Continue with many-body states.

Orthonormal basis in $\mathcal{H} \longmapsto$ orthonormal basis in $\mathcal{H}_N$

$$\{|\alpha\rangle\} \longrightarrow \{|\alpha_1 \ldots \alpha_N) = |\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \ldots \otimes |\alpha_N\rangle\}$$

Wavefunctions corresponding to such basis states:

$$\psi_{\alpha_1\alpha_2\ldots\alpha_N}(\bar{r}_1,\bar{r}_2\ldots\bar{r}_N) = (\bar{r}_1,\bar{r}_2\ldots\bar{r}_N|\alpha_1,\alpha_2,\ldots,\alpha_N) = (\langle\bar{r}_1|\otimes\langle\bar{r}_2|\otimes\ldots\langle\bar{r}_N|)(|\alpha_1\rangle\otimes|\alpha_2\rangle\otimes\ldots|\alpha_N\rangle) =$$
$$= \langle\bar{r}_1|\alpha_1\rangle\langle\bar{r}_2|\alpha_2\rangle\ldots\langle\bar{r}_N|\alpha_N\rangle = \psi_{\alpha_1}(\bar{r}_1)\psi_{\alpha_2}(\bar{r}_2)\ldots\psi_{\alpha_N}(\bar{r}_N)$$

Completeness relation in $\mathcal{H}$: $\sum_\alpha |\alpha\rangle\langle\alpha| = \mathbb{1}$

$\rightarrow$ Completeness relation in $\mathcal{H}_N$: $\sum_{\alpha_1\ldots\alpha_N} |\alpha_1\ldots\alpha_N)(\alpha_1\ldots\alpha_N| = \mathbb{1}$.

$\mathcal{H}_N$ is <u>not</u> the space of physical states (i.e. not all the states of $\mathcal{H}_N$ are physical – see below)

Symmetry requirements

For systems of identical particles we require that

$$\psi(\bar{r}_{P1},\bar{r}_{P2},\ldots\bar{r}_{PN}) = \psi(\bar{r}_1\ldots\bar{r}_N) \qquad \text{(bosons)}$$
$$\psi(\bar{r}_{P1},\bar{r}_{P2},\ldots\bar{r}_{PN}) = (-1)^P\psi(\bar{r}_1\ldots\bar{r}_N) \qquad \text{(fermions)}$$

Here $P: (1,2,\ldots N) \longmapsto (P1, P2, \ldots PN)$ is a permutation and $(-1)^P$ is its sign

Notation: $\psi(\bar{r}_{P1},\bar{r}_{P2}\ldots\bar{r}_{PN}) = (\zeta)^P\psi(\bar{r}_1\ldots\bar{r}_N)$ $\qquad \zeta = \begin{cases} 1 & \text{bosons} \\ -1 & \text{fermions}\end{cases}$

To generate physical states out of $\mathcal{H}_N$, we introduce the symmetrization ($\mathcal{P}_B$) and antisymmetrization ($\mathcal{P}_F$) operators.

In position representation $\mathcal{P}_{\{B,F\}}$ act as follows:

$$\mathcal{P}_{\{B,F\}}\psi(\bar{r}_1\ldots\bar{r}_N) = \frac{1}{N!}\sum_{P}\zeta^P\psi(\bar{r}_{P1},\bar{r}_{P2},\ldots,\bar{r}_{PN})$$

$\hookrightarrow$ sum over permutations of N elements

$\mathcal{P}_{\{B,F\}}$ are
- hermitian
- projections ($\mathcal{P}^2_{\{B,F\}} = \mathcal{P}_{\{B,F\}}$)

$\downarrow$ see the exercise class

Example: $N=2$

$$\mathcal{P}_B\psi(\bar{r}_1,\bar{r}_2) = \frac{1}{2}(\psi(\bar{r}_1,\bar{r}_2) + \psi(\bar{r}_2,\bar{r}_1))$$
$$\mathcal{P}_F\psi(\bar{r}_1,\bar{r}_2) = \frac{1}{2}(\psi(\bar{r}_1,\bar{r}_2) - \psi(\bar{r}_2,\bar{r}_1))$$

Consider a basis of $\mathcal{H}$ $\{|\alpha\rangle\}$ (orthonormal).

