Recall the previous lecture:

Creation operators

Consider an (orthonormal) basis $\{|\alpha_i\rangle\}$ of $\mathcal{H}$. Pick any of its elements $|\alpha\rangle \in \{|\alpha_i\rangle\}$

Define $a_\alpha^+ |\alpha_1 \dots \alpha_N\} := |\alpha\, \alpha_1 \dots \alpha_N\}$ $\qquad a_\alpha^+ : \mathcal{B}_N \longmapsto \mathcal{B}_{N+1}$ (bosons)

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathcal{F}_N \longmapsto \mathcal{F}_{N+1}$ (fermions)

a_α^+ adds a particle in the state $|\alpha\rangle$ and (anti)symmetrizes the obtained state.

(for fermions: $\alpha \in \{\alpha_1 \dots \alpha_N\} \Longrightarrow a_\alpha^+ |\alpha_1 \dots \alpha_N\} = 0$)

In addition to the N-particle states considered so far we define

$|0\rangle$ – vacuum (state with no particles)

$$a_\alpha^+ |0\rangle = |\alpha\rangle$$

In realistic physical systems we often encounter situations involving superpositions of states with different particle numbers

e.g. excited atom $\otimes\, |0\rangle$ photon vacuum $\qquad$ atom in its GS $\otimes\, |1\rangle$ single photon

What is the space, where the operators $\{a_\alpha^+\}$ act?

$$\left.\begin{aligned} \mathcal{B} &:= \mathcal{B}_0 \oplus \mathcal{B}_1 \oplus \dots = \bigoplus_{n=0}^{\infty} \mathcal{B}_n \quad \text{(bosons)} \\ \mathcal{F} &:= \mathcal{F}_0 \oplus \mathcal{F}_1 \oplus \dots = \bigoplus_{n=0}^{\infty} \mathcal{F}_n \quad \text{(fermions)} \end{aligned}\right\} \text{Fock space}$$

$\{\mathcal{B}_0 := \text{span}\{|0\rangle\} \quad \mathcal{B}_1 := \mathcal{H}$
$\;\;\mathcal{F}_0 := \text{span}\{|0\rangle\} \quad \mathcal{F}_1 := \mathcal{H}\}$

(states with different numbers of particles are orthogonal)

<u>Example</u>: $|\phi\rangle = \frac{1}{2}|0\rangle + \frac{1}{\sqrt{2}}|\alpha_1\rangle + \frac{1}{2}|\alpha_1 \alpha_2\} \qquad (\alpha_2 \neq \alpha_1)$

$\{a_\alpha^+\}$ generate the Fock space $\longrightarrow$ any $|\alpha_1 \dots \alpha_N\}$ can be generated by acting with $\{a_\alpha^+\}$ on vacuum

Commutation relations: $a_\lambda^+ a_\mu^+ |\alpha_1 \dots \alpha_N\} = |\lambda \mu \alpha_1 \dots \alpha_N\} = \zeta |\mu \lambda \alpha_1 \dots \alpha_N\} = \zeta a_\mu^+ a_\lambda^+ |\alpha_1 \dots \alpha_N\}$

$\longrightarrow$ $a^+_\lambda a^+_\mu - \zeta a^+_\mu a^+_\lambda = 0$ – creation operators commute for bosons and anticommute for fermions.

$\{$Notation: $[A,B]_\mp := AB \mp BA$; $[A,B]_{-\zeta} = AB - \zeta BA\}$

In the next step we would like to analyze the conjugate (h.c.) of a^+_λ – the annihilation operator $(a^+_\lambda)^+ = a_\lambda$.

Identity resolutions:

In H: $\sum_\alpha |\alpha\rangle\langle\alpha| = \mathbb{1}$; In H_N: $\sum_{\alpha_1 \dots \alpha_N} |\alpha_1 \dots \alpha_N)(\alpha_1 \dots \alpha_N| = \sum_{\alpha_1 \dots \alpha_N} (|\alpha_1\rangle\langle\alpha_1|)\otimes(|\alpha_2\rangle\langle\alpha_2|)\otimes$

$\dots\otimes(|\alpha_N\rangle\langle\alpha_N|) = \left(\sum_{\alpha_1}|\alpha_1\rangle\langle\alpha_1|\right)\otimes\left(\sum_{\alpha_2}|\alpha_2\rangle\langle\alpha_2|\right)\otimes$

$\dots\otimes\left(\sum_{\alpha_N}|\alpha_N\rangle\langle\alpha_N|\right) = (\mathbb{1})_1\otimes(\mathbb{1})_1\otimes\dots\otimes(\mathbb{1})_1 = \mathbb{1}$ in H_N.

