Algebraic and geometric aspects of modern theory of integrable systems

Lecture 13

1 Linear algebra of pairs of bivectors and completeness of families of functions in involution

The Jordan–Kronecker decomposition of a pair of bivectors: A bivector b on a vector space V is an element of $\bigwedge^2 V$. We will view a bivector b sometimes as a skew-symmetric map $V^* \to V$ (then its value at $x \in V^*$ will be denoted by $b(x)$) and sometimes as a skew-symmetric bilinear form on V^* (then its value at $x, y \in V^*$ will be denoted by $b(x, y)$). In particular, $b(x, y) = \langle b(x), y \rangle$.

THEOREM. *(Gelfand–Zakharevich, 1989) Given a finite-dimensional vector space V over $\mathbb{C}$ and a pair of bivectors $(b^{(1)}, b^{(2)})$, $b^{(i)} : \bigwedge^2 V^* \to \mathbb{C}$, there exists a direct decomposition $V^* = \oplus_{m=1}^k V_m^*$ such that $b^{(i)}(V_l^*, V_m^*) = 0$ for $i = 1, 2, l \neq m$, and the triples $(V_m^*, b_m^{(1)}, b_m^{(2)})$, where $b_m^{(i)} := b^{(i)}|_{V_m^*}$, are from the following list:*

1. [the Jordan block $\mathbf{j}_{2j_m}(\lambda)$]: $\dim V_m^* = 2j_m$ *and in an appropriate basis of V_m^* the matrices of $b_m^{(1)}, b_m^{(2)}$ are equal to*
$$\begin{bmatrix} \mathbf{0} & I_{j_m} \\ -I_{j_m} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & J_{j_m}(\lambda) \\ -J_{j_m}(\lambda)^T & \mathbf{0} \end{bmatrix}$$
where I_{j_m} is the unity $j_m \times j_m$-matrix and
$$J_{j_m}(\lambda) := \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}$$
is the Jordan $j_m \times j_m$-block with the eigenvalue λ;

2. [the Jordan block $\mathbf{j}_{2j_m}(\infty)$]: $\dim V_m^* = 2j_m$ *and in an appropriate basis of V_m^* the matrices of $b_m^{(1)}, b_m^{(2)}$ are equal to*
$$\begin{bmatrix} \mathbf{0} & J_{j_m}(0) \\ -J_{j_m}(0)^T & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & I_{j_m} \\ -I_{j_m}^T & \mathbf{0} \end{bmatrix};$$

3. [the Kronecker block $\mathbf{k}_{2k_m+1}$]: $\dim V_m^* = 2k_m+1$ *and in an appropriate basis of* V_m^* *the matrices of* $b_m^{(1)}, b_m^{(2)}$ *are equal to*

$$K_{1,k_m} := \begin{bmatrix} \mathbf{0} & B_{1,k_m} \\ -B_{1,k_m}^T & \mathbf{0} \end{bmatrix}, K_{2,k_m} := \begin{bmatrix} \mathbf{0} & B_{2,k_m} \\ -B_{2,k_m}^T & \mathbf{0} \end{bmatrix},$$

where

$$B_{1,k_m} := \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}, B_{2,k_m} := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

($k_m \times (k_m+1)$-matrices).

Kronecker Poisson pencils: Let $\{\eta^t\}_{t\in\mathbb{R}^2}, \eta^t := t_1\eta_1 + t_2\eta_2$, be a Poisson pencil on M. We say that it is *Kronecker at a point* $x \in M$, if the Jordan–Kronecker decomposition of the pair of bivectors $\eta_1|_x, \eta_2|_x$ (regarded as elements of $\bigwedge^2 T_x^{\mathbb{C}} M$, here $T_x^{\mathbb{C}} M$ is the complexified tangent space) does not contain Jordan blocks.

PROPOSITION. $\{\eta^t\}_{t\in\mathbb{R}^2}$ *is Kronecker at x if and only if*

$$\operatorname{rank}(t_1\eta_1|_x + t_2\eta_2|_x) = const, (t_1,t_2) \in \mathbb{C}^2 \setminus \{0\}.$$

Proof It is easy to see that any nontrivial linear combination of matrices K_{1,k_m}, K_{2,k_m} has constant rank equal to $2k_m$. So the rank can "jump" at some $t \neq 0$ if and only if there are Jordan blocks in the decomposition. □

We say that a Poisson pencil Θ on M is *Kronecker* if there exists an open dense set $U \subset M$ such that Θ is Kronecker at any $x \in U$.

Involutivity of Casimir functions for Kronecker Poisson pencils: We have already proven that, if $t', t'' \in \mathbb{R}^2$ are linearly independent, then $\{f,g\}_{\eta^t} = 0$ for any $f \in \mathcal{C}^{t'}(M), g \in \mathcal{C}^{t''}(M), t \in \mathbb{R}^2$. In the same way one can prove that $\eta^t|_x(\alpha,\beta) = 0$ for any $\alpha \in \ker \eta^{t'}|_x, \beta \in \ker \eta^{t''}|_x, t \in \mathbb{R}^2$.

