Algebraic and geometric aspects of modern theory of integrable systems

Lecture 9

1 The "action-angle" coordinates

Digression: the Darboux theorem: Let ω be a symplectic form on $M, \dim M = 2n$. Then in a neighbourhood of any point there exist local coordinates (q^i, p_i) such that

$$\omega = dp_i \wedge dq^i.$$

The coordinates (q^i, p_i) are called the *Darboux coordinates* (another name: the *canonical coordinates*) for ω. Note that the Darboux coordinates are not unique. Given such coordinates (q, p), any local symplectomorphism F, i.e. a local diffeomorphism preserving ω, will produce new Darboux coordinates (F^*q, F^*p).

Description of the "action-angle" coordinates: In the context of the Arnold–Liouville theorem, we will build specific Darboux coordinates on M, the "action-angle" coordinates. Let $N := M_c$ be a fixed common level set of the functions $g_1, \ldots, g_n$.
The "angles": Note that, although we have defined the angles $\varphi_1, \ldots, \varphi_n$ on a single level set, in fact these functions are defined also on the neighbour levels and depend smoothly on the level. Indeed, we can repeat the construction of the map $\psi : \mathbb{R}^n \to N$ on neighbour levels. As a result we will get a n-parameter family of maps $\psi_c : \mathbb{R}^n \to M_c$ to which there corresponds a n-parameter family of linear maps $A_c : \mathbb{R}^n \to \mathbb{R}^n$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{A_c} & \mathbb{R}^n \\ \downarrow p & & \downarrow \psi_c \\ \mathbb{T}^n & \longrightarrow & M_c \end{array} .$$

The maps ψ_c and A_c smoothly depend on c, consequently so do the angles on M_c.
The (g, φ)-coordinates: We claim that in a neighbourhood of N the functions $g_1, \ldots, g_n$ together with the angles $\varphi_1, \ldots, \varphi_n$ form a system of local coordinates. Indeed, the functions $g_1, \ldots, g_n$ are functionally independent by the assumption. They are also independent of the angles, because they are constant on the vector fields $\eta(g_1), \ldots, \eta(g_n)$ which are linear combinations of $\frac{\partial}{\partial\varphi_1}, \ldots, \frac{\partial}{\partial\varphi_n}$.
The "action-angle" coordinates (I, φ): These are coordinates such that: 1) the functions $(I^1, \ldots, I^n)$ depend only on g; 2) $\omega = dI^i \wedge d\varphi_i$. In particular, (I, φ) are the Darboux coordinates on (M, ω).
The initial differential equation in different coordinate systems: Recall that in the φ-coordinates on N the hamiltonian vector field $\eta(H)|_N$ has the form $\eta(H) = (a_1, \ldots, a_n)$ for some $a_i \in \mathbb{R}$. The corresponding differential equation is of the form:

$$\frac{d\overrightarrow{\varphi}}{dt} = \overrightarrow{a}(c),$$

and its solutions are

$$\overrightarrow{\varphi}(t) = \overrightarrow{\varphi}(0) + t\overrightarrow{a}(c).$$

Thus in the (g, φ)-coordinates the flow of $\eta(H)$ is given by the equation

$$\frac{d\overrightarrow{g}}{dt} = 0, \frac{d\overrightarrow{\varphi}}{dt} = \overrightarrow{a}(g).$$

Analogously, in the (I, φ)-coordinates the initial equation is of the form

$$\frac{d\overrightarrow{I}}{dt} = 0, \frac{d\overrightarrow{\varphi}}{dt} = \overrightarrow{a}(I).$$

However, due to the fact that (I, φ)-coordinates are canonical, we get $\overrightarrow{a}(I) = -\frac{\partial H}{\partial \overrightarrow{I}}, \frac{\partial H}{\partial \overrightarrow{\varphi}} = 0$. Thus, knowing the "action-angle" coordinates, we can easily calculate the vector of "frequences" $\overrightarrow{a}$.

