Algebraic and geometric aspects of modern theory of integrable systems

Lecture 12

1 Poisson pencils and families of functions in involution

A Poisson pencil on M: Let a pair (η_1, η_2) of linearly independent bivectors on a manifold M be given. Assume $\eta^t := t_1\eta_1 + t_2\eta_2$ is a Poisson structure for any $t = (t_1, t_2) \in \mathbb{R}^2$. We say that the Poisson structures η_1, η_2 are *compatible* (or form a *bihamiltonian structure* or a *Poisson pair*) and that the whole family $\Theta := \{\eta^t\}_{t\in\mathbb{R}^2}$ is a *Poisson pencil.*

Exercise: Show that the following conditions are equivalent:

1. η^t is Poisson, i.e. $[\eta^t, \eta^t]_S = 0$, for any $t \in \mathbb{R}^2$ (here $[,]_S$ is the Schouten bracket);
2. $[\eta^t, \eta^t]_S = 0$ for any three pairwise nonproportional values of $t \in \mathbb{R}^2$;
3. $[\eta_1, \eta_1]_S = 0, [\eta_1, \eta_2]_S = 0, [\eta_2, \eta_2]_S = 0.$

Example 1: Let η_1, η_2 be bivectors on $\mathbb{R}^n$ with constant coefficients. Then they form a Poisson pair (recall that, given a bivector $\eta = \eta^{ij}(x)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, we have $[\eta, \eta]_S^{ijk} := \sum_{c.p.\, i,j,k} \eta^{ir}(x)\frac{\partial}{\partial x^r}\eta^{jk}(x)$).

Example 2: Let $\mathfrak{g}$ be a Lie algebra and $\eta_\mathfrak{g}$ the Lie–Poisson structure on $\mathfrak{g}^*$. Let $c : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be a 2-cocycle on $\mathfrak{g}$, i.e. c is skew-symmetric and $\sum_{c.p.\ v,w,u} c([v, w], u) = 0$ for any $v, w, u \in \mathfrak{g}$. Then $c \in (\mathfrak{g} \wedge \mathfrak{g})^* \cong \mathfrak{g}^* \wedge \mathfrak{g}^*$ can be regarded as a bivector on $\mathfrak{g}^*$ with constant coefficients. It turns out that (η_1, η_2), where $\eta_1 := \eta_\mathfrak{g}, \eta_2 := c$, is a Poisson pair.

Indeed, it is easy to see that the bracket $[(v, \alpha), (w, \beta)]' := ([v, w], c(v, w))$ defines a Lie algebra structure on $\mathfrak{g}' := \mathfrak{g} \times \mathbb{R}$ (*Exercise:* check this). The $\mathbb{R}$-component lies in the centre of $\mathfrak{g}'$, we say that $\mathfrak{g}'$ is a *central extension* of $\mathfrak{g}$. The affine subspaces $\mathfrak{g}^*_{x_0} := \mathfrak{g}^* \times x_0 \subset (\mathfrak{g}')^* = \mathfrak{g}^* \times \mathbb{R}$ are Poisson submanifolds of the Poisson manifold $((\mathfrak{g}')^*, \eta_{\mathfrak{g}'})$. The restriction $\eta_{\mathfrak{g}'}|_{\mathfrak{g}^*_{x_0}}$ coincides with $\eta_1 + x_0\eta_2$, i.e. the last bivector is Poisson at least for three different values of x_0. We conclude that (η_1, η_2) is a Poisson pair.

In coordinates this looks as follows. Let $e_1, \dots, e_n$ be a basis of $\mathfrak{g}$ and $[e_i, e_j] = c_{ij}^k e_k, c(e_i, e_j) = c_{ij}, i, j, k = 1, \dots, n$, for some constants $c_{ij}^k, c_{ij} \in \mathbb{R}$. Put $\eta'_0 := (0, 1), \eta'_i := (\eta_i, 0) \in \mathfrak{g}', i = 1, \dots, n$, and let $x'_0, \dots, x'_n$ denote the same elements regarded as coordinates on $(\mathfrak{g}')^*$. Then $\eta_{\mathfrak{g}'} = (c_{ij}^k x'_k + x'_0 c_{ij})\frac{\partial}{\partial x'_i} \wedge \frac{\partial}{\partial x'_j}$ and $\eta^t = (t_1 c_{ij}^k x_k + t_2 c_{ij})\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$. Here $x_1, \dots, x_n$ are coordinates on $\mathfrak{g}^*$ corresponding to $e_1, \dots, e_n$.

Example 3: In a particular case when the cocycle c is trivial, i.e. $c(v,w) = a([v,w])$ for some $a \in \mathfrak{g}^*$ we get a Poisson pencil $\{\eta^t\}, \eta^t := (t_1 c_{ij}^k x_k + t_2 c_{ij}^k a_k)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, here $a_1, \ldots, a_n$ are coordinates of a in the dual basis $e^1, \ldots, e^n$ of $\mathfrak{g}^*$. In the corresponding Poisson pair (η_1, η_2) the first bivector is the Lie-Poisson one, $\eta_{\mathfrak{g}}$, and the second one is $\eta_{\mathfrak{g}}(a)$, the Lie-Poisson bivector "frozen" at a.

