LECTURE XII

SUPERCONDUCTIVITY

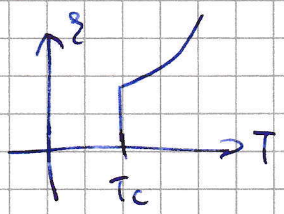
1911 KAMERLINGH ONNES

RESISTIVITY OF MERCURY DROPS SUDDENLY TO 0 FOR $T \to T_c^+$

$T_c \simeq 4.2K$ (FOR Hg)

FOLLOWING YEARS → MANY OTHER SYSTEMS (ELEMENTS AND COMPOUNDS) DISPLAY SIMILAR BEHAVIOR.

Al	1.2K
Hg	4.2K
Nb	9.3K
Pb	7.2K



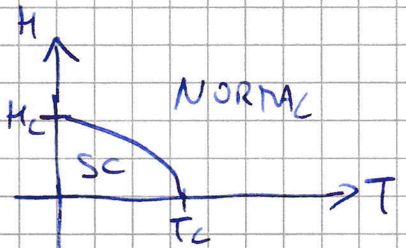
DRUDE THEORY OF CONDUCTIVITY:

$$\rho = \rho_0 + \alpha T^2 + \dots$$

SC RING – ELECTRIC CURRENT FLOWS WITHOUT ANY ATTENUATION

MAGNETIC FIELD DESTROYS SC, SYSTEM RETURNS TO ITS NORMAL STATE FOR $H > H_c(T)$.

$$H_c(T) \simeq H_c(0)\left(1 - \left(\frac{T}{T_c}\right)^2\right)$$

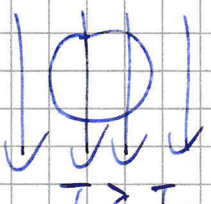


TYPE II → BETWEEN $H_c^1(T)$ AND $H_c^2(T)$ BOTH NORMAL AND SC REGIONS.

ANOTHER CRUCIAL FEATURE → PERFECT DIAMAGNETISM

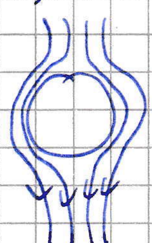
When a material is cooled in presence of magnetic field to $T < T_c$, magnetic flux becomes expelled from the inside of the SC.

→ Meissner effect (1933)



$T > T_c$

(MAGNETIC FIELD PENETRATES THE MATERIAL)



$T < T_c$

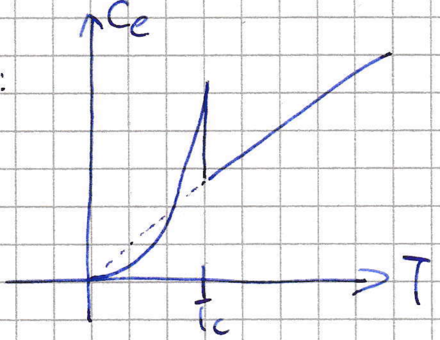
MAGNETIC FIELD EXPELLED FROM THE MATERIAL (DUE TO APPEARENCE OF SURFACE CURRENTS)

→ NOT A CONSEQUENCE OF VANISHING OF RESISTIVITY.

Specific heat:

Metals: $C_e = \alpha T$ (electronic contribution)

SC:



$C_e \sim e^{-\frac{\Delta}{k_B T}}$

(characteristic for systems with excitation gap of 2Δ)

→ compare Ising model.

Isotope effect

Change of ionic mass through use of different isotopes ⇒ shift of T_c.

Changing isotopes should have no effect on energy bands or Coulomb interaction ⇒ electron-phonon interactions play a role in the phenomenon of SC.

Till 1986 highest T_c was 23.3K (Nb_3Ge) (normal p).

1986 Bednorz – Müller high-T_c Sc. La_2CuO_4 $\gtrsim 80$ K.

London theory (phenomenological)

In what way should electrodynamics of S-C differ from that of normal metal in order to account for the Meissner effect!

F. London (Fritz) and H. London (Heinz) (1935).

Normal metal with n conduction electrons (per unit volume)

Apply a static $\vec{E}$ ⇒ electron is accellerated (but also scattered by phonons and impurities).

→ damping of electron motion.