Finally the solutions of this equation are given by

$$\overrightarrow{I} = const, \overrightarrow{\varphi}(t) = \overrightarrow{\varphi}(0) - t\frac{\partial H}{\partial \overrightarrow{I}}.$$

Construction of the "action-angle" coordinates (a sketch): *Geometrically* we can explain this construction as follows.

Let $(M, \omega), \dim M = 2n$, be a symplectic manifold and $g : M \to B$ a lagrangian fibration, i.e. a surjective submersion all fibers of which are lagrangian submanifolds in M. Let $c \in B$. Let us explain how the construction of the action of $8\mathbb{R}^n$ on $M_c := g^{-1}(c)$ described in the previous section can be done simultaneously for all points from some neighbourhood U of c.

Namely, let $\alpha \in T^*_c B$ and let $f \in \mathcal{E}(B)$ be such that $d_c f = \alpha$. It is easy to see that the vector field $\eta(g^* f), \eta := \omega^{-1}$, is tangent to M_c (because $\eta(g^* f) g^* h = 0$ for any $f, h \in \mathcal{E}(B)$) and its restriction $\eta(g^* f)|_B$ is independent of the choice of f (if f' is another function with $d_c f' = \alpha$, we have $(g^* f - g^* f')|_B \equiv 0$ and $\eta(g^* f - g^* f')|_B \equiv 0$). Thus we get a linear mapping $\alpha \mapsto v(\alpha) := \eta(g^* f)|_B : T^*_c B \to \Gamma(TB)$, i.e. an action of the abelian (commutative) Lie algebra $T^*_c B$ on M_c. Integrating this action (i.e. passing from vector fields to their flows) we get an action of the abelian group $T^*_c B$ on M_c. Recall that fixing a point $x_c \in M_c$ we obtain a lattice $\Lambda_{x_c} \subset T^*_c B$, the stabilizer of x_c with respect to this action.

Now allow c to move over U. Repeating this construction for all points in U, we will have to choose $x_c \in M_c$, i.e. a section of g. Let us do this smoothly. As a result our lattice Λ_{x_c} will depend smoothly on c and we will get n one-forms $l_1, \ldots, l_n \in \Gamma(T^* B)$, the generators of this lattice.

It turns out that: 1) the section $c \mapsto x_c$ can be so chosen that its image will be a lagrangian submanifold in M; 2) if it will be chosen in such a way, the corresponding one-forms $l_1, \ldots, l_n$ will be closed, i.e. locally $l_i = dI_i$ for some functions I_i.

These last are the action coordinates we are looking for.

Analytically one can calculate the action coordinates as follows.

FACT. If $U \subset B$ is small enough the symplectic form ω is exact on $g^{-1}(U)$.

Proof We will use the De Rham theorem: $H^k_{DR}(M, \mathbb{R}) \cong (H_k(M, \mathbb{R}))^*$. Here $H^k_{DR}(M, \mathbb{R})$ stands for the space of the k-th De Rham cohomology, i.e. the factor space of closed modulo exact smooth k-forms. $H_k(M, \mathbb{R})$ is the space of the so-called singular k-th homology. It is known that it is isomorphic to its "smooth variant", which can be described as follows. Let $C_k(M,\mathbb{R})$ denote the space of finite formal linear combinations $a^i f_i$, where $a^i \in \mathbb{R}$ and f_i are smooth k-simplices in M, i.e. smooth maps from open neighbourhoods of k-dimensional simplices in $\mathbb{R}^k$ to M. The boundary operator $\partial_k : C_k(M,\mathbb{R}) \to C_{k-1}(M,\mathbb{R})$ satisfies the identity $\partial_{k-1} \circ \partial_k = 0$, so one can set $H_k(M,\mathbb{R}) := \ker \partial_k \big/ \operatorname{im} \partial_{k+1}$.

