Advanced Quantum Mechanics of Many-Body Systems
Homework 1

(11 Oct 2024)

Problem 1

Consider a bosonic particle in one spatial dimension. The hamiltonian is given by $\hat{H} = \hbar\omega\left(a^\dagger a + 1/2\right) + \hbar\omega_0\left(a^\dagger + a\right)$, where $[a, a^\dagger] = 1$, $\omega, \omega_0 > 0$. Find the eigenstates and eigenvalues of the system.
Hint: Introduce $\alpha = a + \omega_0/\omega$.

Problem 2

In the so-called Schwinger representation spin is represented by two bosonic operators a and b such that $\hat{S}^+ = a^\dagger b$, $\hat{S}^- = (\hat{S}^+)^\dagger$, $\hat{S}^z = \frac{1}{2}\left(a^\dagger a - b^\dagger b\right)$. We put $\hbar = 1$.

a) Show that this definition is consistent with the commutation relations for $\hat{S}^+$ and $\hat{S}^-$.

b) Show that $|S, m\rangle = \frac{(a^\dagger)^{S+m}}{\sqrt{(S+m)!}}\frac{(b^\dagger)^{S-m}}{\sqrt{(S-m)!}}|\Omega\rangle$ is consistent with the eigenstates of the total spin $\mathbf{S}^2$ and its z-component operators. Here $|\Omega\rangle$ denotes the vacuum of Schwinger bosons.

Problem 3

Evaluate the following commutators for fermionic creation/annihilation operators:

(a) $[a_i^\dagger, \sum_j a_j^\dagger a_j]$

(b) $[a_i, \frac{1}{2}\sum_{\alpha,\beta} a_\alpha^\dagger a_\beta^\dagger a_\beta a_\alpha]$.

Problem 4

Express the current density operator using any one-particle basis $|\nu\rangle$ and the associated creation/annihilation operators $a_\nu, a_\nu^\dagger$.
Hint: Start in the position basis.

Problem 5

Derive the action of the field operator $\hat{\Psi}(\mathbf{x})$ on a general N-particle ket $|\mathbf{y}_1, \ldots \mathbf{y}_N\}$. For this purpose first show that

$$\{\mathbf{x}_1, \ldots \mathbf{x}_{N-1}|\hat{\Psi}(\mathbf{x}_N)|\mathbf{y}_1, \ldots \mathbf{y}_N\} = \{\mathbf{x}_1, \ldots \mathbf{x}_N|\mathbf{y}_1, \ldots \mathbf{y}_N\} \ .$$

Subsequently write the r.h.s. of the above equation in a form of a determinant/permanent and perform the Laplace expansion along row N to demonstrate that

$$\hat{\Psi}(\mathbf{x})|\mathbf{y}_1, \ldots \mathbf{y}_N\} = \sum_{k=1}^{N} (\zeta)^{N+k}\delta(\mathbf{x} - \mathbf{y}_k)|\mathbf{y}_1, \ldots \mathbf{y}_{k-1}, \mathbf{y}_{k+1} \cdots \mathbf{y}_N\} \ .$$