Algebraic and geometric aspects of modern theory of integrable systems

Lecture 14

1 Lie pencils and completely integrable systems

Digression on semidirect products of Lie algebras: Let $\rho : \mathfrak{g} \to \mathrm{End}(V)$ be a representation of a Lie algebra $\mathfrak{g}$ on a vector space V.

Exercise: Prove that the formula $[(x,v),(y,w)]' = ([x,y], \rho(x)w - \rho(y)v)$ defines a Lie algebra structure on $\mathfrak{g}' := \mathfrak{g} \times V$.

We put $\mathfrak{g} \times_\rho V := (\mathfrak{g} \times V, [,]')$ and say that $\mathfrak{g} \times_\rho V$ is a *semidirect product* of $\mathfrak{g}$ and V.

Note that the subspaces $\mathfrak{g}_0 := \mathfrak{g} \times \{0\} \subset \mathfrak{g}', \mathfrak{g}_1 := \{0\} \times V \subset \mathfrak{g}'$ satisfy the following commutation relations: $[\mathfrak{g}_0, \mathfrak{g}_0]' \subset \mathfrak{g}_0, [\mathfrak{g}_0, \mathfrak{g}_1]' \subset \mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_1]' = \{0\}$ (in particular $\mathfrak{g}_0$ is an abelian ideal of $\mathfrak{g}'$). And it is easy to see that, given any Lie algebra $\mathfrak{g}' = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with the commutation relations as above, we can put $\rho(x) := \mathrm{ad}'_x|_{\mathfrak{g}_1}, x \in \mathfrak{g}_0$ (here $\mathrm{ad}'_x v := [x,v]'$), and get a representation of a Lie algebra $\mathfrak{g}_0$ on the vector space $\mathfrak{g}_1$ and an isomorphism of $\mathfrak{g}'$ with $\mathfrak{g}_0 \times_\rho \mathfrak{g}_1$ (*Exercise:* prove this).

Given a Lie algebra $\mathfrak{g}$, we call the codimension of a regular coadjoint orbit the *index* of $\mathfrak{g}$. In particular, $\mathrm{ind}\, \mathfrak{g} = \mathrm{corank}\, \eta_{\mathfrak{g}}|_x := \dim \mathfrak{g} - \mathrm{rank}\, \eta_{\mathfrak{g}}|_x$ for generic $x \in \mathfrak{g}^*$.

Theorem. *(Raïs, 1978)*

$$\mathrm{ind}(\mathfrak{g} \times_\rho V) = \mathrm{ind}\, \mathfrak{g}_\nu + \mathrm{codim}\, O_\nu.$$

Here $\nu \in V^*$ is a generic element, O_ν is the orbit of this element with respect to the dual (anti) representation $\rho^* : \mathfrak{g} \to \mathrm{End}(V^*), \rho^*(x) := (\rho(x))^*$, and $\mathfrak{g}_\nu := \{x \in \mathfrak{g} \mid \rho^*(x)\nu = 0\}$ is the stabilizer of the element ν with respect to ρ^*.

Example: Let $\mathfrak{g} := \mathfrak{so}(n,\mathbb{R})$ and $\rho : \mathfrak{g} \to \mathrm{End}(\mathbb{R}^n)$ be the standard representation (the skew-symmetric matrices act on vector-columns). Then $\mathfrak{e}(n,\mathbb{R}) := \mathfrak{so}(n,\mathbb{R}) \times_\rho \mathbb{R}^n$ is called the *euclidean* Lie algebra.

