Algebraic and geometric aspects of modern theory of integrable systems

Lecture 4

1 Symplectic and nondegenerate Poisson manifolds

A symplectic form on M: A differential 2-form (2-form for short) ω on M such that

1. ω is nondegenerate, i.e. $\omega^\flat$ is an isomorphism of bundles, or, equivalently, $\omega_{ij}(x)$ is a nondegenerate matrix for any x in some (consequently in any) local coordinate system;

2. $d\omega = 0$.

A nondegenerate Poisson structure on M: A bivector field (bivector for short) η such that $\eta^\sharp : T^*M \to TM$ is inverse to $\omega^\flat : TM \to T^*M$ for some symplectic form ω.

The Poisson bracket on $\mathcal{E}(M)$: Given a bivector field $\eta : T^*M \to TM$ (not necessarily Poisson), put $\{f, g\} := \eta(df)g, f, g \in \mathcal{E}(M)$. (From now on we will often skip $\sharp$ and $\flat$ indices.) Then $\{,\}$ is a bilinear skew-symmetric operation on $\mathcal{E}(M)$. We say that $\eta(f) := \eta(df)$ is a *hamiltonian* vector field corresponding to the function f.

Fact. Let η be a nondegenerate bivector. Then it is Poisson if and only if $\{,\}$ satisfies the Jacobi identity, $\sum_{c.p.\, f,g,h}\{\{f, g\}, h\} = 0$. □

Proof Put $\omega := \eta^{-1}$, i.e. $\omega(\eta(\alpha), v) = \alpha(v)$ for any vector field v and 1-form α. Then $\eta(f)\omega(\eta(g), \eta(h)) = \eta(f)(dg(\eta(h))) = \eta(f)(\eta(h)g) = \eta(f)\{h, g\} = \{f, \{h, g\}\} = -\{f, \{g, h\}\}$ and $\omega([\eta(f), \eta(g)], \eta(h)) = -\omega(\eta(h), [\eta(f), \eta(g)]) = -dh([\eta(f), \eta(g)]) = -[\eta(f), \eta(g)]h = -\eta(f)\eta(g)h + \eta(g)\eta(f)h = -\eta(f)\{g, h\} + \eta(g)\{f, h\} = -\{f, \{g, h\}\} + \{g, \{f, h\}\}$. Thus $d\omega(\eta(f), \eta(g), \eta(h)) = \sum_{c.p.\, f,g,h} \eta(f)\omega(\eta(g), \eta(h)) - \omega([\eta(f), \eta(g)], \eta(h)) = -\sum_{c.p.\, f,g,h}\{g, \{f, h\}\}$. So, if $d\omega = 0$, then $\{,\}$ satisfies the Jacobi identity.

Conversely, if the JI holds, $d\omega$ vanishes on all hamiltonian vector fields. To finish the proof it remains to note that the hamiltonian vector fields span T_xM at any $x \in M$. Indeed, it is enough to take $\eta(x^i)$, where (x^i) are local coordinates.

Example: the canonical symplectic structure on the cotangent bundle T^*Q: Let $\pi_Q : T^*Q \to Q$ be a cotangent bundle to a manifold Q. There is a canonical differential 1-form $\lambda \in \Gamma(T^*M), M := T^*Q$ determined uniquely by the following condition: for any $\alpha \in \Gamma(T^*Q)$, the following equality holds $\alpha^*\lambda = \alpha$, here α in the LHS is regarded as a map $\alpha : Q \to T^*Q$. We call λ the *Liouville* 1-form. If $(U, q^1, \ldots, q^n)$ is a local chart on Q, the 1-forms $dq^1, \ldots, dq^n$ form a basis of the vector space $T^*_xQ, x \in U$, and define the chart $(\pi_Q^{-1}(U), q^1, \ldots, q^n, p_1, \ldots, p_n)$. In these coordinates $\lambda = p_i dq^i$. Indeed, $\alpha : (q^1,\dots,q^n) \mapsto (q^1,\dots,q^n,\alpha_1(q),\dots,\alpha_n(q))$, where $\alpha = \alpha_i(q)dq^i$. Thus $\alpha^*\lambda = \alpha_i(q)dq^i = \alpha$.

The canonical symplectic form ω on M is given by $\omega := d\lambda$, or, locally, $\omega = dp_i \wedge dq^i$.

Hamiltonian differential equation on a symplectic manifold (M,ω)**:** The ODE related to a hamiltonian vector field $\eta(f), f \in \mathcal{E}(M)$, here $\eta = \omega^{-1}$. In the context of the example above (in the canonical coordinates (q,p)): $\eta = -\frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}, \eta(H) = \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i}$, the corresponding equations read:

$$\dot{q}^i = \frac{\partial H(q,p)}{\partial q^i}, \dot{p}_i = -\frac{\partial H(q,p)}{\partial p_i}.$$

2 Poisson structures, their characteristic distributions, symplectic leaves and Casimir functions

A Poisson structure on M**:** A bivector $\eta : T^*M \to TM$ (not necessarily nondegenerate) such that the corresponding bracket $\{,\}$ on $\mathcal{E}(M)$ satisfies the Jacobi identity (JI for short).