Given $\alpha \in \Gamma(\bigwedge^k T^*M)$, $a^i f_i \in C_k(M,\mathbb{R})$, put $\langle \alpha, a^i f_i \rangle := a^i \int_{\operatorname{im}(f_i)} \alpha$. The De Rham theorem says that in fact this pairing 1) induces a pairing between $H^k_{DR}(M,\mathbb{R})$ and $H_k(M,\mathbb{R})$ (this follows from the Stokes formula); 2) the induced pairing is nondegenerate.

In particular, it follows from the De Rham theorem that if the integral of a closed k-form over all the smooth k-cycles (i.e. the elements of $\ker \partial_k$) is zero, then this form is exact.

Now any 2-cycle f in $g^{-1}(U)$ is homotopically equivalent to some cocycle $\tilde{f}$ in $M_c, c \in U$. Thus $\int_{\operatorname{im} f} \omega = \int_{\operatorname{im} \tilde{f}} \omega = 0$. The last equality holds due to the fact that the restriction of ω to M_c is zero. □

Let λ be the corresponding potential, $d\lambda = \omega$. Let $\gamma_{1,c}, \dots, \gamma_{n,c}$ be the smooth closed curves on $M_c \cong \mathbb{T}^N$ representing the basis of $H_1(M_c, \mathbb{R}) \cong \mathbb{R}^n$. Put

$$I_i(c) := (1/2\pi) \int_{\gamma_{i,c}} \lambda.$$

Fact. 1. This does not depend on the choice of the representatives.

2. This does not depend on the choice of the potential.

Proof follows from the Stokes formula.

Example (harmonic oscillator I): Let $M = \mathbb{R}^2, H = (1/2)(p^2+q^2), \omega = dp \wedge dq$. Then in the polar coordinates $q = r\cos\varphi, p = r\sin\varphi$ we have $dp \wedge dq = -\sin\varphi dr \wedge r \sin\varphi d\varphi + \cos\varphi r d\varphi \wedge \cos\varphi dr = -r dr \wedge d\varphi = d(-r^2/2) \wedge d\varphi$. Hence $I = -H$.

Example (harmonic oscillator II): Let $M = \mathbb{R}^2, H = (1/2)(a^2p^2 + b^2q^2), \omega = dp \wedge dq$. The hamiltonian vector field is $\eta(H) = -a^2 p \frac{\partial}{\partial q} + b^2 q \frac{\partial}{\partial p}$, here $\eta = \omega^{-1} = -\frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}$. The level sets $M_c = \{(q,p) \mid H(q,p) = c\}$ are ellipses $\{(q,p) \mid q^2/(2c/b^2) + q^2/(2c/a^2) = 1\}$ with the semiaxes $\sqrt{2c}/b, \sqrt{2c}/a$. Note that the standard parametrization of the ellipse, $\varphi \mapsto (\sqrt{2c}/b \cos\varphi, \sqrt{2c}/a \sin\varphi)$ is not a trajectory of $\eta(H)$

The receipt gives $I(c) = \frac{1}{2\pi} \int_{M_c} p dq = \frac{1}{2\pi} \int_{\overline{M}_c} \omega = -\frac{c}{ab}$, which up to $-\frac{1}{2\pi}$ is the area of the figure $\overline{M}_c := \{(q,p) \mid q^2/(2c/b^2) + q^2/(2c/a^2) \leqslant 1\}$ bounded by the ellipse. From this we conclude that $H = -abI$ and that the solution of the hamiltonian system

$$\dot{q} = -a^2 p, \dot{p} = b^2 q$$

is given by $H = c, \varphi(t) = \varphi(0) - t\frac{\partial H}{\partial I} = \varphi(0) + tab$ or, in other words, by

$$t \mapsto (\sqrt{2c}/b \cos(t_0 + tab), \sqrt{2c}/a \sin(t_0 + tab)).$$