In B_N (F_N): $\sum_{\alpha_1 \dots \alpha_N} |\alpha_1 \dots \alpha_N\}\{\alpha_1 \dots \alpha_N| \underset{P^+_{\{B/F\}} = P_{\{B/F\}}}{=} \sum_{\alpha_1 \dots \alpha_N} N! \sum_{\alpha_1 \dots \alpha_N} P_{\{B/F\}} |\alpha_1 \dots \alpha_N)(\alpha_1 \dots \alpha_N| P_{\{B/F\}} =$

$= N!\, P_{\{B/F\}}\, \mathbb{1}\, P_{\{B/F\}} = N!\, \mathbb{1}_{\{B/F\}}$ (since $P^2_{\{B/F\}} = P_{\{B/F\}}$)

$\Longrightarrow \frac{1}{N!}\sum_{\alpha_1 \dots \alpha_N} |\alpha_1 \dots \alpha_N\}\{\alpha_1 \dots \alpha_N| = \mathbb{1}$ (in B_N/F_N)

Fock space: $\mathbb{1}_{Fock} = \mathbb{1}^{(0)} + \mathbb{1}^{(1)} + \mathbb{1}^{(2)} + \dots = |0\rangle\langle 0| + \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{\alpha_1 \dots \alpha_N} |\alpha_1 \dots \alpha_N\}\{\alpha_1 \dots \alpha_N|$

$\mathbb{1}_{Fock} = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\alpha_1 \dots \alpha_N} |\alpha_1 \dots \alpha_N\}\{\alpha_1 \dots \alpha_N|$

Now we work out the action of a_λ (working in a given basis)

$a_\lambda |\beta_1 \dots \beta_M\} = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\alpha_1 \dots \alpha_N} \{\alpha_1 \dots \alpha_N | a_\lambda | \beta_1 \dots \beta_M\} |\alpha_1 \dots \alpha_N\} =$ $\{\{\alpha_1 \dots \alpha_N|a_\lambda|\beta_1 \dots \beta_M\} = \{\beta_1 \dots \beta_M|a^+_\lambda|\alpha_1 \dots \alpha_N\}^* = \{\lambda\alpha_1 \dots \alpha_N|\beta_1 \dots \beta_M\}\}$

$= \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\alpha_1 \dots \alpha_N} \underbrace{\{\lambda\alpha_1 \dots \alpha_N|\beta_1 \dots \beta_M\}}_{\text{vanishes for } N \neq M-1 \Rightarrow a_\lambda \text{ anihilates a particle}} |\alpha_1 \dots \alpha_N\} = \frac{1}{(M-1)!} \sum_{\alpha_1 \dots \alpha_{M-1}} \{\lambda\alpha_1 \dots \alpha_{M-1}|\beta_1 \dots \beta_M\} |\alpha_1 \dots \alpha_{M-1}\} =$

$\{$scalar product: $\{\alpha'_1 \dots \alpha'_N|\alpha_1 \dots \alpha_N\} = N! \left[(\alpha'_1 \dots \alpha'_N|P_{\{B/F\}}\right]\left[P_{\{B/F\}}|\alpha_1 \dots \alpha_N)\right] = \left\{ \begin{matrix} P^2_{\{B/F\}} = P_{\{B/F\}} \\ P^+_{\{B/F\}} = P_{\{B/F\}} \end{matrix} \right\} =$

$= N!(\alpha'_1 \dots \alpha'_N|P_{\{B/F\}}|\alpha_1 \dots \alpha_N) = \sum_P \zeta^P \langle\alpha'_1|\alpha_{P1}\rangle\langle\alpha'_2|\alpha_{P2}\rangle \dots \langle\alpha'_N|\alpha_{PN}\rangle \}$

$= \frac{1}{(M-1)!} \sum_{\alpha_1 \dots \alpha_{M-1}} \sum_{P(M)} \zeta^{P(M)} \langle\lambda|\beta_{P1}\rangle\langle\alpha_1|\beta_{P2}\rangle \dots \langle\alpha_{M-1}|\beta_{PM}\rangle |\alpha_1 \dots \alpha_{M-1}\} =$

$= \frac{1}{(M-1)!} \sum_{P(M)} \zeta^{P(M)} \langle\lambda|\beta_{P1}\rangle |\beta_{P2} \dots \beta_{PM}\} =$ $\hookrightarrow P1 = i$

$\{$Each P(M) may be expressed as a composition of a permutation taking element "1" to the i-th slot without

$$= \frac{1}{(M-1)!}\sum_{i=1}^{M}\sum_{P(M-1)} \zeta^{i-1}\zeta^{P(M-1)}\langle\lambda|\beta_i\rangle\,|\beta_{P2}\dots\beta_{PM}\} =$$

perturbing the order of the remaining elements, and a permutation of the remaining M-1 elements.