Digression on Lie algebras: A *Lie algebra* is a vector space $\mathfrak{g}$ endowed with a bilinear skew-symmetric operation $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the JI:

1. $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0\ \forall x,y,z \in V$, or, equivalently,
2. $\mathrm{ad}_x[y,z] = [\mathrm{ad}_x y, z] + [y, \mathrm{ad}_x z]\ \forall x,y,z \in V$, where $\mathrm{ad}_x y := [x,y]$, or, equivalently,
3. $\mathrm{ad}_{[x,y]} = [\mathrm{ad}_x, \mathrm{ad}_y]\ \forall x,y \in V$, where the bracket in the RHS denotes the commutator of the operators.

The second condition means that ad_x is a differentiation of the bracket $[,]$. The third one has the following interpretation. A pair $(V,[,])$, where V is a vector space and $[,] : V \times V \to V$ is a bilinear operation, is called an *algebra*. Given algebras $(V_1,[,]_1)$ and $(V_2,[,]_2)$, we say that a linear map $L : V_1 \to V_2$ is a *homomorphism* of algebras, if $L[x,y]_1 = [Lx, Ly]_2\ \forall x,y \in V_1$.

So the third condition means that the map $x \mapsto \mathrm{ad}_x : V \to \mathrm{End}(V)$ a *homomorphism* of algebras $(V,[,])$ and $(\mathrm{End}(V),[,])$. Note that the last algebra is in fact a Lie algebra. A homomorphism of Lie algebras $(\mathfrak{g},[,]) \to (\mathrm{End}(V),[,])$ is called a *representation* of the Lie algebra $(\mathfrak{g},[,])$ in the vector space V (so $x \mapsto \mathrm{ad}_x$ is a representation of $\mathfrak{g}$ in $\mathfrak{g}$).

Consider the Lie algebra $(\mathcal{E}(M),\{,\})$ on a Poisson manifold. The corresponding ad_f-operator, $f \in \mathcal{E}(M)$, coincides with $\eta(f) : \mathcal{E}(M) \to \mathcal{E}(M)$.

The characteristic (generalized) distribution of a Poisson structure $\eta : T^*M \to TM$**:** $\mathcal{D}_\eta := \mathrm{im}\,\eta$ (locally generated by the hamiltonian vector fields $\eta(x^1),\dots,\eta(x^n)$, where $(x^1,\dots,x^n)$ are some local coordinates).

By the third condition above the map $f \mapsto \eta(f), (\mathcal{E}(M),\{,\}) \to (\Gamma(TM),[,])$ is a homomorphism of Lie algebras, here $[,]$ is the commutator of vector fields. This implies involutivity of $\mathcal{D}_\eta$: $[\eta(x^i),\eta(x^j)] = \eta(\{x^i,x^j\}) = \eta(\eta^{ij}(x))$, where $\eta = \eta^{ij}(x)\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$. On the other hand, $\eta(f) = \eta^{ij}(x)\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j} = \frac{\partial f}{\partial x^k}\eta(x^k)$ for any f. In particular, $[\eta(x^i),\eta(x^j)]$ is a linear combination (with smooth coefficients) of $\eta(x^1),\dots,\eta(x^n)$.

Theorem: The characteristic distribution $\mathcal{D}_\eta$ is integrable (we call the corresponding foliation *characteristic* or *symplectic*).

Proof In analytic category this follows from the involutivity of $\mathcal{D}$ by the generalized Frobenius theorem. In the smooth case this is also true, but the proof is more complicated, so we skip it. $\square$

Digression on linear algebra of bivectors: Let V be a vector space and e a bivector on V. Then e can be treated as: 1) an element $e \in \bigwedge^2 V$; 2) a linear skew-symmetric map $e^\sharp : V^* \to V$; 3) a bilinear form $\tilde{e}$ on V^*.

FACT. Let $W := \operatorname{im} e^\sharp \subset V$. Then there exists a correctly defined bivector $e|_W \in \bigwedge^2 W$, called the *restriction* of e to W. Moreover, the restriction $e|_W$ is nondegenerate, i.e. $e|_W^\sharp : W^* \to W$ is an isomorphism.

Proof I. A theorem from linear algebra says that there exists a basis $v_1, \ldots, v_n$ of V such that $e = v_1 \wedge v_2 + \cdots + v_{2k-1} \wedge v_{2k}$ (the number $2k$ is equal to $\dim W$ and is called the *rank* of e). It is easy to see that $v_1, \ldots, v_{2k}$ span W. $\square$

Proof II. e is skew-symmetric, i.e. $(e^\sharp)^* = -e^\sharp$. This implies $\ker e^\sharp = (\operatorname{im} e^\sharp)^\perp$, where $(\cdot)^\perp$ stands for the annihilator of $(\cdot)$. So the natural isomorphism $\hat{e} : V^*/\ker e^\sharp \to \operatorname{im} e^\sharp = W$ induced by $e^\sharp$ can regarded as a map from $W^* \cong V^*/(W^\perp)$ to $W \subset V$. The map $\hat{e}$ being skew-symmetric induces the element of $\bigwedge^2 W$, which we denote by $e|_W$. $\square$

Proof III. Let ω be a skew-symmetric bilinear form on a vector space L. Put $\ker \omega := \{x \in L \mid \omega(x, y) = 0 \ \forall y \in L\}$. The form is called *nondegenerate* if $\ker \omega = \{0\}$.

