Algebraic and geometric aspects of modern theory of integrable systems

Lectures 5-6

1 Lie–Poisson structures

Definition I: Let $(\mathfrak{g}, [,])$ be a finite-dimensional Lie algebra, $\mathfrak{g}^*$ its dual space (space of linear functionals on $\mathfrak{g}$). Given $f, g \in \mathcal{E}(\mathfrak{g}^*)$ define $\{f,g\}_{\mathfrak{g}}(x) := \langle x, [df|_x, dg|_x]\rangle, x \in \mathfrak{g}^*$. Here we identify $T_x^*\mathfrak{g}^*$ with $\mathfrak{g}$, $\langle,\rangle$ stands for the canonical pairing between vectors and covectors.

Digression: Let M be a manifold, $\{,\} : \mathcal{E}(M) \times \mathcal{E}(M) \to \mathcal{E}(M)$ a bilinear operation being a differentiation with respect to each argument: $\{fg, h\} = f\{g,h\} + g\{f,h\}, \{f, gh\} = \{f,g\}h + \{f,h\}g$. Then it can be shown that there is a tensor $\eta \in \Gamma(\bigotimes^2 TM)$ such that $\{f,g\} = \eta(df, dg)$. Let us show this in the case when $\{,\}$ is skew-symmetric.

Indeed, since $\{f,\cdot\}, \{g,\cdot\}$ are differentiations they are vector fields, say X_f, X_g. Let $f \in \mathcal{E}(M)$ be such that $df|_x = 0$ for some $x \in M$. Then $\langle X_f|_x, dg|_x\rangle = (X_f g)(x) = \{f,g\}(x) = -\{g,f\}(x) = -(X_g f)(x) = -\langle X_g|_x, df|_x\rangle = 0$. Here $g \in \mathcal{E}(M)$ is arbitrary, hence $X_f|_x = 0$. Thus the map $df|_x \to X_f|_x$ depends only on the value of df at x, i.e. is given by a morphism $T^*M \to TM$.

Definition II: Let $(\mathfrak{g}, [,])$ be a finite-dimensional Lie algebra, $e_1, \ldots, e_n \in \mathfrak{g}$ its basis, $x_1 = e_1, \ldots, x_n = e_n$ these vectors regarded as linear functions on $\mathfrak{g}^*$ (in particular $x_1, \ldots, x_n$ are linear coordinates on $\mathfrak{g}^*$). Let $[e_i, e_j] = c_{ij}^k e_k$ (c_{ij}^k are called the *structure constants* corresponding to the basis $e_1, \ldots, e_n$). Put $\eta_{\mathfrak{g}} := c_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$.

Fact. The bivector corresponding to the bracket $\{,\}_{\mathfrak{g}}$ coincides with $\eta_{\mathfrak{g}}$.

Proof Let $\eta = \eta^{ij}(x)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ be the bivector corresponding to $\{,\}_{\mathfrak{g}}$. Take $f := x_i, g := x_j$, then $\{f,g\}(x) = \eta^{ij}(x)$. On the other hand, by Definition I, $\{f,g\}(x) = \langle x, [x_i, x_j]\rangle = c_{ij}^k x_k$. $\square$

Exercise: 1) Let $\eta \in \Gamma(\bigwedge^2 TM)$, in local coordinates $\eta = \eta^{ij}(x)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$. Show that the JI for $\{,\}, \{f,g\} = \eta^{ij}(x)\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$ holds if and only if the expression

$$[\eta,\eta]_S^{ijk} := \sum_{c.p.\, i,j,k} \eta^{ir}(x)\frac{\partial}{\partial x^r}\eta^{jk}(x)$$

vanishes for all $i,j,k \in \{1,\ldots,n\}$. 2) Show that, given $\eta, \zeta \in \Gamma(\bigwedge^2 TM), \eta = \eta^{ij}(x)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \zeta = \zeta^{ij}(x)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, the expression