(M-1)! identical terms!

$$\sum_{P(M-1)}\zeta^{P(M-1)}|\beta_{P2}\dots\beta_{PM}\} = (M-1)!\,\underbrace{|\beta_1\dots\beta_{i-1}\beta_{i+1}\dots\beta_M\}}_{=|\beta_1\dots\hat{\beta}_i\dots\beta_M\}\ \text{(notation)}}$$

$$= \sum_{i=1}^{M}\zeta^{i-1}\delta_{\lambda,\beta_i}|\beta_1\dots\hat{\beta}_i\dots\beta_M\}$$

This way we obtained:

$$a_\lambda|\beta_1\dots\beta_M\} = \sum_{i=1}^{M}\zeta^{i-1}\delta_{\lambda,\beta_i}|\beta_1\dots\hat{\beta}_i\dots\beta_M\}$$

Occupation number operator

Define $\hat{n}_\alpha := a^\dagger_\alpha a_\alpha$ How does it act on $|\alpha_1\dots\alpha_N\}$?

$$\hat{n}_\alpha|\alpha_1\dots\alpha_N\} = \sum_{i=1}^{N}\zeta^{i-1}\delta_{\alpha,\alpha_i}a^\dagger_\alpha|\alpha_1\dots\hat{\alpha}_i\dots\alpha_N\} = \sum_{i=1}^{N}\zeta^{i-1}\delta_{\alpha\alpha_i}|\alpha,\alpha_1\dots\hat{\alpha}_i\dots\alpha_N\} = \sum_{i=1}^{N}\delta_{\alpha,\alpha_i}|\alpha_1\dots\alpha\dots\alpha_N\}$$

$$= \sum_{i=1}^{N}\delta_{\alpha,\alpha_i}|\alpha_1\dots\alpha_N\} = n_\alpha|\alpha_1\dots\alpha_N\}$$

occupation number of α (in $|\alpha_1\dots\alpha_N\}$)

$\hat{n}_\alpha = a^\dagger_\alpha a_\alpha$ – occupation number operator (for given $|k\rangle$)

$\hat{N} = \sum_\alpha \hat{n}_\alpha$ – particle number operator.

- Commutation relations $[a_\lambda, a_\mu]_{-\zeta} = [a^\dagger_\lambda, a^\dagger_\mu]_{-\zeta} = 0$ (already shown)

$[a_\lambda, a^\dagger_\mu]_{-\zeta} = ?$

- $a_\lambda a^\dagger_\mu|\alpha_1\dots\alpha_N\} = a_\lambda|\mu\,\alpha_1\dots\alpha_N\} = \delta_{\lambda\mu}|\alpha_1\dots\alpha_N\} + \sum_{i=1}^{N}\zeta^i\delta_{\lambda,\alpha_i}|\mu,\alpha_1\dots\hat{\alpha}_i\dots\alpha_N\}$
- $a^\dagger_\mu a_\lambda|\alpha_1\dots\alpha_N\} = a^\dagger_\mu\sum_{i=1}^{N}\zeta^{i-1}\delta_{\lambda\alpha_i}|\alpha_1\dots\hat{\alpha}_i\dots\alpha_N\} = \zeta\sum_{i=1}^{N}\zeta^i\delta_{\lambda\alpha_i}|\mu\,\alpha_1\dots\hat{\alpha}_i\dots\alpha_N\}$

$$(a_\lambda a^\dagger_\mu - \zeta a^\dagger_\mu a_\lambda)|\alpha_1\dots\alpha_N\} = \delta_{\lambda\mu}|\alpha_1\dots\alpha_N\}$$

$$\boxed{[a_\lambda, a^\dagger_\mu]_{-\zeta} = \delta_{\lambda\mu}}$$

- One-body operators and creation/annihilation operators.

$$\hat{U}|\alpha_1 \dots \alpha_N) = \sum_{i=1}^{N} \hat{U}_i |\alpha_1 \dots \alpha_N) = \sum_{i=1}^{N} U_{\alpha_i} |\alpha_1 \dots \alpha_N)$$

basis where U is diagonal

$$\hat{U}|\alpha_1 \dots \alpha_N\} = \hat{U} \frac{1}{\sqrt{N!}} \sum_P \zeta^P |\alpha_{P1} \dots \alpha_{PN}) = \frac{1}{\sqrt{N!}} \sum_P \zeta^P \underbrace{\sum_{i=1}^{N} \hat{U}_i |\alpha_{P1} \dots \alpha_{PN})}_{\sum_{i=1}^{N} U_{\alpha_i} |\alpha_{P1} \dots \alpha_{PN})} = (*)$$

$$\left\{ \sum_{i=1}^{N} U_{\alpha_i} = \sum_\alpha U_\alpha n_\alpha \right\} \qquad (*) = \frac{1}{\sqrt{N!}} \sum_P \zeta^P \sum_\alpha U_\alpha n_\alpha |\alpha_{P1} \dots \alpha_{PN}) =$$

$$= \sum_\alpha U_\alpha \hat{n}_\alpha |\alpha_1 \dots \alpha_N\}$$

$$\hat{U} = \sum_\alpha U_\alpha \hat{n}_\alpha = \sum_\alpha \langle \alpha | \hat{U}^{(1)} | \alpha \rangle a^\dagger_\alpha a_\alpha$$