PROPOSITION. *Let* $\{\eta^t\}_{t\in\mathbb{R}^2}$ *be Kronecker and let* $t' \in \mathbb{R}^2, t' \neq 0$. *Then* $\{f,g\}_{\eta^t} = 0$ *for any* $f,g \in \mathcal{C}^{t'}(M), t \in \mathbb{R}^2$.

Proof Fix $x \in U$. Let $t_{(n)} \in \mathbb{R}^2$ be such that $t_{(n)}$ is linearly independent with t' and $t_{(n)} \overset{n\to\infty}{\longrightarrow} t'$. The kernel of the map $\eta^t|_x : T_x^*M \to T_xM$ continuously depend on $t \in \mathbb{R}^2 \setminus \{0\}$ and is of constant dimension. Consequently we can find a sequence of covectors $\alpha_n \in \ker \eta^{t_{(n)}}|_x$ such that $\alpha_n \overset{n\to\infty}{\longrightarrow} d_xg$. We get $\eta^t|_x(d_xf, \alpha_n) = 0$ and by continuity we conclude that $\eta^t|_x(d_xf, d_xg) = 0$. In other words, $\{f,g\}_{\eta^t}(x) = 0$ for any $x \in U$. Since U is dense, using again the continuity argument we get the proof. □

Summarizing, we get the following result.

Proposition. *Let $\Theta = \{\eta^t\}_{t\in\mathbb{R}^2}$ be a Kronecker Poisson pencil and let*

$$\mathcal{C}^{\Theta}(M) := \mathrm{Span}\{\bigcup_{t\in\mathbb{R}^2\setminus\{0\}} \mathcal{C}^t(M)\}.$$

Then $\mathcal{C}^{\Theta}(M)$ is a family of functions in involution with respect to any Poisson bivector η^t.

Completeness of Casimir functions for Kronecker Poisson pencils: Let (M,η) be a Poisson structure. We say that an open set $W \subset M$ is *correct* for η if the set $W' := W \setminus (W \cap \mathrm{Sing}\,\eta)$ is nonempty and the common level sets of the functions from $\mathcal{C}^{\eta}(W')$ coincide with the symplectic foliation of η on the set W'. In other words, the set W is correct if the Poisson structure does not have regular symplectic leaves dense in W. Equivalent definition: W is correct if $\{d_xf \mid f \in \mathcal{C}^{\eta}(W)\} = \ker\eta_x$ for any $x \in W'$. Note that in analytic category any sufficiently small open set is correct.

Proposition. *Let $\Theta = \{\eta^t\}_{t\in\mathbb{R}^2}$ be a Kronecker Poisson pencil. Assume $W \subset M$ is an open set that is correct for η^t for a countable set $\{t_{(1)}, t_{(2)}, \ldots\}$ of pairwise linearly independent values of the parameter t and the set $W' := W\setminus\bigcup_{i=1}^{\infty}\mathrm{Sing}\,\eta^{t_{(i)}}$ is nonempty. Then the set of functions in involution $\mathcal{C}^{\Theta}(W')$ is* complete *with respect to any $\eta^t, t \neq 0$.*

Proof Fix $x \in U \cap W'$. Let us first prove that the set $C_x := \{d_xf \mid f \in \mathcal{C}^{\Theta}(W')\} \subset T_x^*M$ coincides with the set $L_x := \mathrm{Span}\{\bigcup_{t\in\mathbb{R}^2\setminus\{0\}}\ker\eta_x^t\}$. Indeed, the vector space L_x is finite-dimensional, hence is generated by a finite number of kernels $\ker\eta_x^t = \{d_xf \mid f \in \mathcal{C}^t(W)\}$. Hence $L_x \subset C_x$. The same considerations show that $C_x \subset L_x$.

It is easy to see that the set L_x is of dimension $(1/2)\mathrm{rank}\,\eta_x^t + \dim M - \mathrm{rank}\,\eta_x^t$. Assume for a moment that the Jordan–Kronecker decomposition of the pair $\eta_1|_x, \eta_2|_x$ consists of one Kronecker block $\mathbf{k}_{2k_m+1}$. The kernel of the matrix $\lambda K_{1,k_m} + K_{2,k_m}$ is 1-dimensional and is spanned by the vector $[0,\ldots,0,1,-\lambda,\ldots,(-\lambda)^{k_m}]$. Taking k_m+1 different values of λ we get $k_m + 1 = (1/2)\mathrm{rank}\,\eta_x^t + \dim M - \mathrm{rank}\,\eta_x^t$ linearly independent vectors (recall the Vandermonde determinant) spanning the set L_x. In the case of several Kronecker blocks you repeat these considerations for each block. $\square$

Remark: In fact it is sufficient to require that W is correct for a finite number of η^t. However, this number depends on the number and dimension of the Kronecker blocks, so we make a bit stronger assumption (which in practice is always satisfied).