$$[\eta,\zeta]_S^{ijk} := \frac{1}{2}\sum_{c.p.\, i,j,k} \eta^{ir}(x)\frac{\partial}{\partial x^r}\zeta^{jk}(x) + \zeta^{ir}(x)\frac{\partial}{\partial x^r}\eta^{jk}(x)$$

is a local representation of a trivector on M (called the *Schouten bracket* of η and ζ). 3) If $\eta = v_1 \wedge v_2, \zeta = w_1 \wedge w_2, v_i, w_i \in \Gamma(TM)$, then

$$[\eta, \zeta]_S \sim [v_1, w_1] \wedge v_2 \wedge w_2 + v_1 \wedge [v_2, w_1] \wedge w_2 + v_2 \wedge [v_1, w_2] \wedge w_1 + v_1 \wedge w_1 \wedge [v_2, w_2].$$

Here $\sim$ means equality up to a constant.

Proof of the Jacobi identity for the Lie–Poisson structure: $[\eta_{\mathfrak{g}}, \eta_{\mathfrak{g}}]_S^{ijk} = \sum_{c.p.\, i,j,k} c_{ir}^l x_l c_{jk}^r$. The last expression vanishes for all i, j, k if and only if $\sum_{c.p.\, i,j,k} c_{ir}^l c_{jk}^r = 0$ for all l, i, j, k, which is equivalent to the JI for $[,]$. □

2 Actions of Lie algebras and symplectic foliations of Lie–Poisson structures

An action of a Lie algebra $\mathfrak{g}$ on a manifold: A homomorphism of Lie algebras $\rho : (\mathfrak{g}, [,]) \to (\Gamma(TM), [,])$ (in the target space $[,]$ stands for the commutator of vector fields) is called a *(right) action* of $\mathfrak{g}$ on M.

Orbits of an action $\rho : (\mathfrak{g}, [,]) \to (\Gamma(TM), [,])$: Put $D_x := \{\rho(v)|_x \mid v \in \mathfrak{g}\}, x \in M$.

FACT. Let $\mathfrak{g}$ be finite-dimensional. Then the generalized distribution $\mathcal{D} := \{D_x\}_{x \in M}$ is integrable.

Proof The distribution $\mathcal{D}$ is involutive: $[\rho(v), \rho(w)] = \rho([v, w])$. Thus in the analytic category the proof follows from the generalized Frobenius theorem (note that in fact the same argument works for infinite-dimensional $\mathfrak{g}$ as well). We skip the proof in the smooth case (roughly it consists in integrating the action of the Lie algebra to a local action of the corresponding Lie group). □

The leaves of the corresponding generalized foliation are called the *orbits* of the action ρ. If the Lie algebra $\mathfrak{g}$ is finite-dimensional, the action can be "integrated" to a local action of a Lie group G such that $\mathfrak{g}$ is its Lie algebra. Then the orbits of the Lie algebra action and of the Lie group action coincide.

Linear representations and actions: Let V be a vector space and $A \in \mathrm{End}(V)$ a linear operator. It induces a uniquely defined vector field $\tilde{A}$ on V given by $x \mapsto (x, Ax) : V \to V \times V \cong TV$. If $e_1, \ldots, e_n$ is a basis of V, $x^1, \ldots, x^n$ the dual basis of V^* and $Ae_i = A_{ji} e_j$, we have $\tilde{A} = A_{ji} x^i \frac{\partial}{\partial x^j}$.

Exercise: The map $A \mapsto \tilde{A} : \mathrm{End}(V) \to \Gamma(TV)$ is an action of the Lie algebra $\mathrm{End}(V)$ on V.

Let $L : (\mathfrak{g}, [,]) \to (\mathrm{End}(V), [,])$ be a representation of a Lie algebra $\mathfrak{g}$ in a vector space V. Then the map $\tilde{L} : \mathfrak{g} \to \Gamma(TM), \tilde{L}(x) := \widetilde{L(x)}$ is an action of $\mathfrak{g}$ on the manifold V.

