Algebraic and geometric aspects of modern theory of integrable systems

Lecture 2

2. Preliminaries on manifolds

A chart on a topological space M: A pair (U, ψ), here $U \subset M$ is an open set, $\psi : U \to \mathbb{R}^n$ is a homeomorphism onto its image. Two charts $(U_1, \psi_1), (U_2, \psi_2)$ are *compatible* if $\psi_1 \circ \psi_2^{-1}|_{\mathrm{im}(U_1 \cap U_2)} : \mathrm{im}(U_1 \cap U_2) \to \mathbb{R}^n$ is smooth (analytical) mapping. The components of the vector $\psi = (\psi_1, \dots, \psi_n)$ are called *local coordinates* on M.

An atlas on a topological space M: A collection of pairwise compatible charts $\mathcal{A} := \{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ such that $M = \bigcup_{\alpha \in A} U_\alpha$. Two atlases are *equivalent or compatible* if ...

A manifold: A topological space endowed with a class of equivalent atlases.

Example: The sphere S^2 with two stereographic projections (from the north and south poles).

A vector bundle $E \to M$ over a manifold M: A surjective map $\pi : E \to M$, here E is a topological space, such that here is a structure of a vector space on each fiber $E_x := \pi^{-1}(x), x \in M$, and there is an atlas $\mathcal{A} := \{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ on M and homeomorphisms $\Psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^m$ with the properties:

1. the following diagram is commutative

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\Psi_\alpha} & U_\alpha \times \mathbb{R}^m \\ \downarrow \pi & & \downarrow \pi_1 \\ U_\alpha & = & U_\alpha \end{array} ;$$

2. the map $\widetilde{\Psi}_{\alpha,x} := \Psi_\alpha|_{E_x}$ is a linear isomorphism of the vector spaces E_x and $\mathbb{R}^m$;

3. the collection $\{(\pi^{-1}(U_\alpha), \Psi_\alpha)\}_{\alpha \in A}$ is an atlas on E, in particular $\Psi_\alpha \circ \Psi_\beta^{-1}(x, y) = (x, \widetilde{\Psi}_{\alpha,x} \circ \widetilde{\Psi}_{\beta,x}^{-1}(y)), x \in U_\alpha \cap U_\beta, y \in \mathbb{R}^m$, and the functions $\widetilde{\Psi}_{\alpha\beta,x} := \widetilde{\Psi}_{\alpha,x} \circ \widetilde{\Psi}_{\beta,x}^{-1}$ are linear isomorphisms of $\mathbb{R}^m$ which smoothly depend on $x \in M$.

The functions $\widetilde{\Psi}_{\alpha\beta,x}$ are called *transition functions* of the vector bundle. Given the base M and the collection of transition functions, one can reconstruct the initial vector bundle (up to an isomorphism).

A section of a vector bundle $E \to M$: A mapping $s : M \to E$ such that $\pi(s(x)) = x$ for any $x \in M$. The space of sections will be denoted by $\Gamma(E)$.

Example 1, the tangent bundle $TM \xrightarrow{\tau_M} M$: Let M be a manifold with an atlas $\mathcal{A} := \{(U_\alpha, \psi_\alpha)\}_{\alpha\in A}$. Put $\widetilde{\Psi}_{\alpha\beta,x} := \frac{\partial\psi_{\alpha\beta}(\varphi_\beta(x))}{\partial\varphi_\beta}$, here $\psi_{\alpha\beta} := \psi_\alpha \circ \psi_\beta^{-1} : \mathbb{R}^n \to \mathbb{R}^n$. Below we give an explicit description of TM.

A tangent vector at x to M: A *curve* in M is a mapping $c : \mathbb{R} \to M$. Two curves c_1, c_2 such that $c_1(0) = c_2(0) = x$ are *equivalent* at x if the derivatives of the functions $f(c_1(t))$ and $f(c_2(t))$ coincide at 0 for any $f \in \mathcal{E}(M)$ ($\mathcal{E}(M)$ is $C^\infty(M)$ or the space of analytic functions on M depending on the category). Note that c_1, c_2 are equivalent at x if and only if $\frac{d}{dt}|_{t=0}(\psi^i \circ c_1)(t) = \frac{d}{dt}|_{t=0}(\psi^i \circ c_2)(t), i = 1, \ldots, n$, for some (consequently for any) chart (U, ψ) with $x \in U$.

A class $v = [c]_x$ of equivalence of curves at x is called a *tangent vector* at x. We say that v is *tangent* to c (and to any other representative of the class) at x. A tangent vector in local coordinates $(\psi^1, \ldots, \psi^n)$ is represented by the n-tuple $(\frac{d}{dt}|_{t=0}(\psi^1 \circ c)(t), \ldots, \frac{d}{dt}|_{t=0}(\psi^n \circ c)(t))$, here c is any representative of the class. Since we can add such n-tuples and multiply them by scalars, the set of tangent vectors inherits a structure of vector space (which is independent of the choice of local coordinates). Given two local coordinate systems ψ_α, ψ_β the corresponding n-tuples are related by

$$\frac{d}{dt}|_{t=0}(\psi_\alpha^i \circ c)(t) = \frac{\partial\psi^i_{\alpha\beta}(\varphi_\beta(x))}{\partial\varphi^j_\beta} \frac{d}{dt}|_{t=0}(\psi^j_\beta \circ c)(t).$$

Tangent vectors as differentiations: A *differentiation* of the ring $\mathcal{E}(M)$ at x is a linear mapping $l : \mathcal{E}(M) \to \mathbb{R}$ such that $l(fg) = l(f)g(x) + f(x)l(g), f, g \in \mathcal{E}(M)$. Given a tangent vector v at x which is represented by a curve c, we construct a differentiation $\tilde{v}$ by $\tilde{v}(f) := \frac{d}{dt}|_{t=0}(f \circ c)(t)$. It does not depend on the choice of representative.

