which, of course, can be shown directly by using the same steps that we used to deal with $\hat{V}_{\lambda 0}$.

We next consider an observable for which the classical analog, $V(\mathbf{r}_{\rm I},\mathbf{P}_{\rm I},\mathbf{r}_2,\mathbf{P}_2)$, can be written as a polynomial or convergent powers series in the Cartesian components of $\mathbf{r}_{\rm I}$, $\mathbf{P}_{\rm I}$, $\mathbf{r}_2$, and $\mathbf{P}_2$.[11] Such an operator can obviously be constructed from a linear combination of terms of the form

$$\hat{V}_G = \prod_{\alpha=1}^{3} (\hat{r}_{{\rm I}\alpha})^{m_\alpha}(\hat{r}_{{\rm II}\alpha})^{n_\alpha}(\hat{P}_{{\rm I}\alpha})^{a_\alpha}(\hat{P}_{{\rm II}\alpha})^{b_\alpha} + \prod_{\alpha=1}^{3} (\hat{r}_{{\rm II}\alpha})^{m_\alpha}(\hat{r}_{{\rm I}\alpha})^{n_\alpha}(\hat{P}_{{\rm II}\alpha})^{a_\alpha}(\hat{P}_{{\rm I}\alpha})^{b_\alpha} + \text{Hermitian adjoint}, \quad (18)$$

where $\alpha = 1,2,3$ designates the x, y, z Cartesian components of the $\hat{\mathbf{r}}$'s and $\hat{\mathbf{P}}$'s, and the 12 numbers $a_\alpha, b_\alpha, m_\alpha$, and n_α for $\alpha = 1,2,3$, are nonnegative integers. The first term on the right side of Eq. (18) is an "almost typical" term in a polynomial or powers series in the Cartesian components of $\hat{\mathbf{r}}_{\rm I}$, $\hat{\mathbf{r}}_{\rm II}$, $\hat{\mathbf{P}}_{\rm I}$, and $\hat{\mathbf{P}}_{\rm II}$; i.e., it is a product of nonnegative, integral powers of the Cartesian components of $\hat{\mathbf{r}}_{\rm I}, \hat{\mathbf{r}}_{\rm II}, \hat{\mathbf{P}}_{\rm I}$, and $\hat{\mathbf{P}}_{\rm II}$. This term is "almost typical" rather than typical because the components of $\hat{\mathbf{P}}_{\rm I}$ and $\hat{\mathbf{P}}_{\rm II}$ stand to the right of the components of $\hat{\mathbf{r}}_{\rm I}$ and $\hat{\mathbf{r}}_{\rm II}$. But, of course, in the Hermitian adjoint of this term, which is also contained in $\hat{V}_G$, the components of $\hat{\mathbf{P}}_{\rm I}$ and $\hat{\mathbf{P}}_{\rm II}$ stand to the left of the components of $\hat{\mathbf{r}}_{\rm I}$ and $\hat{\mathbf{r}}_{\rm II}$. Terms of these two forms are all that are needed because a "mixed" term with powers of $\hat{r}_{{\rm I}\alpha}$ and/or $\hat{r}_{{\rm II}\alpha}$ on both the left and the right of the powers of $\hat{P}_{{\rm I}\alpha}$ and $\hat{P}_{{\rm II}\alpha}$ can always be expressed as a sum of terms of the two forms present in $\hat{V}_G$. This can be accomplished through use of the identity

$$\hat{p}^c\hat{q}^d = \hat{q}^d\hat{p}^c + \sum_{k=1}^{c} \binom{c}{k} [d(d-1)\cdots(d-k+1)] \times (-i\hbar)^k \hat{q}^{d-k}\hat{p}^{c-k} \quad (19)$$

and its Hermitian adjoint.[12] Here, d can be any number and c must be a nonnegative integer. This identity can be obtained by using

$$\hat{p}\hat{q} = \hat{q}\hat{p} - [\hat{q},\hat{p}] = \hat{q}\hat{p} - i\hbar \quad (20)$$

to move the powers of $\hat{q}$ to the left through the powers of $\hat{p}$; it can also be obtained from Leibnitz's rule for the derivative of a product.[13] The second term on the right side of Eq. (18) is the same as the first term with the subscripts I and II interchanged; thus $\hat{V}_G$ is invariant under the interchange of I and II. A moment's thought serves to reveal that precisely the same steps that were used in dealing with $\hat{V}_{\lambda 0}$ and $\hat{V}_0$ suffice to prove that $\hat{V}_G$ satisfies our Eq. (1) and Eq. (28) of Ref. 1.