The adjoint and coadjoint actions: Let $(\mathfrak{g}, [,])$ be a Lie algebra. The homomorphism $v \mapsto \mathrm{ad}_v : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, where $\mathrm{ad}_v w := [v, w]$, gives the *adjoint* representation (of $\mathfrak{g}$ on $\mathfrak{g}$). The corresponding action $x \mapsto \widetilde{\mathrm{ad}_x} : \mathfrak{g} \to \Gamma(T\mathfrak{g})$ is also called *adjoint*. The homomorphism $v \mapsto \mathrm{ad}_v^* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$, where ad_v^* is the transposed operator to ad_v, and the corresponding action $x \mapsto \widetilde{\mathrm{ad}_x^*} : \mathfrak{g} \to \Gamma(T\mathfrak{g}^*)$ are called the *coadjoint* representation and action, respectively.

The symplectic leaves of the Lie-Poisson structure $\eta_{\mathfrak{g}}$ on $\mathfrak{g}^*$ coincide with the orbits of the coadjoint action : Indeed, the tangent spaces to symplectic leaves are spanned by the vector fields $\eta_{\mathfrak{g}}(x_i) = c_{ij}^k x_k \frac{\partial}{\partial x_j}, i = 1, \dots, n$. Since $\mathrm{ad}_v e_i = v^j c_{ji}^k e_k$, where $v = v^j e_j$, the matrix A_{ki} of the operator ad_v of the adjoint representation is given by $A_{ki} = v^j c_{ji}^k$ and that of the coadjoint one by $A_{ki} = v^j c_{jk}^i$. Hence the tangent spaces to the orbits of the coadjoint action are spanned by the vector fields $c_{jk}^i x_i \frac{\partial}{\partial x_k}, j = 1, \dots, n$. $\square$

An invariant symmetric bilinear form on $(\mathfrak{g}, [,])$**:** A symmetric bilinear form $(,) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ satisfying the equality $(\mathrm{ad}_x y, z) = -(y, \mathrm{ad}_x z)$ for any $x, y, z \in \mathfrak{g}$.

FACT. Let $(,)$ be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$. Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by means of the map $v \mapsto (v, \cdot)$. Then the adjoint orbits become coadjoint ones under this identification.

Proof Indeed, if $A : \mathfrak{g} \to \mathfrak{g}$ is a linear operator the transposed operator $A^* : \mathfrak{g}^* \to \mathfrak{g}^*$ becomes the adjoint one under this identification: $(A^* y, z) = (y, Az)$ for any $y, z \in \mathfrak{g}$. Thus ad_x^* becomes $-\mathrm{ad}_x$. $\square$

Notations (for the Lie algebras): $\mathfrak{gl}(n, \mathbb{R}) := \{n \times n -$ matrices with real entries$\}, \mathfrak{sl}(n, \mathbb{R}) := \{x \in \mathfrak{gl}(n, \mathbb{R}) \mid \mathrm{Tr}(x) = 0\}, \mathfrak{so}(n, \mathbb{R}) := \{x \in \mathfrak{gl}(n, \mathbb{R}) \mid x = -x^T\}, \mathfrak{sp}(n, \mathbb{R}) := \{x \in \mathfrak{gl}(2n, \mathbb{R}) \mid xJ + Jx^T = 0\}$, here $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, I_n being the identity $n \times n$-matrix. It is easy to see that $x \in \mathfrak{sp}(n, \mathbb{R})$ if and only if $x = \begin{bmatrix} a & b \\ c & -a^T \end{bmatrix}$, here $a, b, c, \in \mathfrak{gl}(n, \mathbb{R}), b = b^T, c = c^T$.

The sets above are Lie algebras with respect to the commutator of matrices.

Notations (for the Lie Groups): $GL(n, \mathbb{R}) := \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid \det X \neq 0\}, SL(n, \mathbb{R}) := \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid \det X = 1\}, SO(n, \mathbb{R}) := \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid XX^T = I_n\}, SP(n, \mathbb{R}) := \{X \in \mathfrak{gl}(2n, \mathbb{R}) \mid XJX^T = J\}$. All these sets are groups with respect to the matrix multiplication. It is easy to see that if $x \in \mathfrak{g}$, where $\mathfrak{g}$ is one of the Lie algebras above, then $\exp(x) \in G$, where G is the corresponding Lie group. Also $\mathfrak{g} = T_I G$.