(*) $m\dot{\vec{v}} = -e\vec{E} - \frac{m}{\tau}\vec{v}$

τ has dimension of time (~average time between scattering events).

current density $\vec{J} = -ne\vec{v}$

Steady state conditions: $\bar{J} = \text{const} \Rightarrow \dot{\bar{v}} = 0$ AND $\bar{v} = -\frac{e\tau}{m}\bar{E}$

$$\Rightarrow \bar{J} = \frac{ne^2\tau}{m}\bar{E}$$

OHM'S LAW: $\bar{E} = \varrho \bar{J} \Rightarrow \varrho = \frac{m}{ne^2\tau}$

In a perfect conductor $\varrho = 0$, thus $\tau = \infty$ and (*) goes

into $m\dot{\bar{v}} = -e\bar{E} \Rightarrow \dot{\bar{J}} = n\frac{e^2}{m}\bar{E}$ (L1)

TAKE THE CURL OF THIS AND USE $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$

(MAXWELL)

$$\frac{\partial}{\partial t}\nabla \times \bar{J} = -n\frac{e^2}{m}\frac{\partial \bar{B}}{\partial t}$$

$$\frac{\partial}{\partial t}\Big(\underbrace{\nabla \times \bar{J} + n\frac{e^2}{m}\bar{B}}_{(*)}\Big) = 0$$

$$\underbrace{\nabla \times \bar{B} = \mu_0\Big(\bar{J} + \epsilon_0 \frac{\partial \bar{E}}{\partial t}\Big)}_{(**)}, \quad \epsilon_0\frac{\partial \bar{E}}{\partial t} = 0$$

THESE TWO EQ. DETERMINE MAGNETIC FIELD AND CURRENT DENSITY OF A PERFECT CONDUCTOR.

ANY STATIC $\bar{B}$ DETERMINES VIA (**) A STATIC $\bar{J}$ SO THAT (*) IS automatically satisfied.

→ consistent with existence of an arbitrary static magnetic field inside a perfect conductor.

(Vanishing of ϱ does ~~dot~~ not imply Meissner eff.)

LONDON BROTHERS CONJECTURED: $\underbrace{\nabla \times \bar{J} + n\frac{e^2}{m}\bar{B} = 0}_{(***)}$ FOR SC (L2)

(COMPARE (*))

(L1, L2) → LONDON EQ.

TAKE CURL OF (**), USE $\nabla \times (\nabla \times \bar{B}) = \nabla(\nabla \bar{B}) - \Delta \bar{B}$

THEN RECALL $\nabla \bar{B} = 0$ (MAXWELL).

$$\cancel{\nabla \cdot} \; -\Delta \bar{B} = \cancel{\mu_0 \ldots}$$

$$\Rightarrow -\Delta \bar{B} = +\mu_0 \nabla \times \bar{J}$$

USE (***) $\Rightarrow -\Delta \bar{B} = -n\frac{e^2}{m}\bar{B}\mu_0 \qquad \Delta \bar{B} = \frac{ne^2\mu_0}{m}\bar{B}$

TAKE CURL OF (***) $\nabla(\nabla \bar{J}) - \Delta \bar{J} = -\frac{ne^2}{m}\nabla \times \bar{B} \overset{(**)}{=} -\frac{ne^2}{m}\mu_0 \bar{J}$

ALSO FROM (**) $\nabla(\nabla\times\bar{B}) = \mu_0 \nabla\bar{J}$

$\overset{\shortparallel}{0}$ (vector calculus identity)

SO THAT WE HAVE
$$\underline{\Delta\bar{J} = \frac{ne^2}{m}\mu_0\bar{J}}$$

SOLVE $\Delta\bar{B} \overset{(1)}{=} \frac{ne^2\mu_0}{m}\bar{B}$ INSIDE A SUPERCONDUCTING SEMIINFINITE SLAB ($z \geq 0$)

FOR TWO CASES:

- $\bar{B} \parallel \bar{e}_z$, $B = B(z)$ $\Rightarrow$ $\bar{B} = \begin{pmatrix} 0 \\ 0 \\ B(z) \end{pmatrix}$

  IN THIS CASE $\nabla\bar{B} = 0$ IMPLIES $B(z) = \text{const.}$

  SO THAT $\Delta\bar{B} = 0$ AND FROM (1) $\bar{B} = 0$.

