar{k}_4\sigma_4} (\bar{k}_3\sigma_3\, \bar{k}_4\sigma_4 | \hat{V} | \bar{k}_1\sigma_1\, \bar{k}_2\sigma_2)\, a^+_{\bar{k}_3\sigma_3} a^+_{\bar{k}_4\sigma_4} a_{\bar{k}_2\sigma_2} a_{\bar{k}_1\sigma_1}$$

$$\downarrow$$

$$\left\{ \sum_{\sigma_1'\sigma_2'} \int d\bar{r}_1 \int d\bar{r}_2\, |\bar{r}_1\sigma_1'\, \bar{r}_2\sigma_2') V(\bar{r}_1 - \bar{r}_2)(\bar{r}_1\sigma_1'\, \bar{r}_2\sigma_2'| \qquad \langle \bar{k}_i | \bar{r}_i \rangle = \frac{1}{\sqrt{V}} e^{-i\bar{k}_i\bar{r}_i} \right.$$

$$= \frac{1}{2}\frac{1}{V^2} \sum_{\sigma_1\sigma_2} \sum_{\bar{k}_1\bar{k}_2\bar{k}_3\bar{k}_4} \int d\bar{r}_1 \int d\bar{r}_2\, V(\bar{r}_1 - \bar{r}_2)\, e^{i(-\bar{r}_1\bar{k}_3 - \bar{r}_2\bar{k}_4 + \bar{r}_1\bar{k}_1 + \bar{r}_2\bar{k}_2)} a^+_{\bar{k}_3\sigma_1} a^+_{\bar{k}_4\sigma_2} a_{\bar{k}_2\sigma_2} a_{\bar{k}_1\sigma_1} =$$

$$\left\{ \begin{array}{l} \bar{r} := \bar{r}_2 - \bar{r}_1 \\ \bar{q} := \bar{k}_2 - \bar{k}_4 \end{array} \right. \qquad e^{i(\dots)} = e^{i(\bar{r}_2 - \bar{r}_1)(\bar{k}_2 - \bar{k}_4)} e^{i(\bar{k}_1\bar{r}_1 - \bar{k}_3\bar{r}_1 + \bar{k}_2\bar{r}_1 - \bar{k}_4\bar{r}_1)} =$$

$$= e^{i(\bar{k}_2 - \bar{k}_4)(\bar{r}_2 - \bar{r}_1)} e^{i(\bar{k}_1 + \bar{k}_2 - \bar{k}_3 - \bar{k}_4)\bar{r}_1}$$

$$\int d\bar{r}_1 \int d\bar{r}_2 \longrightarrow \int d\bar{r} \int d\bar{r}_1$$

- $\int d\bar{r}\, V(\bar{r}) e^{i\bar{q}\bar{r}} =: V_{\bar{q}}$
- $\int d\bar{r}_1\, e^{i(\bar{k}_1 - \bar{k}_3 + \bar{q})\bar{r}_1} = V \delta_{\bar{k}_1 - \bar{k}_3 + \bar{q}, 0}$

$$= \frac{1}{2\cancel{V^2}} \sum_{\sigma_1\sigma_2} \sum_{\bar{k}_1 \dots \bar{k}_4} \cancel{V} \delta_{\bar{k}_1 - \bar{k}_3 + \bar{q}, 0} V_{\bar{q}}\, a^+_{\bar{k}_3\sigma_1} a^+_{\bar{k}_4\sigma_2} a_{\bar{k}_2\sigma_2} a_{\bar{k}_1\sigma_1} = \left\{ \begin{array}{l} \bar{k}_4 = \bar{k}_2 - \bar{q} \\ \bar{k}_3 = \bar{k}_1 + \bar{q} \ \ (\text{from } \delta \dots) \\ \sum_{\bar{k}_4} \longrightarrow \sum_{\bar{q}} \end{array} \right\} =$$

$$= \frac{1}{2V} \sum_{\sigma_1\sigma_2} \sum_{\bar{k}_1\bar{k}_2\bar{q}} V_{\bar{q}}\, a^+_{\bar{k}_1 + \bar{q}\,\sigma_1} a^+_{\bar{k}_2 - \bar{q}\,\sigma_2} a_{\bar{k}_2\sigma_2} a_{\bar{k}_1\sigma_1}$$