How about other bases? $\{|\tilde{\alpha}\rangle\}$

$$|\tilde{\alpha}\rangle = \sum_\alpha \langle \alpha | \tilde{\alpha} \rangle |\alpha\rangle$$

$$a^\dagger_{\tilde{\alpha}} |0\rangle = |\tilde{\alpha}\rangle = \sum_\alpha \langle \alpha | \tilde{\alpha} \rangle |\alpha\rangle = \sum_\alpha \langle \alpha | \tilde{\alpha} \rangle a^\dagger_\alpha |0\rangle$$

$$a^\dagger_{\tilde{\alpha}} = \sum_\alpha \langle \alpha | \tilde{\alpha} \rangle a^\dagger_\alpha$$

conjugate: $\quad a_{\tilde{\alpha}} = \sum_\alpha \langle \tilde{\alpha} | \alpha \rangle a_\alpha$

Comm. relations:

$$[a_{\tilde{\beta}}, a^\dagger_{\tilde{\alpha}}]_{-\zeta} = \sum_\alpha \sum_\beta \langle \tilde{\beta} | \beta \rangle \langle \alpha | \tilde{\alpha} \rangle \underbrace{[a_\beta, a^\dagger_\alpha]_{-\zeta}}_{\delta_{\alpha\beta}} = \sum_\alpha \langle \tilde{\beta} | \alpha \rangle \langle \alpha | \tilde{\alpha} \rangle$$

$$= \langle \tilde{\beta} | \underbrace{\sum_\alpha |\alpha\rangle\langle\alpha|}_{\mathbb{1}} \tilde{\alpha} \rangle = \langle \tilde{\beta} | \tilde{\alpha} \rangle = \delta_{\tilde{\alpha}, \tilde{\beta}}$$

orthonormal basis

$\hat{U}$ in arbitrary basis

$$\hat{U} = \sum_\alpha \underbrace{\langle \alpha | \hat{U} | \alpha \rangle}_{U_\alpha} a_\alpha^\dagger a_\alpha = \sum_\alpha \sum_{\beta\beta'} U_\alpha \langle \beta | \alpha \rangle a_\beta^\dagger \langle \alpha | \beta' \rangle a_{\beta'} =$$

$$= \sum_{\beta\beta'} \langle \beta | \underbrace{\sum_\alpha U_\alpha | \alpha \rangle\langle \alpha |}_{\hat{U}} \beta' \rangle a_\beta^\dagger a_{\beta'} =$$

$$= \sum_{\beta\beta'} \langle \beta | \hat{U} | \beta' \rangle a_\beta^\dagger a_{\beta'}$$

Example one-body operator – kinetic energy

$$\hat{T} = \sum_i \frac{\hat{p}_i^2}{2m} \qquad \hat{T} = \sum_{\vec{k}} \frac{\hbar^2 k^2}{2m} a_{\vec{k}}^\dagger a_{\vec{k}} = \sum_{\vec{k}} \frac{\hbar^2 k^2}{2m} \hat{n}_{\vec{k}}$$

$$\{ | \vec{k} \rangle \}$$

Field operators

$$a_{\tilde{\alpha}}^\dagger = \sum_\alpha \langle \alpha | \tilde{\alpha} \rangle a_\alpha^\dagger$$

$$a_{\tilde{\alpha}} = \sum_\alpha \langle \tilde{\alpha} | \alpha \rangle a_\alpha$$

Take $\{ |\tilde{\alpha}\rangle \} \to \{ |\vec{r}\rangle \}$

Denote $a_{\tilde{\alpha}}^\dagger = \Psi^\dagger(\vec{r})$, $a_{\tilde{\alpha}} = \Psi(\vec{r})$ $\to$ Field operators

$$\boxed{\begin{aligned} \Psi^\dagger(\vec{r}) &= \sum_\alpha \langle \alpha | \vec{r} \rangle a_\alpha^\dagger = \sum_\alpha \psi_\alpha^*(\vec{r})\, a_\alpha^\dagger \\ \Psi(\vec{r}) &= \sum_\alpha \langle \vec{r} | \alpha \rangle a_\alpha = \sum_\alpha \psi_\alpha(\vec{r})\, a_\alpha \end{aligned}}$$

$$[\Psi(\vec{r}_1), \Psi^\dagger(\vec{r}_2)]_{-\zeta} = \langle \vec{r}_1 | r_2 \rangle = \delta(\vec{r}_1 - \vec{r}_2)$$