A state of N bosons (fermions) with one particle in state $|\alpha_1\rangle$, one particle in state $|\alpha_2\rangle$ ... one particle in state $|\alpha_N\rangle$ can be represented as

$$|\alpha_1 \dots \alpha_N\} := \sqrt{N!}\, P_{\{B/F\}} |\alpha_1 \dots \alpha_N) = \frac{1}{\sqrt{N!}} \sum_P \zeta^P |\alpha_{P1}\rangle \otimes |\alpha_{P2}\rangle \otimes \dots \otimes |\alpha_{PN}\rangle$$

{Normalization factor (see later)}

Remarks: • $|\alpha_1 \dots \alpha_N\}$ is not normalized to unity.

• for fermions $\alpha_j = \alpha_k$ $\Rightarrow |\alpha_1 \dots \alpha_j \dots \alpha_k \dots \alpha_N\} = 0$ (Pauli principle)

$|\alpha_1 \dots \alpha_j \dots \alpha_k \dots \alpha_N\} = \zeta |\alpha_1 \dots \alpha_k \dots \alpha_j \dots \alpha_N\}$

Define: $\mathcal{B}_N := P_B \mathcal{H}_N$

$F_N := P_F \mathcal{H}_N$ $\longrightarrow$ space of physical states

$\mathcal{B}_N, F_N$ inherit the vector space structure from $\mathcal{H}_N$.

Operators

We now discuss classes of operators acting in $\mathcal{B}_N$ or $\mathcal{F}_N$, which will be useful for physics.

We call $\hat{U}$ a one-body operator if $\hat{U}|\alpha_1 \dots \alpha_N) = \sum_{i=1}^{N} \hat{U}_i |\alpha_1 \dots \alpha_N)$, where $\hat{U}_i$ acts only on the i-th state.

{ Formally one might write: $\hat{U}_i \to \mathbb{1} \otimes \mathbb{1} \otimes \dots \mathbb{1} \otimes \hat{U}_i \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1}$ }

{ Example: Kinetic energy op. in the $\vec{p}$-basis:

$$\hat{T}|\vec{p}_1 \dots \vec{p}_N) = \sum_{i=1}^{N} \frac{\hat{\vec{p}}_i^{\,2}}{2m} |\vec{p}_1 \dots \vec{p}_N) = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} |\vec{p}_1 \dots \vec{p}_N)$$ }

Matrix elements: $(\alpha_1 \dots \alpha_N|\hat{U}|\beta_1 \dots \beta_N) = \sum_{i=1}^{N} (\alpha_1 \dots \alpha_N|\hat{U}_i|\beta_1 \dots \beta_N) = \sum_{i=1}^{N} \left[\left(\prod_{k \neq i} \langle \alpha_k | \beta_k \rangle \right) \langle \alpha_i | \hat{U}_i | \beta_i \rangle \right]$

$\hat{U}$ is fully determined by its action in a single-particle space. ($\mathcal{H}$)

We call $\hat{V}$ a two-body operator if $\hat{V}|\alpha_1 \dots \alpha_N) = \sum_{1 \le i < j \le N} \hat{V}_{ij} |\alpha_1 \dots \alpha_N)$, where $\hat{V}_{ij}$ acts only on i-th and j-th state.

{ We additionally require $\hat{V}_{ij} = \hat{V}_{ji}$ }

[Equivalently: $\sum_{1 \le i < j \le N} \longrightarrow \frac{1}{2} \sum_{1 \le i \ne j \le N}$ - summation over pairs]

Matrix elements: $$(\alpha_1\ldots\alpha_N|\hat{V}|\beta_1\ldots\beta_N) = \frac{1}{2}\sum_{1\leq i\neq j\leq N}(\alpha_1\ldots\alpha_N|\hat{V}_{ij}|\beta_1\ldots\beta_N) = \frac{1}{2}\sum_{1\leq i\neq j\leq N}\Big(\prod_{\substack{k\neq i\\ k\neq j}}\langle\alpha_k|\beta_k\rangle\Big)\cdot(\alpha_i\alpha_j|\hat{V}_{ij}|\beta_i\beta_j)$$

$\hat{V}$ is fully determined by its action in $\mathcal{H}_2$

Example (important): Potential energy op. for two-body interaction (position basis)

$$(\bar{r}_1\bar{r}_2|\underbrace{\hat{V}|\bar{r}_3\bar{r}_4)}_{=V(\bar{r}_3-\bar{r}_4)|\bar{r}_3\bar{r}_4)} = V(\bar{r}_3-\bar{r}_4)(\bar{r}_1\bar{r}_2|\bar{r}_3\bar{r}_4) = V(\bar{r}_3-\bar{r}_4)\,\delta(\bar{r}_1-\bar{r}_3)\,\delta(\bar{r}_2-\bar{r}_4)$$

$$\hat{V}|\bar{r}_1\ldots\bar{r}_N) = \frac{1}{2}\sum_{1\leq i\neq j\leq N}V(\bar{r}_i-\bar{r}_j)\,|\bar{r}_1\ldots\bar{r}_N)$$

3-body, 4-body etc. operators may be defined analogously.