- $\bar{B} \parallel \bar{e}_x$, $B = B(z)$ $\rightarrow$ $\bar{B} = \begin{pmatrix} B(z) \\ 0 \\ 0 \end{pmatrix}$

  THEN (1) $\Rightarrow$ $\frac{\partial^2 B(z)}{\partial z^2} = \frac{ne^2\mu_0}{m} B(z)$ $z \geq 0$

  WHICH HAS THE SOLUTION:
$$B(z) = B(0)\, e^{-\frac{z}{\lambda_L}} \quad z \geq 0$$

  WITH $\lambda_L = \sqrt{\frac{m}{ne^2\mu_0}}$

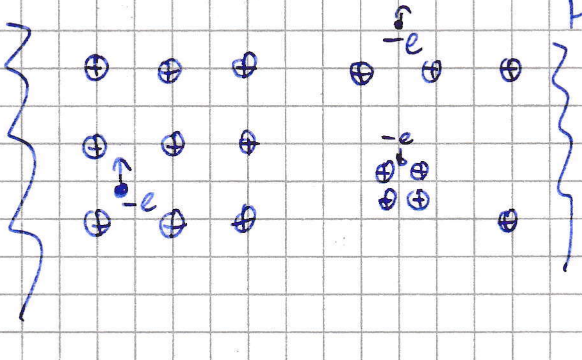
IN MOST SC $\lambda_L \simeq 10^2 - 10^3$ Å

MAGNETIC FIELD DECAYS EXPONENTIALLY INSIDE THE SC AND ONLY PENETRATES SMALL DISTANCE INTO THE SC.

EFFECTIVE ELECTRON-ELECTRON INTERACTION

METAL:
- (SCREENED) COULOMB INTERACTION — REPULSIVE
- INTERACTION FROM COUPLING ELECTRONS TO LATTICE VIBRATIONS — MAY BE ATTRACTIVE.

$\rightarrow$ FIRST NOTED AND DISCUSSED IN CONTEXT OF SC H. FRÖHLICH, 1950

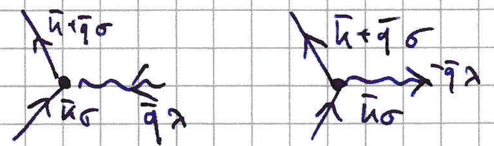


FRÖHLICH HAMILTONIAN: $H = H_0 + H'$

$$H_0 = \sum_{\bar{k}\sigma} e_{\bar{k}} c^{+}_{\bar{k}\sigma} c_{\bar{k}\sigma} + \sum_{\bar{q}\lambda} \hbar\omega_{\bar{q}\lambda}\left(a^{+}_{\bar{q}\lambda} a_{\bar{q}\lambda} + \frac{1}{2}\right)$$

(ASSUMING THE METAL HAS ONLY 1 PARTIALLY FILLED BAND)

$$H' = \sum_{\bar{k}\sigma}\sum_{\bar{q}\lambda} M_{\bar{q}\lambda}\left(c^{+}_{\bar{k}+\bar{q}\sigma} c_{\bar{k}\sigma} a_{\bar{q}\lambda} + c^{+}_{\bar{k}+\bar{q}\sigma} c_{\bar{k}\sigma} a^{+}_{-\bar{q}\lambda}\right)$$



H' CAN BE OBTAINED BY CONSIDERING THE COULOMB INTERACTION BETWEEN THE ELECTRONS AND THE ~~IONS~~ IONS (WHEN DISPLACED FROM THEIR EQUILIBRIUM POSITIONS)

HOW TO OBTAIN AN EXPRESSION FOR THE ELECTRON-ELECTRON INTERACTION MEDIATED BY PHONONS?

CANONICAL TRANSFORMATION METHOD:

$H \rightarrow$ OBTAINED BY USING THE BASIS SET OF STATES $|n\rangle = |\bar{k}\sigma\rangle|\bar{q}\lambda\rangle$

TRANSFORM TO A NEW BASIS SET OF STATES: $|\tilde{n}\rangle = U|n\rangle$
$(U^{+}U = 1)$.

IN THE NEW BASIS MATRIX EL. OF HAMILTONIAN:

$$\langle\tilde{m}|H|\tilde{n}\rangle = \langle m|U^{+}HU|n\rangle = \langle m|\tilde{H}|n\rangle$$

$\Rightarrow$ CHANGE OF BASIS EQUIVALENT TO APPLYING A SIMILARITY TRANSFORMATION TO H

$H \rightarrow \tilde{H} = U^{+}HU$. LET $U = e^{S}$ $(S^{+} = -S$ SO THAT U IS UNITARY)

$\cancel{\tilde{H}}$ CHOOSE S SO AS TO ELIMINATE EL-PHONON INTERACTION IN 1ST ORDER.