The standard euclidean scalar product $(|)$ on $\mathbb{R}^n$ is invariant with respect to ρ, i.e. $(\rho(x)v|w) = -(v|\rho(x)w) = 0$. In particular, we can identify the orbits of ρ and ρ^*. Thus the orbit of ρ^* through an element $\nu \in (\mathbb{R}^n)^* \cong \mathbb{R}^n$ is the sphere $S^{n-1}_{|\nu|}$ of radius $|\nu|$. The stabilizer $\mathfrak{g}_\nu$ is the Lie algebra of rotations "around" ν (i.e. preserving ν) and is isomorphic to the Lie algebra $\mathfrak{so}(n-1,\mathbb{R})$ (of rotations "around" $(1,0,\dots,0)$). Finally, $\mathrm{ind}\, \mathfrak{e}(n,\mathbb{R}) = \mathrm{ind}\, \mathfrak{so}(n-1,\mathbb{R}) + 1$.

Recall that the ring of Casimir functions of $\eta_{\mathfrak{g}}$ is generated by $\operatorname{Tr}(x^2), \operatorname{Tr}(x^4)\ldots, \operatorname{Tr}(x^{2k})$ for $n = 2k+1$ and by $\operatorname{Tr}(x^2), \operatorname{Tr}(x^4)\ldots, \operatorname{Tr}(x^{2k-2}), \operatorname{Pf}(x)$ for $n = 2k$. Hence $\operatorname{ind}\mathfrak{so}(n,\mathbb{R}) = [n/2]$. In particular, $\operatorname{ind}\mathfrak{e}(n-1,\mathbb{R}) = [(n-2)/2]+1 = [n/2] = \operatorname{ind}\mathfrak{so}(n,\mathbb{R})$.

Digression on contractions of Lie algebras: Assume $(\mathfrak{g}, [,])$ is a Lie algebra and that there exists a family of Lie brackets $[,]^\lambda$ on $\mathfrak{g}$ continuously depending on the parameter $\lambda \in U \setminus \{\lambda_0\}$, here $U \subset \mathbb{R}^k$ is an open set, $\lambda_0 \in U$ is a fixed element. Assume that $[,] = [,]^\lambda$ for some $\lambda \in U \setminus \{\lambda_0\}$ and that for any $x, y \in \mathfrak{g}$ there exists $\lim_{\lambda\to\lambda_0}[x,y]^\lambda =: [x,y]_0$. Then by the continuity the bracket $[,]_0$ will be a Lie bracket on $\mathfrak{g}$. We will say that $(\mathfrak{g}, [,]_0)$ is a *contraction* of a Lie algebra $(\mathfrak{g}, [,])$.

Example: Let $(\mathfrak{g}, [,])$ be any Lie algebra and let $[,]^\lambda := \lambda[,], \lambda \in \mathbb{R}\setminus\{0\}$. Then $\lim_{\lambda\to 0}[x,y]^\lambda =: [x,y]_0$ exists and gives an abelian Lie bracket on $\mathfrak{g}$.

Lie pencils and complete families of functions in involution: Let $\mathfrak{g} = \mathfrak{so}(n,\mathbb{R}), \mathfrak{g}^t := (\mathfrak{g}, [,]^t)$, where $[,]^t := [,]_{t_1 I + t_2 A}$, $A = \operatorname{diag}(a_1,\ldots,a_n)$ is a fixed diagonal matrix with a simple spectrum. The linear map given by $L^t : X \mapsto \sqrt{t_1 I + t_2 A}X\sqrt{t_1 I + t_2 A}$ is an isomorphism of the Lie algebras $\mathfrak{g}^{(1,0)}$ and $\mathfrak{g}^t$ for t nonproportional to $(a_1,-1),\ldots,(a_n,-1)$. Indeed, $[L^tX, L^tY] = \sqrt{t_1 I + t_2 A}X(t_1 I + t_2 A)Y\sqrt{t_1 I + t_2 A} - \sqrt{t_1 I + t_2 A}Y(t_1 I + t_2 A)X\sqrt{t_1 I + t_2 A} = L^t[X,Y]_{t_1 I + t_2 A}$.

