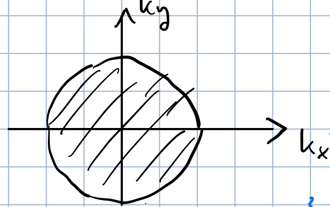
GS of a non-interacting electron gas : $|FS\rangle$ (states filled up to k_f, empty above)



Imagine adding two electrons and turning on an attractive interaction between these two electrons. Assume this interaction exists only when the two electrons occupy states within a shell of energy width $\hbar\omega_D$ around the FS, $\hbar\omega_D \ll \epsilon_F$

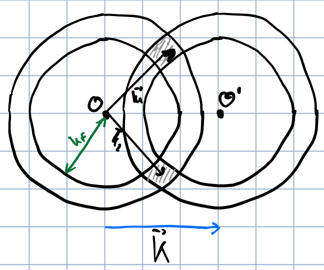
{the role of the other electrons is only to prevent the two extra electrons from occupying any lower energy states.}

In absence of interaction the GS of the two electrons has energy $2\epsilon_F$. How does this change when the interaction is switched on? $\to$ Cooper 1956

$\vec{K}$ - total momentum of the pair

Interaction - electrons can scatter from $|\bar{k}_1\sigma_1, \bar{k}_2\sigma_2\rangle$ to $|\bar{k}_1+\bar{q}\,\sigma_1, \bar{k}_2-\bar{q}\,\sigma_2\rangle$

$$\vec{k}_1+\vec{k}_2=\vec{K} \quad \text{unchanged.}$$



If we pick an arbitrary $\vec{K}$, $\vec{k}_1$ and $\vec{k}_2$ are restricted to tiny regions of the shell $\to$ the region of intersection of two shells centered at points separated by $\vec{K}$

This is however different for $\vec{K}\approx 0$, for which case electrons from the entire shell may interact. Henceforth we assume that the two added electrons have total momentum O ($\vec{k}_1\to\vec{k}$, $\vec{k}_2\to-\vec{k}$).

$\vec{r}_1, \vec{r}_2 \to$ positions of the two added electrons, their wavefunction $\psi(\vec{r}_1\sigma_1, \vec{r}_2\sigma_2)$

Hamiltonian: $H=\frac{\vec{p}_1^{\,2}}{2m}+\frac{\vec{p}_2^{\,2}}{2m}+\mathcal{U}(\bar{r}_1-\bar{r}_2)$ {spin independent, translation invariance}

$\to$ Eigenstates can be written as product of spatial and spin components

$$\psi \longrightarrow e^{i\vec{K}\vec{R}}\varphi(\vec{r})\,\chi_{\sigma_1\sigma_2}$$

$\to$ centre of mass

The spin function can be taken as one of the eigenstates of the total spin op.:

$$\chi_\sigma=\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle\right) \quad \text{- singlet}$$

$$\chi_\sigma=\begin{cases}|\uparrow\uparrow\rangle\\ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle)\\ |\downarrow\downarrow\rangle\end{cases} \quad \text{- triplet}$$

{If we take the singlet state $\to$ the spin component is antisymmetric and then the entire state is antisymmetric if the spatial component $\varphi(\bar{r})$ is symmetric

$\downarrow$

take the singlet state

Translational symmetry $\to$ $$\psi=\sum_{\bar{k}}g(\bar{k})\underbrace{\frac{1}{V}e^{i\bar{k}(\bar{r}_1-\bar{r}_2)}}_{\langle\bar{r}_1\bar{r}_2|\bar{k},-\bar{k}\rangle}\chi_{\sigma_1\sigma_2} \qquad (*)$$

$g(\bar{k})$ - to be determined (antisymmetry under $(\bar{r}_1,\sigma_1)\longleftrightarrow(\bar{r}_2,\sigma_2)$ requires $g(-\bar{k})=g(\bar{k})$

{for triplet we'd have $g(-\bar{k})=-g(\bar{k})$}

Restriction of the states to the shell requires $g(\bar{k})$ nonvanishing only for $\bar{k}$ such that $\epsilon_F<\epsilon_{\bar{k}}<\epsilon_F+\hbar\omega_D$

Substitute $(*)$ into $H\psi=E\psi$ {$\epsilon_{\bar{k}}=\frac{\hbar^2\bar{k}^2}{2m}$}

$$\to\left[\sum_{\bar{k}'}2\epsilon_{\bar{k}'}g(\bar{k}')e^{i\bar{k}'\bar{r}}+\sum_{\bar{k}'}g(\bar{k}')\,\mathcal{U}(\bar{r})e^{i\bar{k}'\bar{r}}\right]=E\sum_{\bar{k}'}g(\bar{k}')e^{i\bar{k}'\bar{r}} \longrightarrow \sum_{\bar{k}'}(2\epsilon_{\bar{k}'}-E)g(\bar{k}')e^{i\bar{k}'\bar{r}}+\sum_{\bar{k}'}g(\bar{k}')\,\mathcal{U}(\bar{r})e^{i\bar{k}'\bar{r}}=0$$