As we have noted, an observable, $\hat{V}(\text{I,II})$, for which the classical analog, $V(\mathbf{r}_{\rm I},\mathbf{P}_{\rm I},\mathbf{r}_2,\mathbf{P}_2)$, can be written as a polynomial or convergent powers series in the Cartesian components of $\mathbf{r}_{\rm I}$, $\mathbf{P}_{\rm I}$, $\mathbf{r}_2$, and $\mathbf{P}_2$, can be written as a linear combination of terms of the form of $\hat{V}_G$. Obviously, such an operator satisfies our Eq. (1) and Eq. (28) of Ref. 1.

We have thus shown that for a very general two-body operator, one which is a physical observable and a polynomial or convergent powers series in the Cartesian components of $\hat{\mathbf{r}}_{\rm I}, \hat{\mathbf{P}}_{\rm I}, \hat{\mathbf{r}}_{\rm II}$, and $\hat{\mathbf{P}}_{\rm II}$, invariance under interchange of I and II is all that is necessary to obtain Eq. (1). To the best of our knowledge, this is the first time that such a derivation has been given.

[1] N. I. Greenberg and S. Raboy, "One-body and two-body operators on systems of identical particles," Am. J. Phys. **50**, 148–155 (1982).

[2] As usual, by an *observable* we mean a Hermitian operator possessing a complete set of eigenvectors. See, e.g., Albert Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1958), Secs. V.9, VII. 9, and VII.13.

[3] This is Eq. (28) of Ref. 1 written out somewhat more explicitly. Note that we reserve the caret ($\hat{}$) for operators in the abstract state vector space.

[4] Reference 1, p. 155, the last paragraph.

[5] See Eq. (29) of Ref. 1.

[6] Albert Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1958), Secs. VIII.6 and VIII.16.

[7] J. J. Sakurai, *Modern Quantum Mechanics* (Benjamin/Cummings, Menlo Park, CA, 1985), Sec. 1.7.

[8] Reference 6, Secs. VII.6 and VIII.7.

[9] Jean-Paul Blaizot and Georges Ripka, *Quantum Theory of Finite Systems* (MIT, Cambridge, 1986), Sec. 1.1.

[10] See Eq. (27) of Ref. 1.

[11] For the powers series case, it may be necessary to use different powers series for different ranges of the Cartesian components of $\mathbf{r}_{\rm I},\mathbf{P}_{\rm I},\mathbf{r}_2$, and $\mathbf{P}_2$.

[12] Of course, we could use this identity to put all the terms in $\hat{V}_{\lambda 0}$ into the form of the terms in $\hat{V}_\lambda$, i.e., with all the powers of $\hat{q}_{\rm I},\hat{q}_{\rm II},\hat{q}'_{\rm I},\hat{q}'_{\rm II}$ to the left of the powers of $\hat{p}_{\rm I},\hat{p}_{\rm II},\hat{p}'_{\rm I},\hat{p}'_{\rm II}$.

[13] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products, Corrected and Enlarged Edition* (Academic, New York, 1980), p. 19.

Are sound waves isothermal or adiabatic?

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(Received 17 January 1989; accepted for publication 14 April 1989)

In preparing a first-year physics course, I was attracted by the new organization of the third edition of *Fundamentals of Physics* by Halliday and Resnick;[1] especially by the fascinating essays.