We claim that the Lie algebra $(\mathfrak{g}, [,]^t)$ for $t \neq (0,0)$ proportional to one of the vectors $(a_1,-1),\ldots,(a_n,-1)$ is isomorphic to $\mathfrak{e}(n-1,\mathbb{R})$ (hence $\mathfrak{e}(n-1,\mathbb{R})$ is a contraction of $\mathfrak{so}(n,\mathbb{R})$). For instance, take $t = (a_1,-1)$. The map $L' : X \mapsto \sqrt{A'}X\sqrt{A'}$, where $A' := \operatorname{diag}(1, 1/\sqrt{a_1-a_2},\ldots,1/\sqrt{a_1-a_n})$, gives the isomorphism of $[,]^{(a_1,-1)}$ with $[,]_B, B := (0,1,\ldots,1)$.

Let us prove, that $(\mathfrak{g}, [,]_B)$ is isomorphic to $\mathfrak{e}(n-1,\mathbb{R})$. Put

$$\mathfrak{g}_0 := \left\{ \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & y_{12} & \cdots & y_{1,n-1} \\ 0 & -y_{12} & 0 & \cdots & y_{2,n-1} \\ & & & \cdots & \\ 0 & -y_{1,n-1} & -y_{2,n-1} & \cdots & 0 \end{bmatrix} \mid y_{ij} \in \mathbb{R}, i<j \right\}, \mathfrak{g}_1 := \left\{ \begin{bmatrix} 0 & -y_1 & \cdots & -y_n \\ y_1 & 0 & \cdots & 0 \\ & & \cdots & \\ y_n & 0 & \cdots & 0 \end{bmatrix} \mid y_i \in \mathbb{R} \right\}.$$

Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and it is easy to see that $[\mathfrak{g}_0,\mathfrak{g}_0] \subset \mathfrak{g}_0, [\mathfrak{g}_0,\mathfrak{g}_1] \subset \mathfrak{g}_1, [\mathfrak{g}_1,\mathfrak{g}_1] \subset \mathfrak{g}_0$. In particular, $\mathfrak{g}_0$ is a Lie subalgebra (isomorphic to $\mathfrak{so}(n-1,\mathbb{R})$). On the other hand we obviously have: 1) $[\mathfrak{g}_0,\mathfrak{g}_0]_B = [\mathfrak{g}_0,\mathfrak{g}_0]$; 2) $[\mathfrak{g}_0,\mathfrak{g}_1]_B = [\mathfrak{g}_0,\mathfrak{g}_1]$; 3) $[\mathfrak{g}_1,\mathfrak{g}_1]_B = \{0\}$. So to finish the proof it remains to notice that the representation $\rho : \mathfrak{g}_0 \to \operatorname{End}(\mathfrak{g}_1), \rho(x) := \operatorname{ad}_x|_{\mathfrak{g}_1}$ is isomorphic to the standard representation of $\mathfrak{so}(n-1,\mathbb{R})$ on $\mathbb{R}^n$ (*Exercise:* check this).

Now we are ready to prove the kroneckerity of the Poisson pencil $\Theta := \{t_1\eta_1 + t_2\eta_2\}_{(t_1,t_2)\in\mathbb{R}^2}$ on $\mathfrak{g}^*$ associated to the Lie pencil $\{(\mathfrak{g},[,]^t\}_{t\in\mathbb{R}^2}$. Here $\eta_1 := \eta_{\mathfrak{g}}$ is the canonical Lie–Poisson structure on $\mathfrak{so}(n,\mathbb{R})$ and η_2 is the Lie–Poisson structure corresponding to the modified commutator $[,]_A$. We need to prove that for a generic point $x \in \mathfrak{g}^*$ we have $\operatorname{rank}(t_1\eta_1|_x + t_2\eta_2|_x) = const$ for $(t_1,t_2) \in \mathbb{C}^2 \setminus \{0\}$.