Any ω induces a nondegenerate skew-symmetric bilinear form on the vector space $L/\ker \omega$.

Treating e as a skew-symmetric bilinear form $\tilde{e}$ on V^* we have $\ker \tilde{e} = \ker e^\sharp$. The restriction $e|_W$ treated as a skew-symmetric bilinear form on $W^* \cong V^*/\ker \tilde{e}$ is the above mentioned nondegenerate form induced from $\tilde{e}$. $\square$

Symplectic leaves of a Poisson structure η on M: The leaves of the characteristic foliation $\mathcal{D}_\eta$. Since $D_{\eta,x} = \operatorname{im} \eta_x^\sharp$ for any $x \in M$, the bivector η admits a restriction $\eta|_S$ to any symplectic leaf $S \subset M$, which is a nondegenerate bivector on S. Moreover, since any hamiltonian vector field $\eta(f)$ is tangent to S at points of S, the value $\{f, g\}(x) = (\eta(f)g)(x), x \in S$, depends only of $g|_S$ and by the skew-symmetry the same is true with respect to f. In other words, $\{f|_S, g|_S\}_{\eta|_S} = (\{f, g\}_\eta)|_S$ for any $f, g \in \mathcal{E}(M)$ and the operation $\{,\}_{\eta|_S}$ satisfies the JI, hence $\eta|_S$ is a nondegenerate Poisson structure on S. This explains the term "symplectic leaf" ($(\eta|_S)^{-1}$ is a symplectic form).

Example 1: Let $M := \mathbb{R}^2, \eta = x^1 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}$. On the open set $U := \{x^1 \neq 0\}$ the form $(\eta|_U)^{-1} = -(1/x^1) dx^1 \wedge dx^2$ is symplectic. Thus the JI holds for $\{,\}_\eta$ on U and by continuity it holds also on the whole M. The symplectic leaves are U and all the points on the line $\{x^1 = 0\}$.

Example 2: Let $M := \mathbb{R}^3, \eta = \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}$. On each plane $P_c := \{x^3 = c\}$ the form $(\eta|_{P_c})^{-1} = -dx^1 \wedge dx^2$ is symplectic. The JI holds for $\{,\}_\eta$ on P_c for any $c \in \mathbb{R}$. Since P_c sweep the whole space M as c runs through $\mathbb{R}$, the JI holds for $\{,\}_\eta$ globally. The symplectic leaves are the planes P_c.

Example 3: Let $M := \mathbb{R}^3, \eta = x^1 \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} + x^2 \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}$ (we will prove that this is a Poisson bivector later). The symplectic leaves are ...

Example 4: Let $M = \mathbb{T}^2 \times \mathbb{R}$, let y be a coordinate on the second component. Put $\eta = \tilde{v}_{a,b} \wedge \frac{\partial}{\partial y}$, where $\tilde{v}_{a,b}$ is the generator of winding line. η is Poisson because locally it looks like the bivector from Example 2. If b/a is irrational, the symplectic leaves (which are two-dimensional) are dense in M.

Casimir functions of a Poisson structure η on M: Let $U \subset M$ be an open set. We say that $f \in \mathcal{E}(U)$ is a *Casimir function* if $\eta(f) \equiv 0$ on U. In particular, since $\{f, g\} = \eta(f)g$ on U

the Casimir functions constitute the centre of the Lie algebra $(\mathcal{E}(U), \{,\}|_U)$. The space of Casimir functions over U will be denoted by $\mathcal{C}(U)$.

FACT. The Casimir functions are constant on the leaves of the symplectic foliation.

Proof We have $\eta(f)g = -\eta(g)f = 0$ for any $f \in \mathcal{C}(U), g \in \mathcal{E}(U)$. So, since $\eta(g)$ span the characteristic distribution, f is constant along its leaves. □

Example 1': $\mathcal{C}(M) = \mathbb{R}$, the space of constant functions.

Example 2': $\mathcal{C}(M) = \mathcal{F}un(x^3)$, the space of functions functionally generated by x^3.

Example 3': $\mathcal{C}(M) = \mathcal{F}un((x^1)^2 + (x^2)^2 + (x^3)^2)$. Hence the symplectic leaves are the concentric spheres and the point $\{(0,0,0)\}$.

Example 4': If b/a is irrational $\mathcal{C}(M) = \mathbb{R}$. However, for sufficiently small U the space $\mathcal{C}(U)$ will be functionally generated by one nonconstant function. So "local Casimirs" are not obtained as the restriction of the "global Casimirs".