Algebraic and geometric aspects of modern theory of integrable systems

Lecture 15

1 Introduction to the KdV equation and infinite-dimensional argument translation method

The Gelfand–Fuchs cocycle: Let $\mathfrak{g} := \Gamma(TS^1)$ be the Lie algebra of vector fields on a circle. Elements of $\mathfrak{g}$ can be viewed as $v(x)\partial_x$, where v is a function on S^1 and x is a coordinate. The bracket will be expressed as $[v(x)\partial_x, w(x)\partial_x] = (-vw_x + wv_x)(x)\partial_x$.

PROPOSITION. *The expression $c(v\partial_x, w\partial_x) := \int_{S^1} vw_{xxx}dx$ is a cocycle on $\mathfrak{g}$.*

Proof $c([v\partial_x, w\partial_x], u\partial_x) = \int_{S^1}(-vw_x+wv_x)u_{xxx}dx = \int_{S^1}(-vw_x+wv_x)du_{xx} =$ [integration by parts] $= -\int_{S^1} u_{xx}(-vw_x + wv_x)_x dx = -\int_{S^1} u_{xx}(-v_xw_x + w_xv_x - v_{xx}w + w_{xx}v)dx = \int_{S^1} u_{xx}(v_{xx}w - w_{xx}v)dx$. Summing the last expression over cyclic permutations of v, w, u gives zero. *Exercise:* Prove the skew-symmetry. □

The Virasoro Lie algebra: The central extension $\mathfrak{g}' := \mathfrak{g}\oplus\mathbb{R}$ of $\mathfrak{g}$ with respect to the Gelfand–Fuchs cocycle: $[(v(x)\partial_x, a), (w(x)\partial_x, b)]' := ((-vw_x + wv_x)(x)\partial_x, c(v\partial_x, w\partial_x))$.

The "$H^1_{\alpha\beta}$-energy" on $\mathfrak{g}'$: The quadratic form

$$\langle (v(x)\partial_x, a), (w(x)\partial_x, b)\rangle := \int_{S^1} (\alpha vw + \beta v_xw_x)dx + ab.$$

If $\alpha = 1, \beta = 0$ we get the L^2 scalar product, if $\alpha = 1, \beta = 1$, this is the Sobolev one.

The Virasoro group: This is a central extension $G' := G \times \mathbb{R}$ of the group $G := Diff(S^1)$ of diffeomorphisms of a circle by means of the Bott cocycle

$$B(\psi, \varphi) := \int_{S^1} \log((\psi\circ\varphi)_x)d\log(\varphi_x).$$

The group operation on G' is given by

$$(\psi(x), a)\circ(\varphi(x), b) := ((\psi\circ\varphi)(x), a + b + B(\psi, \varphi)).$$

Remark: If $\alpha \neq 0$ the $H^1_{\alpha\beta}$-energy can be extended to a right-invariant metric on G.

The dual space $\mathfrak{g}^*$: It can be naturally identified with the space of quadratic differentials $\{u(x)(dx)^2\}$ on the circle. The pairing is given by the formula:

$$\langle u(x)(dx)^2, v(x)\partial_x\rangle := \int_{S^1} u(x)v(x)dx.$$

The coadjoint orbits coincide with the orbits of the action of diffeomorphisms on quadratic differentials:

$$\mathrm{Ad}^*_\varphi : u(dx)^2 \mapsto u(\varphi)\cdot\varphi_x^2(dx)^2 = u(\varphi)(d\varphi)^2.$$

Remark: If $u(x) > 0$ for any $x \in S^1$, the square root $\sqrt{u(x)(dx)^2}$ transforms as a 1-form. In particular, $\Phi(u(x)(dx)^2) := \int_{S^1}\sqrt{u(x)}dx$ is a Casimir function: the value of Φ is stable under the diffeomorphism action. The corresponding orbit has codimension one: a diffeomorphism action sends the quadratic differential $u(x)(dx)^2$ to the constant quadratic differential $C(dx)^2$, where $C := (1/2\pi)\int_{S^1}\sqrt{u(x)}dx$.

If u changes sign, the integral $\int_a^b \sqrt{u(x)}dx$ between two consecutive zeroes a, b of u is invariant. Thus the codimension of the orbit is greater than 1 in this case.

The dual space $(\mathfrak{g}')^*$: It can be naturally identified with the space of pairs $\{(u(x)(dx)^2, a)\}$ with the natural pairing

$$\langle (u(x)(dx)^2, a), (v(x)\partial_x, b)\rangle := \int_{S^1} u(x)v(x)dx + ab.$$

Generic coadjoint orbits are of codimension 2 (they are contained in the hyperplanes $a = \mathit{const}$).