Example (method of the argument translation): Let $M := \mathfrak{g}^*, \eta_1 := \eta_{\mathfrak{g}}, \eta_2 := \eta_{\mathfrak{g}}(a)$, where $a \in \mathfrak{g}^* \setminus \mathrm{Sing}\,\eta_{\mathfrak{g}}$. Assume that $\mathrm{codim}\ \mathrm{Sing}\,\eta_{\mathfrak{g}} \geqslant 2$ (if $\mathfrak{g}$ is semisimple it is known that $\mathrm{codim}\ \mathrm{Sing}\,\eta_{\mathfrak{g}} \geqslant 3$). Note that $\mathrm{Sing}\,\eta_{\mathfrak{g}}$ is an algebraic set, i.e. it is defined by a finite number of algebraic equations $f_1(x) = 0, \ldots, f_m(x) = 0$ on $\mathfrak{g}^*$. Any algebraic set in a neighbourhood of its generic point is diffeomorphic to a manifold, hence its dimension is correctly defined.

If $e_1, \ldots, e_n$ is a basis of $\mathfrak{g}$ and the corresponding structure constants are defined by $[e_i, e_j] = c_{ij}^k e_k$, the polynomials $f_1, \ldots, f_m$ are the $r \times r$-minors of the matrix $c_{ij}(x) = c_{ij}^k x_k$, where $r = \max_x \mathrm{rank}\,[c_{ij}(x)]$. Here $x_1 = e_1, \ldots, x_n = e_n$ are the corresponding coordinates on $\mathfrak{g}^*$.

In order to check the condition of Kroneckerity we need to consider the complexification $\mathfrak{g}^{\mathbb{C}}$ of the initial Lie algebra. It can be regarded as a vector space $\mathrm{Span}_{\mathbb{C}}\{e_1, \ldots, e_n\} \cong \mathbb{C}^n$ with the Lie bracket defined by the same structure constants. The set $S := \{(z_1, \dots, z_n) \in (\mathfrak{g}^{\mathbb{C}})^* \cong \mathbb{C}^n \mid \operatorname{rank} c_{ij}^k z_k < \max_{z\in\mathbb{C}^n} \operatorname{rank} c_{ij}^k z_k\}$ is a complex algebraic set defined by the equations $f_1(z) = 0, \dots, f_m(z) = 0$, where $f_1, \dots, f_m$ are the same polynomials as above. In particular, the set S is of complex codimension at least 2.

We know that $t_1\eta_1|_x + t_2\eta_2|_x = c_{ij}^k(t_1x_k + t_2a_k), t_1, t_2 \in \mathbb{C}$. Thus $\operatorname{rank}(t_1\eta_1|_x + t_2\eta_2|_x)$ is maximal (over t) and independent of $t \in \mathbb{C}^2 \setminus \{0\}$ if and only if $t_1x + t_2a \in (\mathfrak{g}^{\mathbb{C}})^* \setminus S$ if and only if $x \notin \overline{a, S}$, where $\overline{a, S} := \{z \in (\mathfrak{g}^{\mathbb{C}})^* \mid \exists (t_1, t_2) \in \mathbb{C}^2 \setminus \{0\} \colon t_1z + t_2a \in S\}$.

Note that the set S is homogeneous (stable under rescaling). Passing to the projectivization the set $\overline{a, S}$ becomes a cone in $\mathbb{CP}^{n-1}$ over the projectivization of S. This shows that the set $\overline{a, S}$ is also algebraic (by the standard arguments from algebraic geometry) and, moreover, $\dim_{\mathbb{C}} \overline{a, S} = \dim_{\mathbb{C}} S + 1$. In particular $\operatorname{codim}_{\mathbb{C}} \overline{a, S} \geqslant 1$ and we can put $U := \mathfrak{g}^* \setminus (\mathfrak{g}^* \cap \overline{a, S}) = \mathfrak{g}^* \setminus (\overline{a, \operatorname{Sing} \eta_{\mathfrak{g}}})$. Here $\overline{a, \operatorname{Sing} \eta_{\mathfrak{g}}} := \{x \in \mathfrak{g}^* \mid \exists (t_1, t_2) \in \mathbb{R}^2 \setminus \{0\} \colon t_1x + t_2a \in \operatorname{Sing} \eta_{\mathfrak{g}}\}$ and $\operatorname{codim}_{\mathbb{R}} \overline{a, \operatorname{Sing} \eta_{\mathfrak{g}}} \geqslant 1$. The set U is an open dense set in $\mathfrak{g}^*$ such that $\{\eta^t\}$ is Kronecker at any $x \in U$.

Finally assume that $\mathfrak{g}$ is semisimple. Then $\eta_{\mathfrak{g}}$ has enough global Casimir functions and the whole space $\mathfrak{g}^*$ is a corrrect set for $\eta_{\mathfrak{g}}$. In particular, the assumptions of the proposition above are satisfied and we get a complete set $\mathcal{C}^{\Theta}(\mathfrak{g}^*)$ of functions in involution (with respect to any η^t). This set is generated by the "translations" $f(x + \lambda a), \lambda \in \mathbb{R}$, of the Casimir functions f of $\eta_{\mathfrak{g}}$.