Algebraic and geometric aspects of modern theory of integrable systems

Lecture 7

1 Symplectic and Poisson reduction

Digression on linear algebra of skew-symmetric bilinear forms: Let V be a vector space, $\omega \in \bigwedge^2 V^*, W \subset V$ a subspace. We put $W^{\perp\omega} := \{v \in V \mid \omega(v, W) = 0\}$ and $\ker\omega := V^{\perp\omega} = \{v \in V \mid \omega(v, w) = 0\ \forall w \in V\}$. We say that W is *isotropic* (*coisotropic*) if $W \subset W^{\perp\omega}$ (respectively $W \supset W^{\perp\omega}$). In case when ω is nondegenerate, or, in other words, symplectic, we call W *lagrangian*, if it is maximal isotropic (i.e. W is isotropic and for any isotropic $W' \supset W$ we have $W' = W$). Equivalently, W is lagrangian if it is minimal coisotropic.

Examples: Let $e_1, \ldots, e_{2n}$ be a basis of V, $e^1, \ldots, e^{2n}$ be the dual basis of V^*, $\omega = e^1 \wedge e^{n+1} + \cdots e^n \wedge e^{2n}$. Then $W_l := \langle e_1, \ldots, e_l\rangle$ is isotropic for any $l \leqslant n$, $W_l^{\perp\omega} = \langle e_1, \ldots e_n, e_{n+l+1}, \ldots e_{2n}\rangle$ is coisotropic, W_n is lagrangian.

A coisotropic submanifold of a symplectic manifold (M, ω): A submanifold $N \subset M$ such that T_xN is a coisotropic subspace of the sympectic vector space (T_xM, ω_x) for any $x \in M$.

Fact. Let $f_1, \ldots, f_k \in \mathcal{E}(M)$ be such that $N = \{x \in M \mid f_1(x) = 0, \ldots f_k(x) = 0\}$. Then N is coisotropic if and only if $\{f_i, f_j\}|_N \equiv 0, i, j = 1, \ldots, k$.

Proof Let $\eta := \omega^{-1}$, then $(T_xN)^{\perp\omega_x} = \langle \eta(f_1)|_x, \ldots, \eta(f_k)|_x\rangle$. Indeed, if $w \in T_xN$, we have $\omega_x(w, \eta(f_i)|_x) = -\omega_x(\eta(f_i)|_x, w) = -d_xf_i(w) = 0$. So the inclusion $(T_xN)^{\perp\omega_x} \subset T_xN$ is equivalent to the equality $d_xf_j(\eta(f_i)|_x) = 0, i, j = 1, \ldots, k$. On the other hand, $d_xf_j(\eta(f_i)|_x) = (\eta(f_i)f_j)|_x = \{f_i, f_j\}|_x$. $\square$

A coisotropic foliation of a symplectic manifold (M, ω): A foliation $\mathcal{F}$ on M such that each leaf locally is a coisotropic submanifold.

Fact. Let $U \subset M$ be an open set such that $\mathcal{F}$ on U is given by $\{x \in U \mid f_1(x) = c_1, \ldots f_k(x) = c_k\}$ for some $f_1, \ldots, f_k \in \mathcal{E}(U)$. Then N is coisotropic if and only if $\{f_i, f_j\} \equiv 0$ on U for any $i, j = 1, \ldots, k$.

Linear version of symplectic reduction: Let (V, ω) be a symplectic vector space, $W \subset V$ a coisotropic subspace (i.e. $W^{\perp\omega} \subset W$). Put $W' := W/W^{\perp\omega}$ and let $p : W \to W'$ be the natural projection. Then there exists a unique symplectic form ω' on W' such that

$$p^*\omega' = \omega|_W,$$

i.e. $\omega(v,w)=\omega'(pv,pw)$ for any $v,w\in W$. To show this we observe that $W^{\perp\omega}=\ker\omega|_W$, so we can put $\omega'(v+W^{\perp\omega},w+W^{\perp\omega}):=\omega(v,w)$.

Digression on factor manifolds: Let M be a manifold and $\mathcal{K}$ a foliation on M. The relation "$x\sim y\Leftrightarrow$ (x and y belong to the same leaf)" is an equivalence relation on M and we shall denote by $M/\mathcal{K}$ the topological quotient space $M/\sim$. We say that $M/\mathcal{K}$ is *good* if the space $M/\mathcal{K}$ has a structure of a smooth (analytic) manifold whose underlying topology is the quotient one such that the canonical projection $M\to M/\mathcal{K}$ is a smooth (analytic) submersion (recall that a smooth map is a *submersion* if the tangent map i s surjective at each point). If such a smooth (analytic) structure exists it is unique.