~~ALTERNATIVELY WE MAY DEFINE NEW ELECTRON AND PHONON OP. THROUGH A CANONICAL TRANSFORMATION:~~

$\cancel{\tilde{c}_{\bar{k}\sigma} = e^{-S} c}$ (ASSUME H' SMALL)

$$\tilde{H} = \left(1 - S + \frac{1}{2}S^2 + \dots\right)(H_0 + H')\left(1 + S + \frac{1}{2}S^2 + \dots\right)$$
$$= H_0 + \left([H_0, S] + H'\right) + [H', S] + \frac{1}{2!}[S,[S,H_0]] + \frac{1}{2!}[S,[S,H']]$$

CHOOSE S SO THAT $[H_0, S] = -H'$

(THIS REQUIRES THAT $S \sim H'$) WITH THIS CHOICE

$$\tilde{H} = H_0 + \frac{1}{2}[H', S] + \dots$$

CONSIDER TWO EIGENSTATES $|m\rangle, |n\rangle$ OF H_0 $(\epsilon_m \neq \epsilon_n)$

$$(*) \quad [S, H_0] = H' \Rightarrow \langle m|S|n\rangle = \frac{\langle m|H'|n\rangle}{\epsilon_n - \epsilon_m}$$

THE EFFECT OF H' IS TO SCATTER FROM A STATE ~~$\epsilon_{\bar{k}}$~~ WITH $\epsilon_{\bar{k}}$ INTO A STATE OF $\epsilon_{\bar{k}+\bar{q}}$ EITHER BY ABSORBING A PHONON $(\bar{q}\lambda)$ OR EMITTING A PHONON $(-\bar{q}, \lambda)$ $\Rightarrow$ $\epsilon_n - \epsilon_m = \epsilon_{\bar{k}} - \epsilon_{\bar{k}+\bar{q}} \pm \hbar\omega_{\bar{q}\lambda}$.

ONE MAY CHECK THAT FULFILLS (*) $\Leftarrow$ $S = \sum_{\bar{k}\sigma}\sum_{\bar{q}\lambda} M_{\bar{q}\lambda} c^\dagger_{\bar{k}+\bar{q}\sigma} c_{\bar{k}\sigma}\left(\frac{a_{\bar{q}\lambda}}{\epsilon_{\bar{k}} - \epsilon_{\bar{k}+\bar{q}} + \hbar\omega_{\bar{q}\lambda}} + \frac{a^\dagger_{-\bar{q}\lambda}}{\epsilon_{\bar{k}} - \epsilon_{\bar{k}+\bar{q}} - \hbar\omega_{\bar{q}\lambda}}\right)$

NEXT STEP: WRITE DOWN $\tilde{H} \simeq H_0 + \frac{1}{2}[H', S]$

OUT OF MANY TERMS IN THE COMMUTATOR THERE ARE TERMS INVOLVING FOUR ELECTRON OPERATORS. (THE OTHER TERMS CONTAIN TWO ELECTRON AND TWO PHONON OP.)

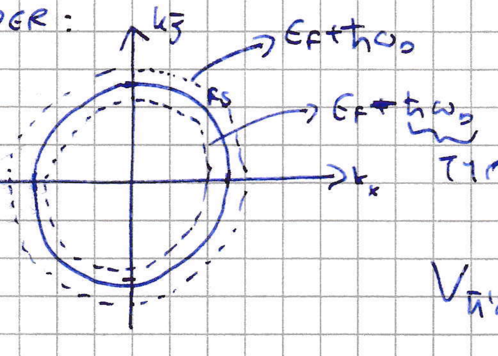
THE EL-EL INTERACTION MEDIATED BY PHONONS:

~~LEAD TO RENORMALIZATION OF SINGLE-PARTICLE ENERGY.~~

$$H'_{INT} = \sum_{\bar{k}\sigma}\sum_{\bar{k}'\sigma'}\sum_{\bar{q}} V_{\bar{k}'\bar{q}}\, c^\dagger_{\bar{k}+\bar{q}\sigma} c^\dagger_{\bar{k}'-\bar{q}\sigma'} c_{\bar{k}'\sigma'} c_{\bar{k}\sigma}$$

$$V_{\bar{k}'\bar{q}} = \sum_\lambda |M_{\bar{q}\lambda}|^2 \frac{\hbar\omega_{\bar{q}\lambda}}{(\epsilon_{\bar{k}'} - \epsilon_{\bar{k}'-\bar{q}})^2 - (\hbar\omega_{\bar{q}\lambda})^2}$$

IN PARTICULAR: CONSIDER:



TYPICAL PHONON EN.

