# Advanced Quantum Mechanics of Many-Body Systems
## Homework 4

(23 Nov 2024)

**Problem 1**

Consider a gas of $N$ noninteracting fermions in a $d$-dimensional box of volume $V$. The temperature $T = 0$ and the number density $n = N/V$. Find: a) the Fermi vector $k_F$, b) the Fermi energy $\epsilon_F$, and c) the energy of the system $E^{(0)}$. Show that $E^{(0)}/N = [d/(d+2)]\epsilon_F$. What is this ratio in the infinite dimensional limit $d \to \infty$? How would you interpret this result? The dispersion relation is $\epsilon_{\mathbf{k}} = \hbar^2 k^2/2m$.

**Problem 2**

The density of states $D(\epsilon)$ describes the number of accessible states of given energy (per unit volume). Calculate $D(\epsilon)$ for free electrons (where $\epsilon_{\vec{k}} = \hbar^2 \vec{k}^2/2m$) in dimensionality $d = 1$, 2, and 3. Compare the dependence on $\epsilon$ in these three cases. Use the expression: $D(\epsilon) = \frac{1}{V}\sum_\sigma \sum_i \delta(\epsilon - \epsilon_{\vec{k}_i}) \to \sum_\sigma \int \frac{d^d k}{(2\pi)^d}\delta(\epsilon - \epsilon_{\vec{k}})$ valid in the thermodynamic limit where $V \to \infty$, $N \to \infty$, $N/V = n \to$ const.

**Problem 3**

In the ground state of an unpolarized noninteracting electron gas, there are $N/2$ spin-up electrons and $N/2$ spin-down electrons. A spin-polarized state has $N_\uparrow$ spin-up electrons and $N_\downarrow$ electrons: $N_\uparrow = (1+p)N/2$, $N_\downarrow = (1-p)N/2$, where $p \in ]-1, 1[$ is the fractional spin polarization. Show that, in three dimensions, the energy per electron in the ground state of given $p$ is

$$\frac{E}{N} = \frac{E_0}{2N}\left[(1+p)^{5/3} + (1-p)^{5/3}\right] ,$$

where $E_0/N$ is the energy per electron in the unpolarized ground state.

**Problem 4**

*Correlation function.* For a system of noninteracting electrons at $T = 0$, define the following correlation function:

$$G_\sigma(\boldsymbol{r}, \boldsymbol{r}') = \langle \mathrm{FS}|\Psi_\sigma^\dagger(\boldsymbol{r})\Psi_\sigma(\boldsymbol{r}')|\mathrm{FS}\rangle \tag{1}$$

This is the probability amplitude for the state $|\alpha\rangle = \Psi_\sigma(\boldsymbol{r}')\,|\mathrm{FS}\rangle$, in which a particle at $\boldsymbol{r}'$ is removed from the system in the ground state, to be found in state $|\beta\rangle = \Psi_\sigma(\boldsymbol{r})\,|\mathrm{FS}\rangle$. $|\beta\rangle$ is the state obtained by removing a particle, with coordinates $(r, \sigma)$, from the ground state. Obtain an expression for $G_\sigma(\boldsymbol{r}, \boldsymbol{r}')$ in terms of $x = |\boldsymbol{r} - \boldsymbol{r}'|$.

**Hint:** Expand the field operators in terms of annihilation/creation operators: $\Psi_\sigma(\boldsymbol{r}) = \frac{1}{L^d}\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} a_{\boldsymbol{k}\sigma}$.

**Problem 5**

Consider a system of electrons which stem from two different energy levels $\epsilon_1$ and $\epsilon_2$. They are described by the following hamiltonian: $H = \sum_\sigma \left[\epsilon_1 a_{1,\sigma}^\dagger a_{1,\sigma} + \epsilon_2 a_{2,\sigma}^\dagger a_{2,\sigma} + V\left(a_{1,\sigma}^\dagger a_{2,\sigma} + a_{2,\sigma}^\dagger a_{1,\sigma}\right)\right]$ with $\sigma \in \{\uparrow, \downarrow\}$.

a) Show that $H$ commutes with the total particle number operator.

b) Develop a general procedure for computing the energy eigenvalues for arbitrary total electron number $N \leq 4$ making use of the Fock states (i.e. states of given $\{n_{i,\sigma}\}$).

c) Calculate the energy eigenvalues for $N = 0$ and $N = 1$.

d) How many Fock states of given $N = 2$ are there? Show that two of them are eigenstates of $H$. Find all the eigenenergies for $N = 2$.