Note that for any foliation $\mathcal{K}$ and small enough open sets $U\subset M$ the factor space $U/\mathcal{K}$ is good.

Symplectic reduction on a symplectic manifold (M,ω): Let $N\subset M$ be a coisotropic submanifold. Put $D_x:=(T_xN)^{\perp\omega_x}\subset T_xM$.

FACT. $\mathcal{D}:=\{D_x\}_{x\in N}$ is a integrable distribution on N.

Proof Let $f_1,\ldots,f_k\in\mathcal{E}(U)$ be local functions defining N. Then $D_x=\langle\eta(f_1)|_x,\ldots,\eta(f_k)|_x\rangle$ and $[\eta(f_i),\eta(f_j)]|_x=\eta_x(\{f_i,f_j\}|_x)=0$. □

Put $\mathcal{K}$ for the foliation such that $T\mathcal{K}=\mathcal{D}$ Assume that $N':=N/\mathcal{K}$ is good and put $p:N\to N'$ for the natural projection.

FACT. There exists a unique symplectic form ω' on N' such that

$$p^*\omega'=\omega|_N.$$

Proof Perform the linear symplectic reduction at each point. □

Example: Let $M:=T^*\mathbb{R}^2\cong\mathbb{R}^4$ and let $\omega=dp\wedge dq$ be the canonical form. Let $N:=\{(q,p)\mid H(q,p)=1\}, H(q,p)=q_1^2+q_2^2+(p^1)^2+(p^2)^2$. Then $T\mathcal{K}$ is generated by $\eta(H)=2(q_1\frac{\partial}{\partial p^1}-p^1\frac{\partial}{\partial q_1}+q_2\frac{\partial}{\partial p^2}-p^2\frac{\partial}{\partial q_2})$. This vector field has 3 first integrals: $H, f_1:=(q_1^2+(p^1)^2)-(q_2^2+(p^2)^2), f_2:=2(q_1p^2-q_2p^1)$. Put $f_3:=2(q_1q_2+p^1p^2)$ and consider the map $\varphi:\mathbb{R}^4\to\mathbb{R}^4$ given by $(q,p)\mapsto(H,f_1,f_2,f_3)$. Restricting the map φ to the sphere $N=S^3$, we get the map $\psi:N\to\mathbb{R}^3$. In fact, because of the relation $f_2^2+f_3^2=H^2-f_1^2$ the image of ψ lies in the 2-dimensional sphere $N':=S^2$ in $\mathbb{R}^3$.

Exercise: 1) If $f':=(f_1',f_2',f_3')\in N'$, the preimage $\psi^{-1}(f)$ is a "great circle" on N contained in the plane

$$\begin{aligned}(1+f_1')q_2-f_3'q_1+f_2'p^1&=0\\(1+f_1')p^2-f_2'q_1-f_3'p^1&=0\end{aligned}$$

for $f\neq(0,0,-1)$ and in the plane $\{q_1=0,p^1=0\}$ for $f=(0,0,-1)$. 2) This plane is a complex one dimensional subspace of the space $\mathbb{C}^2\cong\mathbb{R}^4$, where the complex coordinates are given by q_1+ip^1,q_2+ip^2.

The fibration $\psi:S^3\to S^2$ is called the *Hopf fibration*. As a result of the symplectic reduction we get a symplectic structure on $S^2\cong\mathbb{CP}^1$. Analogous construction gives a symplectic structure on $\mathbb{CP}^n$.

A particular case of Poisson reduction (informally): Let $\mathcal{F}$ be a coisotropic foliation on (M, ω) and let $\mathcal{D} := \{D_x\}_{x \in M}, D_x := (T_x\mathcal{F})^{\perp \omega_x}$. Then $\mathcal{D}$ is an integrable distribution, put $\mathcal{K}$ for the foliation such that $T\mathcal{K} = \mathcal{D}$. Assume that $M' := M/\mathcal{K}$ is good.

Now perform the symplectic reduction with respect to each leaf of $\mathcal{F}$. As a result we will get a foliation of M' by a symplectic (immersed) submanifolds. In fact this is a symplectic foliation of some degenerate Poisson structure η' on M'.