On closer examination, however, I found that in at least two places in the text discussions of the thermodynamic properties of sound waves are misleading. In Sec. 20-6 (p. 470), when adiabatic processes are discussed, it is written that "... the compressions and rarefactions of air as a sound wave passes through are adiabatic. There is simply no time for heat to flow back and forth in synchronism with the rapidly oscillating sound waves... ." In Sec. 21-11 (p. 498),

Table I. The results of a textbook survey.[a]

Title	Authors	Publisher	Where the subject is discussed	Contain correct or wrong statements
Fundamentals of Physics, 3rd ed.	Halliday & Resnick	Wiley (1988)	p. 470 & p. 498	Wrong
Physics For Scientists and Engineers with Modern Physics, 2nd ed.	Giancoli	Prentice-Hall (1989)	p. 475	Wrong
Physics	Gettys, Keller, & Skove	McGraw-Hill (1989)	p. 748	Wrong
Principles of Physics	Marion & Hornyak	Saunders College Publishing (1984)	p. 326	Correct

[a] The subject is not discussed in the following books: (1) H. Ohanian, *Physics* (Norton, New York, 1985); (2) R. Weidner and M. Browne, *Physics* (Allyn & Bacon, Boston, 1989), rev. ed.; (3) R. Serway, *Physics for Scientists and Engineers with Modern Physics* (Saunders, Philadelphia, 1986), 2nd ed.; (4) F. Blatt, *Principles of Physics*, (Allyn & Bacon, Boston, 1986), 2nd ed.

it is repeated that "... sound waves are propagated through air and other gases as a series of compressions and rarefactions that take place so rapidly that there is no time for heat to flow from one part of the medium to another. Volume changes for which $Q = 0$ are called adiabatic processes... ."

These statements suggest that sound waves are more nearly adiabatic at high frequencies than at low frequencies, since the time for heat flow is less. In fact, however, the opposite is true, namely, at low frequencies the speed of sound is the adiabatic ($pv^\gamma = \text{const}$) speed,

$$c_s = (\gamma R_m T)^{1/2}, \tag{1}$$

where R_m is the gas constant per unit mass, γ is the ratio of the specific heat at constant pressure to that at constant volume, and T is the absolute temperature. On the other hand, at very high frequencies, the speed of sound is the isothermal ($pv = \text{const}$) speed,

$$c_T = (R_m T)^{1/2}, \tag{2}$$

the process being isothermal.

The erroneous statement made in the textbook is, in fact, quite common. In four out of eight introductory physics textbooks published after 1984 in the USA, which we found in the library of our department, this subject is discussed; only one has a reasonably correct statement (Table I).

Historically, a similar statement was first proposed by Laplace[2] in 1816. To resolve the discrepancy that the experimental value of sound speed at audible frequencies was about 20% higher than that calculated from c_T, Laplace suggested that the compressions and expansions in a sound wave are adiabatic. This was in contrast to Newton's[3] proposal that they are isothermal. Laplace's theory works well at low frequencies. But the argument provided by Laplace to interpret his results is very similar to that given in the textbook. The point that is missed in the argument is that although at low frequencies there is sufficient time available between successive variations of temperature for thermal equilibrium to be established, the wavelength of the disturbance is large, requiring the heat to flow over a great distance. At high frequencies, on the other hand, the time for the heat to flow is short, but so is the distance the heat must traverse. This can be seen by the following simple calculations: For example, $f = 100$ Hz, the corresponding wavelength λ is given by $\lambda = c_s/f = (340)$ m/s/100 Hz = 3.4 m; at $f = 10\,000$ Hz, however, the corresponding wavelength λ becomes 0.034 m.

The crucial question is which of the two effects, time or space, predominate? The paradox was resolved by the classical theory of absorption and dispersion of sound waves, which was first developed by Kirchhoff[4] and elaborated by Langevin[5] and many others.[6,7] The theory shows that both viscosity and heat conductivity play important roles in acoustic processes, and consequently the speed of acoustic waves is frequency dependent, a phenomenon called dispersion in physics. Lindsay, in his book, *Mechanical Radiation*[8] gave a similar but brief derivation of the speed of sound when only heat conduction is considered. In the passage of an acoustic wave, the temperature in the compression of the fluid is raised and a temperature gradient is locally established. This leads to a heat flow by conduction, which can be significant, before the subsequent rarefaction can be accomplished. By simultaneously solving the continuity equation, the equation of motion, and the energy equation including heat conduction, he has derived the phase velocity $c(\omega)$ of a one-dimensional plane harmonic wave as follows:

$$c(\omega) = c_s[(1 + \omega^2\tau^2)/(1 + \gamma\omega^2\tau^2)]^{1/2}, \tag{3}$$

where τ represents the so-called relaxation time for heat conduction. For air at room temperature this is of the order of 10^{-10} s. It is interesting to look at the two extreme cases. At low frequencies, where $\omega\tau$ is much smaller than unity, $c(\omega)$ approaches c_s. However, at high frequencies, $\omega\tau$ is much larger than unity and $c(\omega)$ approaches c_T. Since τ for air is so small, it is safe to say that sound waves in the audible frequency range ($20\text{ Hz} < f < 20\,000\text{ Hz}$) are adiabatic.