Let $\psi = (\psi^1, \ldots, \psi^n) : U \to \mathbb{R}^n$ be local coordinates on M such that $\psi(x) = 0$. Then $c := \psi^{-1}(L^i)$, where L^i is the i-th coordinate line in $\mathbb{R}^n$, gives (a local) curve with $c(0) = x$. The corresponding vector is denoted $\frac{\partial}{\partial\psi^i}$. The vectors (differentiations) $\frac{\partial}{\partial\psi^i}, i = 1, \ldots, n$, form a basis of the vector space T_xM.

A vector field on M: A section of the tangent bundle TM, i.e. a tangent vector $v(x) \in T_xM$ (smoothly, analytically) depending on $x \in M$. In a local chart (U, ψ) can be expressed as $v(x) = v^i(x)\frac{\partial}{\partial\psi^i}$, here $v^i(x)$ are functions.

Any vector field v is a differentiation of the ring $\mathcal{E}(M)$, i.e. a linear endomorphism of $\mathcal{E}(M)$ such that $v(fg) = v(f)g + fv(g), f, g \in \mathcal{E}(M)$. In local coordinates $(vf)(x) = v^i(x)\frac{\partial f}{\partial\psi^i}(x)$.

The space $\Gamma(TM)$ of vector fields is a vector field over $\mathbb{R}$ and a module over the ring $\mathcal{E}(M)$.

The commutator of vector fields on M: Given two differentiations v_1, v_2 of the ring $\mathcal{E}(M)$, the commutator $[v_1, v_2] := v_1v_2 - v_2v_1$ is again a differentiation: $v_1v_2(fg) = v_1((v_2f)g + f(v_2)g) = (v_1v_2f)g + (v_2f)(v_1g) + (v_1f)(v_2g) + f(v_1v_2g)$, so $[v_1, v_2](fg) = ([v_1, v_2]f)g - f([v_1, v_2]g)$. In local coordinates $[v_1, v_2]^i(x) = v_1^j(x)\frac{\partial v_2^i(x)}{\partial\psi^j} - v_2^j(x)\frac{\partial v_1^i(x)}{\partial\psi^j}$.

A bivector field on M: A section η of the second exterior power of the tangent bundle $\bigwedge^2 TM$. Locally $\eta = \eta^{ij}(x)\frac{\partial}{\partial\psi^i} \wedge \frac{\partial}{\partial\psi^j}$.

Example 2, the cotangent bundle $T^*M \xrightarrow{\pi_M} M$: The bundle dual to TM. The transition functions: $\widetilde{\Psi}^{-1}_{\alpha\beta,x}$. We denote by $d\psi^1, \ldots, d\psi^n$ the basis of T^*_xM dual to the basis $\frac{\partial}{\partial\psi^1}, \ldots, \frac{\partial}{\partial\psi^n}$.

A covector field on M (differential 1-form): A section γ of the bundle T^*M. Locally $\gamma = \gamma_i(x)d\psi^i$.

A differential 2-form on M: A section ω of the second exterior power of the cotangent bundle $\bigwedge^2 T^*M$. Locally $\omega = \omega_{ij}(x) d\psi^i \wedge d\psi^j$.

A morphism of vector bundles $E_1 \xrightarrow{\pi_1} M, E_2 \xrightarrow{\pi_2} M$ over M: A map $\mu : E_1 \to E_2$ such that the following diagram is commutative

$$\begin{array}{ccc} E_1 & \xrightarrow{\mu} & E_2 \\ {\scriptstyle \pi_1}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi_2} \\ M & = & M \end{array}$$

and the induced mappings $\mu_x : E_{1,x} \to E_{2,x}$ are linear for any $x \in M$.

Differential k-forms as morphisms $\bigotimes^k TM \to M \times \mathbb{R}$: any differential k-form σ can be interpreted as such a morphism which is skew-symmetric. In other words, σ is a map form $\mathit{\Gamma}(TM) \times \cdots \times \mathit{\Gamma}(TM) \to \mathcal{E}(M)$ which is multilinear over the ring $\mathcal{E}(M)$ and skew-symmetric.

The exterior derivative $d : \mathit{\Gamma}(\bigwedge^k T^*M) \to \mathit{\Gamma}(\bigwedge^{k+1} T^*M)$: The Cartan formula gives $(d\gamma)(X,Y) = X\gamma(Y) - Y\gamma(X) - \gamma([X,Y]), X, Y \in \mathit{\Gamma}(TM)$ for $\gamma \in \mathit{\Gamma}(TM)$ and $(d\omega)(X,Y,Z) = \sum_{c.p.X,Y,Z} X\omega(Y,Z) - \omega([X,Y],Z)$.

Bivector fields and 2-forms as morphisms: Let $\eta \in \mathit{\Gamma}(\bigwedge^2 TM)$ and $\gamma \in \mathit{\Gamma}(T^*M)$. The *contraction* $\gamma \lrcorner \eta =: \eta(\gamma)$ (in the first index) is a vector field defined by $v = v^j(x)\frac{\partial}{\partial \psi^j}, v^j(x) := \gamma_i(x)\eta^{ij}(x)$. Since this operation is pointwise it defines a morphism of bundles $\eta^\sharp : T^*M \to TM$. Note that it is skew-symmetric, i.e. $(\eta^\sharp)^* = -\eta^\sharp$. Conversely, given such a morphism, we can construct a bivector field.

Analogously, a differential 2-form ω defines a skew-symmetric morphism $\omega^\flat : TM \to T^*M$.