Advanced Quantum Mechanics of Many-Body Systems
Homework 3

(9 Nov 2024)

Problem 1

Consider a simple three-dimensional model of a crystal, in which the hamiltonian is given by $H = \sum_j \frac{p_j^2}{2m} + \sum_{j,\vec{a}} \frac{m\omega_0^2}{2} \left(\phi_j - \phi_{j+\vec{a}}\right)^2$, where $j \in \{1 \ldots N\}$ labels the sites of a cubic lattice, while $\vec{a} \in \{a\vec{e}_x, a\vec{e}_y, a\vec{e}_z\}$. In the considered simplified model the displacement (ϕ_j) and momentum (p_j) variables are treated as one-dimensional quantities. Express the hamiltonian by the phonon creation and annihilation operators. Find the ground-state energy. Consider periodic boundary conditions and assume the crystal volume $L^3 \gg a^3$.

Problem 2

Consider a hypercubic crystal lattice in $d \geq 1$ spatial dimensions. Derive a general expression for the specific heat of the corresponding gas of (noninteracting) phonons. Discuss the limits of low and high temperatures.

Problem 3

Consider a one-dimensional lattice and assume that the Bloch states are just plane waves. Calculate the Wannier functions $\phi_{n\sigma}(x - ma)$ for a given lattice vector $R = ma$ (m being integer). Plot your result.

Problem 4

Identical atoms occupy the sites of a square lattice with lattice constant a. Assume that there is only one Wannier orbital on each site, so that one band is formed from these orbitals. Neglecting electron-electron interaction, and assuming that an electron can hop from one site to only one of the nearest-neighboring sites, the hopping matrix element being $-t$, the Hamiltonian is $H = -t \sum_{<i,j>,\sigma} a_{i,\sigma}^\dagger a_{j,\sigma}$. Calculate the dispersion of the energy band.

Problem 5

In course of the analysis of the Bogoliubov approximation for weakly interacting bosons, we introduced the operators $\alpha_{\vec{k}} = \cosh\left(\theta_{\vec{k}}\right) a_{\vec{k}} - \sinh\left(\theta_{\vec{k}}\right) a_{-\vec{k}}^\dagger$. Show that these operators fulfill the bosonic commutation relations.