The identical conclusion can be reached by the following simple argument. Heat flow is a diffusive phenomenon, and a diffusive wave velocity v_h can be introduced to describe the distance over which heat diffuses in a unit of time, which is given by[8]

$$v_h = (2\pi f K / C_v \rho)^{1/2}, \tag{4}$$

where K is the thermal conductivity, ρ is the density, and C_v is the specific heat at constant volume. Notice that v_h is an increasing function of frequency. When $v_h \gg c_s$, or $f \gg f_0 = c_s^2 C_v / 2\pi K$, the process is isothermal. On the other hand, when $f \ll f_0$, it is adiabatic.

The classical theory works well with the rare gases, which have only translational kinetic energy. This was confirmed by Greenspan's modern measurements[10] of the dispersion of sound speed for five monatomic gases. Analysis of the experiments shows that acoustic waves observed are adiabatic at low frequencies and approach isothermal with increasing frequency.[11] The situation becomes a little complicated with polyatomic molecules; relaxation processes of other degrees of freedom such as rotation and vibration become important as well. Nevertheless, the above result is still true. For the detailed discussion, the reader is referred to Ref. 8.

ACKNOWLEDGMENT

I thank Professor W. L. Nyborg for his valuable discussions.

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A one-dimension collision experiment

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(Received 31 March 1989; accepted for publication 4 May 1989)

Here is a simple piece of apparatus that illustrates one kind of 1-D collision in a most effective manner. The idea behind the demonstration has recently been discussed[1,2] for the case of two balls, and is far from new. A massive ball is dropped simultaneously with a light ball just above it: It is found that the light ball rises considerably higher than its original height.

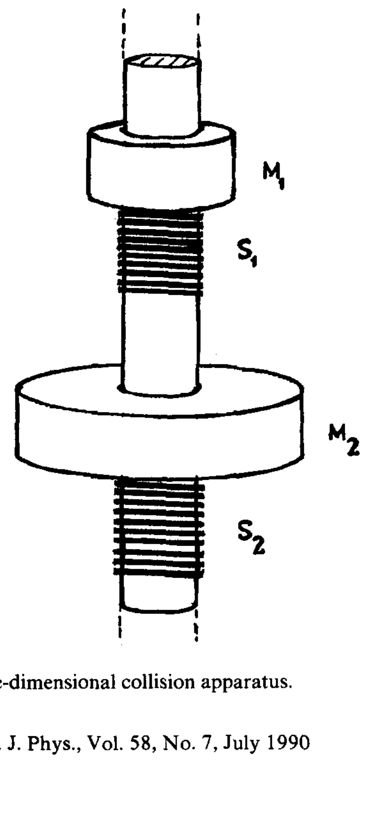

Fig. 1. The one-dimensional collision apparatus.

A more practical arrangement is shown in the diagram. The two masses M_1 and M_2 (typically 100 and 500 g, respectively), are dropped simultaneously with a small gap between them. The larger mass has an almost elastic collision with a bottom support via the spring S_2, and collides via S_1 with the smaller mass, which is still descending. A little practice is needed to get these conditions just right. The mass M_1 rises to a considerable height, typically five times as high for the masses quoted. An easy calculation shows that if M_2 is much larger than M_1 then M_1 could rise as high as nine times as far as dropped, if friction is minimal. One has more control over the masses than with balls.

In the apparatus used, M_1 and M_2 were brass, with a steel rod some 2 m long as the central support, secured in a heavy metal base. The spring S_2 should be the stronger, to withstand the impact of M_2. Experiment to find a good spring combination. The experiment can be performed as a "Gee whiz, explain that!" type demonstration as an introduction to 1-D collisions. A stairwell is a good location, where the height risen can be readily found.

ACKNOWLEDGMENT

Helpful comments by Dr. Lee Torop in the above physics department are hereby acknowledged.

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