Multiply by $\frac{1}{V}e^{-i\vec{k}\vec{r}}$ and integrate over V, use $\int_V e^{i(\vec{k}-\vec{r}')\vec{r}}d\vec{r} = V\delta_{\vec{k}\vec{k}'}$ $\longrightarrow (2\epsilon_{\vec{k}}-E)\,g(\vec{k}) + \frac{1}{V}\sum_{\vec{k}'}\underbrace{\int_V e^{-i(\vec{k}-\vec{k}')\vec{r}}\,U(\vec{r})}_{=:\,U_{\vec{k}\vec{k}'}}\;g(\vec{k}')$

To proceed consider a simple model where $U_{\vec{k}\vec{k}'} = \begin{cases} -U_0 & \text{if } \epsilon_F < \epsilon_{\vec{k}}, \epsilon_{\vec{k}'} < \epsilon_F + \hbar\omega_D \\ 0 & \text{otherwise} \end{cases}$ $\left\{ \left.\begin{matrix} U(\vec{r}) \in \mathbb{R} \\ U(-\vec{r}) = U(\vec{r}) \end{matrix}\right\} \Rightarrow U_{\vec{k}\vec{k}'} \in \mathbb{R} \right\}$

$(U_0 > 0)$

$(2\epsilon_{\vec{k}}-E)\,g(\vec{k}) = \frac{U_0}{V}\sum_{\vec{k}'} g(\vec{k}')$ Divide by $\frac{1}{2\epsilon_{\vec{k}}-E}$ and $\sum_{\vec{k}}$

$\longrightarrow \cancel{\sum_{\vec{k}} g(\vec{k})} = \frac{U_0}{V}\sum_{\vec{k}}\frac{1}{2\epsilon_{\vec{k}}-E}\cancel{\sum_{\vec{k}'} g(\vec{k}')}$

{ sum over states with one spin projection }

$\sum_{\vec{k}} \longrightarrow \int d\epsilon\, \underbrace{D_\sigma(\epsilon)}_{\text{DENSITY of STATES (PER SPIN)}}$ ($D_\sigma(\epsilon)\,d\epsilon$ = number of states of given σ and energy in $[\epsilon, \epsilon + d\epsilon]$)

$$1 = \frac{U_0}{V}\int_{\epsilon_F}^{\epsilon_F+\hbar\omega_D}\frac{D_\sigma(\epsilon)}{2\epsilon-E}\,d\epsilon$$

{ TYPICAL METAL $\hbar\omega_D \simeq 20\,\text{meV}$, $\epsilon_F \simeq 5\,\text{eV}$ }

$\hbar\omega_D \ll \epsilon_F \longrightarrow D_\sigma(\epsilon) \simeq D_\sigma(\epsilon_F)$

$$\longrightarrow 1 = \frac{U_0 D_\sigma(\epsilon_F)}{V}\int_{\epsilon_F}^{\epsilon_F+\hbar\omega_D}\frac{d\epsilon}{2\epsilon-E} = \frac{1}{2}U_0\underbrace{d_\sigma(\epsilon_F)}_{=:\,\frac{D_\sigma(\epsilon_F)}{V}}\ln\frac{2\epsilon_F+2\hbar\omega_D-E}{2\epsilon_F-E}$$

$\left\{ \int_{\epsilon_F}^{\epsilon_F+\hbar\omega_D}\frac{d\epsilon}{2\epsilon-E} = \frac{1}{2}\ln\left(\epsilon-\frac{E}{2}\right)\Big|_{\cdots}^{\cdots} \right\}$

$\left\{ e^{\frac{2}{U_0 d_\sigma}} = \frac{2\epsilon_F+2\hbar\omega_D-E}{2\epsilon_F-E} = 1+\frac{2\hbar\omega_D}{2\epsilon_F-E} \right\}$

$\left\{ 2\epsilon_F - E = \frac{2\hbar\omega_D}{e^{\frac{2}{\hbar\omega_D}}-1} \right\}$

Solve this for E,

$$\longrightarrow E = 2\epsilon_F - \frac{2\hbar\omega_D\, e^{-\frac{2}{U_0 d_\sigma(\epsilon_F)}}}{1-e^{-\frac{2}{U_0 d_\sigma(\epsilon_F)}}}$$, for the weak-coupling limit $U_0 d_\sigma(\epsilon_F) \ll 1$

$$E \simeq 2\epsilon_F - 2\hbar\omega_D\, e^{-\frac{2}{U_0 d_\sigma(\epsilon_F)}}$$

Remarks

- No matter how weak the interaction, the two electrons form a bound state (known as a Cooper pair), the energy of which is lower than $2\epsilon_F$.
- The energy of the bound state is not an analytic function of U_0 for $U_0 \longrightarrow 0$. E cannot be expanded in powers of U_0. The result for E <u>cannot</u> be obtained by a perturbative expansion in powers of U_0.
- The binding energy increases is $\rightarrow U_0$ increases
  $\rightarrow$ DOS at Fermi level increases.
- $|FS\rangle$ will be unstable to any such attractive interaction between electrons.