Algebraic and geometric aspects of modern theory of integrable systems

Lecture 8

1 Hamiltonian reduction and the Arnold-Liouville theorem

Reduction of a hamiltonian system on (M, ω): Let $v = \eta(H), \eta := \omega^{-1}, H \in \mathcal{E}(M)$. Assume that $p : M \to M'$ is surjective submersion such that η is projectable with respect to p and H is constant along the fibers of p. Then v is also projectable with respect to p. Indeed, $p_* v = \eta'(H')$, where $\eta' := p_*\eta, H' \in \mathcal{E}(M')$ is the unique function such that $H = p^* H'$. The hamiltonian system on (M', η') given by the hamiltonian vector field $v' := p_* v$ is called the *reduction* of the initial hamiltonian system with respect to p.

First integrals and reduction: Assume that $g \in \mathcal{E}(M)$ is a first integral of the hamiltonian vector field $v = \eta(H)$, i.e. $vg = 0$. The last can be rewritten as $\eta(H)g = 0$, or, equivalently, $\{H, g\}_\eta = 0$. Consider the foliation $\mathcal{F} := \{g = \mathit{const}\}$ of the level sets of the function g (we assume that $dg \neq 0$ everywhere) and the dual 1-dimensional foliation $\mathcal{K}$ generated by $\eta(g)$. Then H is constant along the leaves of $\mathcal{K}$ (because $\eta(g)H = -\eta(H)g = 0$) and the system can be reduced with respect to the projection $M \to M/\mathcal{K}$ (at least locally, since locally the factor space $M/\mathcal{K}$ is good).

Conclusion: any first integral allows to reduce a bihamiltonian system on $(M, \omega), \dim M = 2n$, to a new hamiltonian system which is defined on a symplectic manifold of dimension $2n - 2$.

More generally: k first integrals $g_1, \ldots, g_k$ in involution (i.e. such that $\{g_i, g_j\} = 0$) allow to reduce the number of independent variables by $2k$.

Even more generally: Assume there exists $S \subset \mathcal{E}(M)$, a Lie subalgebra with respect to $\{,\}_\eta$ consisting of the first integrals of a hamiltonian system. Then it can be reduced to a hamiltonian system on a smaller symplectic manifold. The last is a symplectic leaf of the reduced Poisson structure obtained by the reduction of η with respect to the action of the Lie algebra $\eta(S) \subset \Gamma(TM)$. The dimension of this manifold depends on the structure of the Lie algebra $(S, \{,\}_\eta|_S)$.

The Arnold–Liouville theorem: Let (M, ω) be symplectic, $\dim M = 2n$. Assume a hamiltonian vector field $v(H)$ admits n functionally independent integrals $g_1 = H, g_2, \ldots, g_n$ in involution. Then

1. if the common level sets $M_c := \{x \in M \mid g_i = c_i, i = 1, \ldots, n\}$ of these integrals are compact and connected, they are diffeomorphic to (n-dimensional) tori $\mathbb{T}^n = \{(\varphi_1, \ldots, \varphi_n) \mathrm{mod} 2\pi\}$;

2. the restriction of the initial hamiltonian equation to $\mathbb{T}^n$ gives an almost periodic motion on $\mathbb{T}^n$, i.e. in the "angle coordinates" φ the equation has the form

$$\frac{d\overrightarrow{\varphi}}{dt} = \overrightarrow{a},$$

here $\overrightarrow{a} = (a_1, \dots, a_n)$ is a constant vector depending only on the level;

3. the initial equation can be integrated in "quadratures", i.e. the solutions can be obtained by means of a finite number of algebraic operations and operations of taking integral.

Proof. The functional independence of $g_1, \dots, g_n$ means linear independence of the differentials $dg_1, \dots, dg_n$ at each point of M_c. By the implicit function theorem M_c is a submanifold of M.

Lemma 1 *The vector fields $\eta(g_1), \dots, \eta(g_n)$ are commuting, tangent to M_c and linearly independent at each point of M_c.*

Proof The linear independence follows from that of $dg_1, \dots, dg_n$ and from nondegeneracy of η. The vector fields are tangent to M_c because $\eta(g_i)g_j = \{g_i, g_j\} = 0$. The equality $\eta(\{g_i, g_j\}) = [\eta(g_i), \eta(g_j)]$ shows the commuting property. □

Lemma 2 *Let N be a compact connected n-dimensional manifold which has n linearly independent (at each point) commuting vector fields $v_1, \dots, v_n$. Then N is diffeomorphic to n-dimensional torus.*

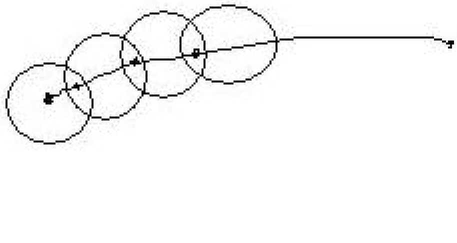
Sketch of proof Let $G_i^t, i = 1, \dots, n$, be the corresponding 1-parametric groups of diffeomorphisms of N. In other words, $\frac{d}{dt}|_{t=0} G_i^t x = v_i|_x$ for any $x \in N$ and $G_i^{t_1+t_2} = G_i^{t_1} \circ G_i^{t_2} = G_i^{t_2} \circ G_i^{t_1}$ for any $t_1, t_2 \in \mathbb{R}$ (and $G_i^0 = \mathrm{Id}, G_i^{-t} = (G_i^t)^{-1}$). Note that G_i^t exist since by compactness of N the vector fields v_i are complete.