Creation operators

Consider an (orthonormal) basis $\{|\alpha_i\rangle\}$ of $\mathcal{H}$. Pick any of its elements $|\alpha\rangle\in\{|\alpha_i\rangle\}$

Define $a_\alpha^+|\alpha_1\ldots\alpha_N\} := |\alpha\,\alpha_1\ldots\alpha_N\}$ $\qquad a_\alpha^+ : \mathcal{B}_N \longmapsto \mathcal{B}_{N+1}$ (bosons)

$\qquad\qquad\qquad\qquad\qquad\qquad\;\;\mathcal{F}_N \longmapsto \mathcal{F}_{N+1}$ (fermions)

a_α^+ adds a particle in the state $|\alpha\rangle$ and (anti)symmetrizes the obtained state.

(for fermions: $\alpha\in\{\alpha_1\ldots\alpha_N\} \Longrightarrow a_\alpha^+|\alpha_1\ldots\alpha_N\} = 0$)

In addition to the N-particle states considered so far we define

$|0\rangle$ - vacuum (state with no particles)

$$a_\alpha^+|0\rangle = |\alpha\rangle$$

In realistic physical systems we often encounter situations involving superpositions of states with different particle numbers

e.g. $\odot \otimes |0\rangle$ $\qquad\qquad\qquad\qquad \cdot \otimes |1\rangle$

excited atom, photon vacuum $\qquad\qquad$ atom in its GS, single photon

What is the space, where the operators $\{a_\alpha^+\}$ act?

$$\left.\begin{array}{ll} \mathcal{B} := \mathcal{B}_0 \oplus \mathcal{B}_1 \oplus \ldots = \bigoplus_{n=0}^{\infty} \mathcal{B}_n & (\text{bosons}) \\ \mathcal{F} := \mathcal{F}_0 \oplus \mathcal{F}_1 \oplus \ldots = \bigoplus_{n=0}^{\infty} \mathcal{F}_n & (\text{fermions}) \end{array}\right\} \text{Fock space} \qquad \left\{\begin{array}{ll} \mathcal{B}_0 := \text{span}\{|0\rangle\} & \mathcal{B}_1 := \mathcal{H} \\ \mathcal{F}_0 := \text{span}\{|0\rangle\} & \mathcal{F}_1 := \mathcal{H} \end{array}\right\}$$

(states with different numbers of particles are orthogonal)

Example: $|\phi\rangle = \frac{1}{2}|0\rangle + \frac{1}{\sqrt{2}}|\alpha_1\rangle + \frac{1}{2}|\alpha_1 \alpha_2\}$ $\quad (\alpha_2 \neq \alpha_1)$

$\{a_\alpha^+\}$ generate the Fock space $\longrightarrow$ any $|\alpha_1 \ldots \alpha_N\}$ can be generated by acting with $\{a_\alpha^+\}$ on vacuum

Commutation relations: $a_\lambda^+ a_\mu^+ |\alpha_1 \ldots \alpha_N\} = |\lambda \mu \alpha_1 \ldots \alpha_N\} = \zeta |\mu \lambda \alpha_1 \ldots \alpha_N\} = \zeta a_\mu^+ a_\lambda^+ |\alpha_1 \ldots \alpha_N\}$

$\longrightarrow$ $\underline{a_\lambda^+ a_\mu^+ - \zeta a_\mu^+ a_\lambda^+ = 0}$ – creation operators commute for bosons and anticommute for fermions.

$\{$Notation: $[A,B]_{\mp} := AB \mp BA$; $[A,B]_{-\zeta} = AB - \zeta BA\}$

In the next step we would like to analyze the conjugate (h.c.) of a_λ^+ – the annihilation operator $(a_\lambda^+)^+ = a_\lambda$.