$V_{\bar{k}'\bar{q}} < 0$ FOR ELECTRONS IN STATES WITHIN THE SHELL!

$\rightarrow$ ~~COMPARE~~ SEE COOPER ~~INSTABILITY~~ PROBLEM (EXERCISE). COOPER PAIRS FORMED BY FERMIONS WITH $(\bar{k},\uparrow), (-\bar{k},\downarrow)$.

BCS THEORY (1957) BARDEEN, COOPER, SCHRIEFFER.

(MICROSCOPIC THEORY OF THE SC STATE). (NOBEL 1972)

$$H_{BCS} = \sum_{\bar{k}\sigma} \epsilon_{\bar{k}} c^\dagger_{\bar{k}\sigma} c_{\bar{k}\sigma} + \sum_{\substack{\bar{k}\bar{k}' \\ (\bar{k}\neq\bar{k}')}} U_{\bar{k}\bar{k}'} c^\dagger_{\bar{k}'\uparrow} c^\dagger_{-\bar{k}'\downarrow} c_{-\bar{k}\downarrow} c_{\bar{k}\uparrow}$$

(SCATTERS A PAIR OF ELECTRONS FROM $|\bar{k}\uparrow, -\bar{k}\downarrow\rangle$ TO $|\bar{k}'\uparrow, -\bar{k}'\downarrow\rangle$)

TO DETERMINE THE GS: VARIATIONAL APPROACH WITH THE TRIAL

WAVE FUNCTION: $|\Psi\rangle = \prod_{\vec{k}} (u_{\vec{k}} + v_{\vec{k}} c^{+}_{\vec{k}\uparrow} c^{+}_{-\vec{k}\downarrow}) |0\rangle$

VARIATIONAL PARAMETERS, ASSUMED REAL AND $u_{\vec{k}}^2 + v_{\vec{k}}^2 = 1$ (WHICH IMPLIES NORMALIZATION)

v_k^2 — PROBABILITY THAT THE PAIR STATE IS OCCUPIED $|\vec{k}\uparrow, -\vec{k}\downarrow\rangle$

$u_{\vec{k}}^2$ — PROBABILITY THAT IT IS EMPTY.

$|\Psi\rangle$ CONTAINS THE FERMI SEA $\longrightarrow$ TAKE $u_{\vec{k}} = 0$, $v_{\vec{k}} = 1$ FOR $|\vec{k}| < k_F$

AND $u_{\vec{k}} = 1$, $v_{\vec{k}} = 0$ FOR $|\vec{k}| > k_F$

$|\Psi\rangle$ IS A LINEAR COMB. OF STATES WITH VARYING NUMBER OF PAIRS
$\rightarrow$ IT IS NOT AN EIGENSTATE OF $\hat{N}$
WE REQUIRE HOWEVER THAT $\langle\Psi|\hat{N}|\Psi\rangle = N$

TASK: MINIMIZE $\langle\Psi|H_{BCS}|\Psi\rangle$ SUBJECT TO THE CONSTRAINT

$$\langle\Psi|\hat{N}|\Psi\rangle = N.$$

$\rightarrow$ INTRODUCE THE LAGRANGE MULTIPLIER (μ).

$$\mathcal{H} = H_{BCS} - \mu\hat{N}$$

$$\bar{\epsilon}_{\vec{k}} = \epsilon_{\vec{k}} - \mu$$

$$\bar{E} = \langle\Psi|\mathcal{H}|\Psi\rangle = 2\sum_{\vec{k}} v_{\vec{k}}^2 \bar{\epsilon}_k + \sum_{\vec{k}\vec{k}'} U_{\vec{k}\vec{k}'} u_k v_k u_{k'} v_{k'}$$

EXERCISE

$$u_{\vec{k}}^2 + v_{\vec{k}}^2 = 1 \implies u_{\vec{k}} = \cos\theta_{\vec{k}} \qquad v_{\vec{k}} = \sin\theta_{\vec{k}}$$

$$\bar{E} = 2\sum_{\vec{k}} \bar{\epsilon}_{\vec{k}} \sin^2\theta_{\vec{k}} + \frac{1}{4}\sum_{\vec{k}\vec{k}'} U_{\vec{k}\vec{k}'} \sin 2\theta_{\vec{k}} \sin 2\theta_{\vec{k}'}$$

$$\frac{\partial\bar{E}}{\partial\theta_{\vec{k}_0}} = 0 \xRightarrow{k_0 \to k} 2\bar{\epsilon}_{\vec{k}} \sin 2\theta_{\vec{k}} + \cos 2\theta_{\vec{k}} \sum_{\vec{k}'} U_{\vec{k}\vec{k}'} \underbrace{\sin 2\theta_{\vec{k}'}}_{u_{k'}v_{k'}} = 0$$

$$\left(2\bar{\epsilon}_{\vec{k}} u_{\vec{k}} v_{\vec{k}} + (u_{\vec{k}}^2 - v_{\vec{k}}^2)\sum_{\vec{k}'} U_{kk'}\right)$$