Due to the commuting property of vector fields the diffeomorphisms also commute: $G_i^t \circ G_j^{t'} = G_j^{t'} \circ G_i^t$. Thus the n-parametric family of diffeomorphisms $G^{\mathbf{t}} : N \to N, G^{\mathbf{t}} := G_1^{t_1} \cdots G_n^{t_n}, \mathbf{t} := (t_1, \dots, t_n)$, has the property $G^{\mathbf{t}+\mathbf{t}'} = G^{\mathbf{t}} \circ G^{\mathbf{t}'} = G^{\mathbf{t}'} \circ G^{\mathbf{t}}$ (and $G^{\mathbf{0}} = \mathrm{Id}, G^{-\mathbf{t}} = (G^{\mathbf{t}})^{-1}$). In other words, we get an action of the commutative group $\mathbb{R}^n$ on N.

Lemma 3 *This action is transitive, i.e. for any two points $x_0, x_1 \in N$ there exists $\mathbf{t}$ such that $G^{\mathbf{t}} x_0 = x_1$.*

Proof Fix x_0 and consider the map $\psi : \mathbb{R}^n \to N, \psi(\mathbf{t}) := G^{\mathbf{t}} x_0$. This map is a local diffeomorphism: there exists an open set $U, \mathbf{0} \subset U \subset \mathbb{R}^n$ such that $\psi|_U$ is a diffeomorphism onto $V := \psi(U)$. Indeed, the derivative $\psi'(\mathbf{0})$ sends the standard basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$ to $\{v_1|_{x_0}, \dots, v_n|_{x_0}\}$. The last vectors are independent, hence $\psi'(\mathbf{0})$ is nondegenerate and we can use the inverse function theorem.

Now connect x_0 with x_1 by a curve and cover this curve by a finite number of sets similar to V.



Choose a point y_i in each of the pairwise intersections of these sets and put $y_0 := x_0, y_m := x_1$. It is clear that there exist $\mathbf{t}_i$ such that $G^{\mathbf{t}_i} y_i = y_{i+1}$. Finally, put $\mathbf{t} := \mathbf{t}_0 + \cdots + \mathbf{t}_{m-1}$. $\square$

Lemma 4 *The stabilizer $G_{x_0} := \{\mathbf{t} \in \mathbb{R}^n \mid G^{\mathbf{t}} x_0 = x_0\} \subset \mathbb{R}^n$ of the point $x_0 \in N$ with respect to this action is a discrete additive subgroup of $\mathbb{R}^n$, independent of the choice of x_0.*

Proof Given any action of a group G on a set X, one proves immediately that the stabilizers are subgroups and the stabilizers of points lying on one orbit are conjugate. Here N consists of one orbit and the group is commutative. Thus G_{x_0} is a subgroup, the same for any point.

To prove that it is discrete, observe that the set U can not contain any point of G_{x_0} different from $\mathbf{0}$. $\square$

Lemma 5 *For any discrete subgroup $H \subset \mathbb{R}^n$ there exist linearly independent vectors $l_1, \ldots, l_k \in \mathbb{R}^n, k \leqslant n$, such that $H = \{\sum_{i=1}^k z_i l_i \mid z_i \in \mathbb{Z}\}$.*

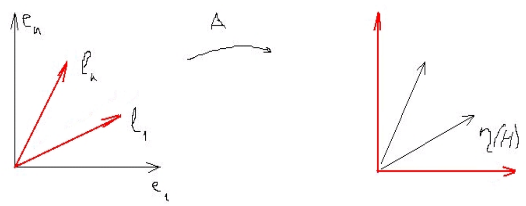
For the proof see the book: V. I. Arnold "Metody matematyczne mechaniki klasycznej", Chapter 49.

Now we are ready to finish the proof of Lemma 2. Any orbit O of a (smooth) action of a Lie group G on a manifold is diffeomorphic to the factor manifold G/G_{x_0}, where $x_0 \in O$ is any element. In our case $O = N$ is diffeomorphic $\mathbb{R}^n/G_{x_0} \cong \mathbb{T}^k \times \mathbb{R}^{n-k} = \{(\varphi_1, \ldots, \varphi_k; y_1, \ldots, y_{n-k})\}, \varphi_i \mathrm{mod} 2\pi$. By compactness of N we conclude that $k = n$ and $N \cong \mathbb{T}^n$. $\square$

So we have proven the first item of the A–L theorem. To show item 2 fix c and observe that the diffeomorphism $\mathbb{T}^n \to N := M_c$ can be included to the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^n \\ \downarrow p & & \downarrow \psi \\ \mathbb{T}^n & \longrightarrow & N \end{array} .$$

Here p is the natural projection and A is the linear isomorphism mapping the vectors $2\pi e_1, \ldots$, $2\pi e_n$, where $e_1, \ldots, e_n$ is the standard base in $\mathbb{R}^n$, to $l_1, \ldots, l_n$.



Obviously, $\eta(H) = v_1 = \psi_*(E_1)$, where E_1 is the constant vector field on $\mathbb{R}^n$ equal to e_1 at $\mathbf{0}$. Thus in the φ-coordinates on N the hamiltonian vector field $\eta(H)$ has the form $\eta(H) = (a_1, \ldots, a_n)$ for some $a_i \in \mathbb{R}$. $\square$

In order to prove item 3 of the A–L theorem we will build special coordinates on M, the "action-angle" coordinates.