Digression on projectability of tensor fields: Let $p : M \to M'$ be a surjective submersion. Put $p_{*,x} : T_xM \to T_{p(x)}M'$ for the tangent map. A vector field $v \in \Gamma(TM)$ is said to be *projectable* with respect to p if there exists a vector field $v' \in \Gamma(TM')$ such that $v'_{p(x)} = p_{*,x}v_x$ for any $x \in M$.

Let $(x^1, \ldots, x^k, y^1, \ldots, y^l)$ be local coordinates on M such that $(x^1, \ldots, x^k)$ are local coordinates on M' and p is given by $p(x, y) = x$. Then v is projectable if and only if $v = u^i(x)\frac{\partial}{\partial x^i} + w^j(x, y)\frac{\partial}{\partial y^j}$ (and if v is so, $v' = u^i(x)\frac{\partial}{\partial x^i}$).

Analogously, one can define the projectability of bivector fields and show that $\eta = \eta^{ij}(x, y)\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + \zeta^{tu}(x, y)\frac{\partial}{\partial x^t} \wedge \frac{\partial}{\partial y^u} + \xi^{rs}(x, y)\frac{\partial}{\partial y^r} \wedge \frac{\partial}{\partial y^s}$ is projectable if and only if $\eta^{ij}(x, y) = \eta^{ij}(x)$ is independent of y.

Poisson reduction formally: Let $p : M \to M'$ be a surjective submersion, $\mathcal{K}$ be the foliation of the fibers of p. Let $\eta \in \Gamma(\bigwedge^2 TM)$ be a nondegenerate Poisson structure and let $\omega := \eta^{-1}$.

FACT. (Liebermann, Weinstein) The following conditions are equivalent: 1) η is projectable with respect to p; 2) the distribution $\mathcal{D}, \mathcal{D} := \{D_x\}_{x \in M}, D_x := (T_x\mathcal{K})^{\perp \omega_x}$, is integrable; 3) the set of functions $S := p^*(\mathcal{E}(M'))$ constant along $\mathcal{K}$ is a Lie subalgebra with respect to $\{,\}_\eta$.

Moreover, if η is projectable, $\eta' := p_*\eta$ is Poisson and the map p is Poisson, i.e. $p^* : (\mathcal{E}(M'), \{,\}_{\eta'}) \to (\mathcal{E}(M), \{,\}_\eta)$ is a homomorphism of Lie algebras.

Proof Locally the leaves of $\mathcal{K}$ are given by $\{x_1 = c_1, \ldots, x_k = c_k\}$ in the (x, y)-coordinates, so $D_x = \langle \eta(x_1)|_x, \ldots, \eta(x_k)|_x \rangle$ (we do not assume $\{x_i, x_j\}_\eta = 0$). $\mathcal{D}$ is integrable if and only if $[\eta(x_i), \eta(x_j)] = \eta(\{x_i, x_j\})$ is a linear combination of $\eta(x_1), \ldots, \eta(x_k)$. Let us show that this last condition is equivalent to condition 3). Indeed, put $f(x, y) := \{x_i, x_j\}_\eta$ and observe, that $\eta(f) = \frac{\partial f}{\partial x^t}\eta(x^t) + \frac{\partial f}{\partial y^u}\eta(y^u)$. Thus $\eta(f)$ is a linear combination of $\eta(x_1), \ldots, \eta(x_k)$ if and only if the function f does not depend on y, i.e. belongs to S. So we have proven 2) $\Longleftrightarrow$ 3).

Since $\eta^{ij}(x, y) = f(x, y)$ (see the previous subsection), we see that η is projectable if and only if f does not depend on y, hence 1) $\Longleftrightarrow$ 3).

Finally, if η is projectable, the Poisson bracket corresponding to η' is the restriction of $\{,\}_\eta$ to S, hence satisfies the JI. $\square$

Dual pairs of foliations (Poisson maps): Note that in the construction above we get a foliation $\mathcal{F}$ such that $T\mathcal{F} = (T\mathcal{K})^{\perp\omega}$. Since taking the skew-orthogonal complement of a subspace twice gives the initial subspace, we also have $T\mathcal{K} = (T\mathcal{F})^{\perp\omega}$. In such a situation we say that the foliations $\mathcal{K}, \mathcal{F}$ (and the natural projections $M \to M/\mathcal{K}, M \to M/\mathcal{F}$) form a *dual pair*. In the particular case discussed in the context of symplectic reduction $\mathcal{F}$ was a coisotropic foliation and $\mathcal{K}$ an isotropic one (since $T\mathcal{K} = (T\mathcal{F})^{\perp\omega} \subset T\mathcal{F}$).