The Lie algebras from Examples 1-5, below, have an invariant nondegenerate symmetric form $(x, y) = \mathrm{Tr}(xy)$ by means of which we can make an identification $\mathfrak{g} \cong \mathfrak{g}^*$. The coadjoint orbits are identified with the adjoint ones, which can be described as the orbits of the corresponding Lie group with respect to the conjugation of matrices: $\{XxX^{-1} \mid X \in G\}, x \in \mathfrak{g}$.

Example 1: $\mathfrak{g} := \mathfrak{gl}(n, \mathbb{R}), \mathcal{C}_{\eta_{\mathfrak{g}}}(\mathfrak{g}) = \mathcal{F}un(\mathrm{Tr}(x), \mathrm{Tr}(x^2), \dots, \mathrm{Tr}(x^n))$.

Example 2: $\mathfrak{g} := \mathfrak{sl}(n, \mathbb{R}), \mathcal{C}_{\eta_{\mathfrak{g}}}(\mathfrak{g}) = \mathcal{F}un(\mathrm{Tr}(x^2), \dots, \mathrm{Tr}(x^n))$. In particular, for $n = 2$ we have a basis $e_1 := e_{11} - e_{22}, e_2 := e_{12}, e_2 := e_{21}$ and the commutation relations $[e_1, e_2] = 2e_2, [e_1, e_3] = -2e_3, [e_2, e_3] = e_1$. Hence $\eta_{\mathfrak{g}} = x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + 2x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} - 2x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3}$. The Casimir function $\mathrm{Tr}(x^2)$ reads as $x_1^2/2 + 2x_2x_3$. The symplectic leaves are the 1-sheet hyperboloids, sheets of 2-sheet hyperboloids, two sheets of the cone (without zero) and the point 0.

Example 3: $\mathfrak{g} := \mathfrak{so}(2n, \mathbb{R}), \mathcal{C}_{\eta_{\mathfrak{g}}}(\mathfrak{g}) = \mathcal{F}un(\mathrm{Tr}(x^2), \mathrm{Tr}(x^4) \dots, \mathrm{Tr}(x^{2n-2}), \mathrm{Pf}(x))$.

Example 4: $\mathfrak{g} := \mathfrak{so}(2n+1, \mathbb{R}), \mathcal{C}_{\eta_{\mathfrak{g}}}(\mathfrak{g}) = \mathcal{F}un(\mathrm{Tr}(x^2), \mathrm{Tr}(x^4) \dots, \mathrm{Tr}(x^{2n}))$.

Example 5: $\mathfrak{g} := \mathfrak{sp}(n, \mathbb{R}), \mathcal{C}_{\eta_{\mathfrak{g}}}(\mathfrak{g}) = \mathcal{F}un(\mathrm{Tr}(x^2), \mathrm{Tr}(x^4) \dots, \mathrm{Tr}(x^{2n}))$.

Example 6 (the Heisenberg algebra): $\mathfrak{g} := \mathbb{R}^3, [e_1, e_2] = e_3$, here e_1, e_2, e_3 is the standard basis of $\mathbb{R}^3$. We have $\eta_{\mathfrak{g}} = x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}, \mathcal{C}_{\eta_{\mathfrak{g}}}(\mathfrak{g}) = \mathcal{F}un(x_3)$, so the coadjoint orbits consist of the planes $\{x_3 = c\}, c \neq 0$ and of the points of the plane $\{x_3 = 0\}$. The adjoint orbits are generated by the vector fields $c_{ij}^k x^i \frac{\partial}{\partial x^k}$, where $\{x^i\}$ is the basis dual to $\{x_i\}$, i. e. by $x^1 \frac{\partial}{\partial x^3}, x^2 \frac{\partial}{\partial x^3}$, so they are the lines parallel to the x_3-axis and the points of this axis.