DEFINE $\Delta_{\vec{k}} = -\sum_{\vec{k}'} U_{\vec{k}\vec{k}'} u_{k'} v_{k'}$ (GAP PARAMETER)

$$\rightarrow \begin{cases} 2\bar{\epsilon}_{\vec{k}} u_{\vec{k}} v_{\vec{k}} - \Delta_{\vec{k}} (u_{\vec{k}}^2 - v_{\vec{k}}^2) = 0 \\ u_{\vec{k}}^2 + v_{\vec{k}}^2 = 1 \end{cases}$$

IS SOLVED BY:

$$u_{\vec{k}}^2 = \frac{1}{2}\left[1 + \frac{\bar{\epsilon}_{\vec{k}}}{\sqrt{\bar{\epsilon}_{\vec{k}}^2 + \Delta_{\vec{k}}^2}}\right], \qquad v_{\vec{k}}^2 = \frac{1}{2}\left[1 - \frac{\bar{\epsilon}_{\vec{k}}}{\sqrt{\bar{\epsilon}_{\vec{k}}^2 + \Delta_{\vec{k}}^2}}\right]$$

PLUGGING THIS INTO (*)

$$\Rightarrow \quad \Delta_{\bar{k}} = -\frac{1}{2}\sum_{\bar{k}'} U_{\bar{k}\bar{k}'} \frac{\Delta_{\bar{k}'}}{\sqrt{\epsilon_{\bar{k}'}^2+\Delta_{\bar{k}'}^2}} \qquad (**)$$

THIS DETERMINES $\Delta_{\bar{k}}$.

A SIMPLE SOLUTION FROM ADOPTING A SIMPLE MODEL OF THE ATTRACTIVE INTERACTION:

$$U_{\bar{k}\bar{k}'} = \begin{cases} -U_0 & -\hbar\omega_D < \epsilon_{\bar{k}}, \epsilon_{\bar{k}'} < \hbar\omega_D \\ 0 & \text{OTHERWISE.} \end{cases}$$

THEN (*) $\Rightarrow$ $\Delta_{\bar{k}}$ DOES NOT DEPEND ON $\bar{k}$ FOR $-\hbar\omega_D < \epsilon_{\bar{k}} < \hbar\omega_D$ AND IS ZERO OTHERWISE.

$$\Delta_{\bar{k}} \to \Delta_0$$

$$(**) \Rightarrow \quad 1 = \frac{U_0}{2} \sum_{\bar{k}:\, -\hbar\omega_D < \epsilon_{\bar{k}} < \hbar\omega_D} \frac{1}{\sqrt{\epsilon_{\bar{k}}^2+\Delta_0^2}} \qquad (\text{ASSUMING } \Delta_0 > 0).$$

$$\sum_{\bar{k}} \longmapsto \int_{-\hbar\omega_D}^{\hbar\omega_D} d\epsilon\, \underbrace{D_\sigma(\epsilon)}_{\text{DOS FOR ONE SPIN PROJECTION}} \simeq D_\sigma(\epsilon_F) \int_{-\hbar\omega_D}^{\hbar\omega_D} d\epsilon$$

$$1 \simeq \frac{U_0 D_\sigma(\epsilon_F)}{2} \int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{d\epsilon}{\sqrt{\epsilon^2+\Delta_0^2}} = U_0 D_\sigma(\epsilon_F)\, \mathrm{arsh}\frac{\hbar\omega_D}{\Delta_0}$$

$$\Delta_0 \simeq \hbar\omega_D \frac{1}{\mathrm{sh}\left[\frac{1}{U_0 D_\sigma(\epsilon_F)}\right]}$$

WEAK COUPLING LIMIT: $U_0 D_\sigma(\epsilon_F) \ll 1$

$$\Delta_0 \simeq 2\hbar\omega_D e^{-\frac{1}{U_0 D_\sigma(\epsilon_F)}}$$