Let $e_1,\ldots,e_n$ be a basis of $\mathfrak{g}$ and let the corresponding structure constants are defined by $[e_i,e_j] = c_{ij}^k e_k, [e_i,e_j] = C_{ij}^k e_k$. The condition above can be rewritten as $\operatorname{rank}(t_1 c_{ij}^k x_k + t_2 C_{ij}^k x_k) = const, (t_1,t_2) \in \mathbb{C}^2 \setminus \{0\}$. To prove it let us pass to the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(n,\mathbb{C})$ (skew-symmetric matrices with complex entries). The same considerations as above show that the map $L^t : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}, X \mapsto \sqrt{t_1 I + t_2 A}X\sqrt{t_1 I + t_2 A}$, where $(t_1,t_2) \in \mathbb{C}^2$ is nonproportional to $(a_1,-1),\ldots,(a_n,-1)$, is an isomorphism of the corresponding Lie algebras. In other words, $t_1 c_{ij}^k x_k + t_2 C_{ij}^k x_k =$

$L^t_{ii'}L^t_{jj'}(L^t)^{-1}_{kk'}c^k_{i'j'}x_k$, here the matrix $L^t_{ii'}$ is defined as $L^t e_i = L^t_{ii'}e_{i'}$ and similarly $(L^t)^{-1}_{kk'}$. Thus we conclude that the rank of $t_1 c^k_{ij}x_k + t_2 C^k_{ij}x_k$ is constant as far as t belongs to $T := \mathbb{C}^2 \setminus (\mathrm{Span}_{\mathbb{C}}\{(a_1,-1)\} \cup \dots \cup \mathrm{Span}_{\mathbb{C}}\{(a_n,-1)\})$ and x belongs to $V := \mathfrak{g}_{\mathbb{C}} \setminus (\bigcup_{t\in T}(L^t)^{-1}S_{\mathbb{C}})$. Recall that $S := \mathrm{Sing}\,\eta_{\mathfrak{g}}$ is the set $\{x \in \mathfrak{g} \mid \mathrm{rank}\,(c^k_{ij}x_k) < \max_x \mathrm{rank}\,(c^k_{ij}x_k)\}$ and $S_{\mathbb{C}}$ is its complexification.

Finally we use the fact that $\mathrm{ind}\,\mathfrak{e}(n-1,\mathbb{C}) = \mathrm{ind}\,\mathfrak{so}(n,\mathbb{C})$ (which can be proved in the same way as in real case) to conclude that Θ is Kronecker at any point $x \in U := \mathfrak{g} \cap V \setminus (V_1 \cup \dots \cup V_n)$. Here $V_i := \mathrm{Sing}\,\eta_{\mathfrak{g}_i}, \mathfrak{g}_i := (\mathfrak{g}, [,]^{(a_i,-1)}), i = 1,\dots,n$. The set U is dense because $\mathfrak{g} \cap V = \mathfrak{g} \setminus (\bigcup_{t\in T'}(L^t)^{-1}S)$, where $T' := \mathbb{R}^2 \setminus (\mathrm{Span}_{\mathbb{R}}\{(a_1,-1)\} \cup \dots \cup \mathrm{Span}_{\mathbb{R}}\{(a_n,-1)\})$, and $\mathrm{codim}(\bigcup_{t\in T'}(L^t)^{-1}S) \geqslant 2$ due to the condition $\mathrm{codim}_{\mathbb{R}} S \geqslant 3$.

The corresponding complete family $\mathcal{C}^{\Theta}(\mathfrak{g}^*)$ of functions in involution is generated by the functions $f((L^t)^{-1}x), t \in \mathbb{R}^2$, where f is a Casimir function of $\eta_{\mathfrak{g}}$.

One can show that the hamiltonian $\mathrm{Tr}((L^{-1}x)x), Lx = Dx + xD$, of the Euler–Manakov top is contained in the family $\mathcal{C}^{\Theta}(\mathfrak{g}^*)$ (with $A := D^2$), but this is a little bit technical question and we will skip it.