Digression on the Euler equations: *Recall:* Let $\mathfrak{g}$ be a Lie algebra with a positively defined scalar product b. Extend b to the right invariant contravariant metric $b_r : T^*G \times_G T^*G \to \mathbb{R}$, denote by $B : T^G \to \mathbb{R}$ the corresponding quadratic form. The hamiltonian equation on T^*G with the hamiltonian $H := B$ is right invariant, hence can be reduced to a hamiltonian equation on $\mathfrak{g}^*$.

The last is called the *Euler equation* and is given by a vector field $\eta_{\mathfrak{g}}(b(v,v))$. Let $A : g \to \mathfrak{g}^*$ be defined by $\langle v, A(w)\rangle = b(v,w)$. Call A *the inertia operator*. It turns out (*Exercise:* prove this) that this equation is of the form

$$\frac{dx}{dt} = -\mathrm{ad}^*_{A^{-1}x}x, x \in \mathfrak{g}^*.$$

The Euler equation related to the "$H^1_{\alpha\beta}$-energy":

THEOREM. *(Khesin–Misiołek) The Euler equation on $x := (v(x)(dx)^2, a)$ corresponding to the "$H^1_{\alpha\beta}$"-scalar product with $\alpha \neq 0$ has the form*

$$\alpha(v_t + 3vv_x) - \beta(v_{xxt} + 2v_xv_{xx} + vv_{xxx}) - bv_{xxx} = 0, a_t = 0.$$

Remark: By choosing $\alpha = 1, \beta = 0$ one obtains the *Korteweg–de Vries equation.* For $\alpha = \beta = 1$ one recovers the *Camassa–Holm equation.*

Proof Let us calculate the ad* operator. We have

$$\langle \mathrm{ad}^*_{(v\partial_x,b)}(u(dx)^2,a),(w\partial_x,c)\rangle = \langle (u(dx)^2,a),[(v\partial_x,b),(w\partial_x,c)]'\rangle =$$
$$\int_{S^1} u(-vw_x+wv_x)dx + a\int_{S^1} vw_{xxx}dx = \int_{S^1} uwv_xdx - \int_{S^1} uvdw - a\int_{S^1} wv_{xxx}dx =$$
$$\int_{S^1} uwv_xdx + \int_{S^1} w(u_xv+uv_x)dx - a\int_{S^1} wv_{xxx}dx = \int_{S^1} w(2uv_x+u_xv-av_{xxx})dx.$$

Hence $\mathrm{ad}^*_{(v\partial_x,b)}(u(dx)^2,a) = ((2uv_x+u_xv-av_{xxx})(dx)^2,0)$.

Now let us look at the inertia operator $A:\mathfrak{g}'\to(\mathfrak{g}')^*$ given by $\langle (v\partial_x,b),A((w\partial_x,a))\rangle = \int_{S^1}(\alpha vw+\beta v_xw_x)dx+ba = \int_{S^1} v\Lambda w dx + ba$, where $\Lambda : \alpha-\beta\partial_x^2$ is a second order differential operator. We have $A((w\partial_x,a)) = ((\Lambda w)(dx)^2,a)$. This operator is nondegenerate for $\alpha\neq 0$.

The corresponding Euler equation is

$$\frac{d}{dt}(u(dx)^2,a) = -\mathrm{ad}^*_{A^{-1}(u(dx)^2,a)}(u(dx)^2,a) = -\mathrm{ad}^*_{((\Lambda^{-1}u)(dx)^2,a)}(u(dx)^2,a),$$

or, using the formula for ad*

$$\frac{d}{dt}(u(dx)^2,a) = -((2u\Lambda^{-1}u_x + u_x\Lambda^{-1}u - a\Lambda^{-1}u_{xxx})(dx)^2,0).$$

Putting $v := \Lambda^{-1}u$ we get

$$\frac{d}{dt}(\Lambda v) = -2(\Lambda v)v_x - (\Lambda v_x)v + av_{xxx},\ a_t = 0.$$

Substituting $\Lambda = \alpha-\beta\partial_x^2$ we get the proof. □

Bihamiltonian property of the KdV and C–H equations:

THEOREM. *(Khesin–Misiołek) The Euler equation corresponding to the "$H^1_{\alpha\beta}$"-scalar product with $\alpha\neq 0$ is bihamiltonian: it is hamiltonian with respect to the Lie-Poisson structure $\eta_{\mathfrak{g}'}$ on $(\mathfrak{g}')^*$ (this is standard fact) and it is also hamiltonian with respect to the constant Poisson structure obtained by "freezing" of $\eta_{\mathfrak{g}'}$ at the point $((\alpha/2)(dx)^2,\beta)$.*

Remark: We leave this theorem without proof. The last but not least remark: one can apply the general Magri–Lenard scheme to obtain an infinite sequence of "first integrals" of the (α,β)-Euler equation (in fact Magri invented his scheme having in mind the KdV equation).