Algebraic and geometric aspects of modern theory of integrable systems

Lecture 3

1 Ordinary differential equations on manifolds

Ordinary differential equation:

$$\frac{dc}{dt}(x) = v(x), (\text{or } \dot{x} = v(x) \text{ for short})$$

here $v \in \Gamma(TM)$ is given, c is unknown. A *solution* of this equation (or a *trajectory* of v) with an initial condition $x_0 \in M$ is a curve $c : \mathbb{R} \to M$ such that $c(0) = x_0$ and the vector $v(x)$ is tangent to c at any $x \in c(\mathbb{R})$.

A solution always exists *locally* and is unique: in local coordinates $(\psi^1, \ldots, \psi^n)$ we have $v = v^i(x)\frac{\partial}{\partial \psi^i}$ and the initial equation is equivalent to the system of ODE

$$\frac{dc^i(t)}{dt} = \psi^i(c^1(t), \ldots, c^n(t)), i = 1, \ldots, n$$

with the initial condition $c^i(0) = x_0^i, i = 1, \ldots, n$, and we can use the corresponding existence-uniqueness theorem.

Globally, if $\operatorname{supp} v := \overline{\{x \in M \mid v(x) \neq 0\}}$ is compact (eg. M is compact itself) one can extend any local solution to a global (in time and space) solution. [1]

Example 1: "nonextendability in time": $M :=]0, 1[, \dot{x} = 1$.

Example 2: "nonextendability in space": $M := \mathbb{R}, \dot{x} = x^2$.

Example 3: "Winding line on a torus": $M := \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, the vector field $v_{a,b} := a\frac{\partial}{\partial x^1} + b\frac{\partial}{\partial x^2}$, where $a, b \in]0, \infty[$ are fixed, can be projected onto the vector field $\tilde{v}_{a,b}$ on $\mathbb{T}^2$. Its trajectories are the projections $t \to P(x^1 + at, x^2 + bt)$ of the lines $t \to (x^1 + at, x^2 + bt)$.

Rational case: b/a is a rational number, $b = m\lambda, a = n\lambda$ for some $\lambda \in \mathbb{R}$. Then for $t := 1/\lambda$ we have $(x^1 + at, x^2 + bt) = (x^1 + m, x^2 + n)$ and $P(x^1 + at, x^2 + bt) = P(x^1, x^2)$ (the trajectory is closed, i.e. periodic).

Irrational case: b/a is an irrational number (any trajectory is dense in M).

[1] see V. I. Arnold, *Równania różniczkowe zwyczajne*, PWN, 1975, Theorem 1,§35.

2 Submanifolds, foliations and distributions

A submanifold S of M of codimension r: A subset $N \subset M$ such that there exists an atlas $\mathcal{A} := \{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ on M with $N \cap U_\alpha = \{x \in U_\alpha \mid \psi^1(x) = 0, \ldots \psi^r(x) = 0\}$ for those $\alpha \in A$ for which $N \cap U_\alpha \neq \emptyset$.

Smooth maps and submanifolds: A smooth map $F : M_1 \to M_2$ is called an *immersion* if $T_m F : T_m M_1 \to T_{F(m)} M_2$ is injective for any $m \in M_1$. The image of an injective immersion is called an *immersed submanifold*. An injective immersion F is an *embedding* if F is a homeomorphism onto $F(M_1)$, where $F(M_1)$ is endowed with the topology induced from M_2.

Remarks: 1. The image $N := F(M_1)$ of an embedding is a submanifold in M_2 and, vice versa, given a submanifold $N \subset M$, the inclusion $N \hookrightarrow M$ is an embedding. 2. If $N \subset M$ is an immersed submanifold, then for any $x \in N$ there exists an open neighbourhood U of x in M such that the connected component of $N \cap U$ containing x is a submanifold in U. Vice versa, ...

Example of an immersed submanifold, which is not a submanifold: "The irrational torus winding" $\mathbb{R} \to \mathbb{T}^2$.

A foliation $\mathcal{F}$ of codimension r on M : A collection $\mathcal{F} = \{F_\beta\}_{\beta \in B}$ of path-connected sets on M such that there exists an atlas $\mathcal{A} := \{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ on M with the following properties:

1. $M = \bigcup_{\beta \in B} F_\beta$;
2. $F_\beta \cap F_\gamma = \emptyset$ for any $\beta, \gamma, \beta \neq \gamma$;
3. for any $\alpha \in A$ and any $(c_1, \ldots, c_r) \in \psi_\alpha(U_\alpha)$ the set $\{x \in U_\alpha \mid \psi^1(x) = c_1, \ldots \psi^r(x) = c_r\}$ coincides with one of the path-connected components of the set $U_\alpha \cap \mathcal{F}_\beta$ if it is nonempty.

By the remark above the sets F_β are immersed submanifolds.

A distribution $\mathcal{D}$ on M of codimension r: A subbundle of the tangent bundle TM with the r-codimensional fiber, or in other words a collection of subspaces $D_x \subset T_x M$ smoothly (analytically) depending on $x \in M$. Such a distribution is locally spanned by $n - r$ linearly independent (at each point) vector fields.

Example: the distribution tangent to a foliation: $\mathcal{D} = T\mathcal{F} := \{v \in TM \mid v \text{ is tangent to } \mathcal{F}\}$.

Integrable distribution: A distribution which is tangent to some foliation.

Involutive distribution: A distribution $\mathcal{D}$ such that for any two vector fields $X, Y \in \Gamma(TM)$ which are tangent to $\mathcal{D}$ (i.e. $X(x), Y(x) \in D_x$ for any $x \in M$) their commutator $[X, Y]$ is also tangent to $\mathcal{D}$ (equivalently, locally there exist $v_1, \ldots, v_m, v_i \in \Gamma(TM)$, and functions f_{ij}^k such that $\mathrm{Span}\{v_1, \ldots, v_m\} = \mathcal{D}$ and $[v_i, v_j] = f_{ij}^k v_k$; *Exercise: prove the equivalence*).

The Frobenius theorem: A distribution $\mathcal{D}$ is integrable if and only if it is involutive.

Example of nonintegrable distribution: $X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, Y = \frac{\partial}{\partial y}$.

A generalized distribution $\mathcal{D}$ on M of codimension r: A collection of subspaces $D_x \subset T_x M$ locally spanned by $n - r$ vector fields linearly independent at least at one point (but not necessarily linearly independent at other points).

A generalized foliation $\mathcal{F}$ on M: ...

Example of a generalized foliation which is not a foliation: The trajectories of a vector field $x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}$.

The generalized Frobenius theorem (Nagano 1966): An analytic generalized distribution $\mathcal{D}$ is integrable if and only if it is *involutive*, i.e. for any two vector fields $X, Y \in \Gamma(TM)$ which are tangent to $\mathcal{D}$ (i.e. $X(x), Y(x) \in D_x$ for any $x \in M$) their commutator $[X, Y]$ is also tangent to $\mathcal{D}$.

An example of smooth involutive nonintegrable distribution: Let $\varphi(x^1)$ be a smooth function on $\mathbb{R}$ such that $\varphi(x^1) \equiv 0$ for $x^1 \leqslant 0$ and $\varphi(x^1) > 0$ for $x^1 > 0$. Take $X = \frac{\partial}{\partial x^1}, Y = \varphi\frac{\partial}{\partial x^2}$ on $\mathbb{R}^2$. Then $[X, Y] := \frac{\partial \varphi}{\partial x^1}\frac{\partial}{\partial x^2}$ can be expressed as a linear combination of X, Y. But it is nonintegrable: look at its "leaves".