Example 4: Let $\mathfrak{g} : \mathfrak{gl}(n, \mathbb{R})$ and $A \in \mathfrak{g}$. Put $[x,y]_A := xAy - yAx$. It is easy to see that $[,]_A$ is a Lie bracket on $\mathfrak{g}$ for any A (*Exercise*: check this). In particular, for a fixed $A \in \mathfrak{g}$ the bracket $[,]^t := t_1[,] + t_2[,]_A = [,]_{t_1 I + t_2 A}$ is a Lie bracket for any $t \in \mathbb{R}^2$ (any family of Lie brackets linearly spanned by two fixed brackets will be called a *Lie pencil*). Denote $\mathfrak{g}^t := (\mathfrak{g}, [,]^t)$. The Lie–Poisson structures $\eta_{\mathfrak{g}^t}$ form a Poisson pencil on $\mathfrak{g}^*$.

We get a generalization of this example taking $\mathfrak{g} := \mathfrak{so}(n, \mathbb{R})$ and A a symmetric $n \times n$-matrix.

I mechanism of constructing functions in involution (the Magri–Lenard scheme): Let (η_1, η_2) be a pair of Poisson structures (not necessarily compatible). Assume we can found a sequence of functions $H_0, H_1, \ldots \in \mathcal{E}(M)$ satisfying

$$\begin{aligned} \eta_1(H_0) &= \eta_2(H_1) \\ \eta_1(H_1) &= \eta_2(H_2) \\ &\vdots \quad . \end{aligned} \tag{1}$$

Proposition. *For any indices i, j the following equality holds:*

$$\{H_i, H_j\}_{\eta_1} = \{H_{i+1}, H_{j-1}\}_{\eta_1}.$$

Proof $\eta_1(H_i)H_j = \eta_2(H_{i+1})H_j = -\eta_2(H_j)H_{i+1} = -\eta_1(H_{j-1})H_{i+1} = \eta_1(H_{i+1})H_{j-1}$ □

Now assume $i < j$. If $j - i = 2k$, we can apply the proposition k times and get $\{H_i, H_j\}_{\eta_1} = \{H_{i+k}, H_{j-k}\}_{\eta_1} = \{H_{i+k}, H_{i+k}\}_{\eta_1} = 0$. If $j - i = 2k+1$, we get $\{H_i, H_j\}_{\eta_1} = \{H_{i+k}, H_{j-k}\}_{\eta_1} = \{H_{i+k}, H_{i+k+1}\}_{\eta_1} = \eta_1(H_{i+k})H_{i+k+1} = \eta_2(H_{i+k+1})H_{i+k+1} = 0$. Hence the sequence $H_0, H_1, \ldots$ is a family of first integrals in involution for any of vector fields $v_i := \eta_1(H_i), i = 0, 1, \ldots$ Note that all these vector fields are "bihamiltonian", i.e. hamiltonian with respect to both the Poisson structures η_1, η_2.

In general it is hard to find the sequences of functions $H_0, H_1, \ldots$ with the required properties. However, if we assume additionally that (η_1, η_2) is a Poisson pair, there are some cases, when such sequences naturally appear. For instance, assume that all the bivectors $\eta^t := t_1\eta_1 + t_2\eta_2$ of the corresponding Poisson pencil are degenerate. Let $\eta^\lambda := \lambda\eta_1 + \eta_2, \lambda := t_1/t_2$, and let f^λ be a Casimir function of η^λ. It turns out that f^λ depends smoothly, let $f^\lambda = f_0 + \lambda f_1 + \lambda^2 f_2 + \cdots$ be the corresponding Tailor expansion. Then we deduce from the equality $\eta^\lambda(f^\lambda) = 0$ that $0 = \eta_2(f_0), \eta_1(f_0) + \eta_2(f_1), \eta_1(f_1) + \eta_2(f_2), \ldots$ (coefficients of different powers of λ). Thus we can put $H_0 := f_0, H_1 := -f_1, H_2 := f_2, \ldots$ Note that such a Magri–Lenard chain starts from a Casimir function of η_2. If $g^\lambda = g_0 + \lambda g_1 + \cdots$ is another Casimir function of η^λ, we get another sequence of functions in involution. A question arises, is it true that $\{f_i, g_j\}_{\eta_k} = 0$? Another important question concerns the *completeness* of the obtained family of functions.

II mechanism of constructing functions in involution (based on the Casimir functions of a Poisson pencil): Let $\{\eta^t\}_{t \in \mathbb{R}^2}$ be a Poisson pencil on M. Denote by $\mathcal{C}^t(M)$ the space of Casimir functions of η^t.

Proposition. *Let $t', t'' \in \mathbb{R}^2$ be linearly independent and let $f \in \mathcal{C}^{t'}(M), g \in \mathcal{C}^{t''}(M)$. Then*

$$\{f, g\}_{\eta^t} = 0$$

for any $t \in \mathbb{R}^2$.

Proof Indeed for any $t \in \mathbb{R}^2$ there exist $c', c'' \in \mathbb{R}$ such that $t = c't' + c''t''$. Then $\{f, g\}_{\eta^t} = \eta^t(f)g = (c'\eta^{t'} + c''\eta^{t''})(f)g = c''\eta^{t''}(f)g = -c''\eta^{t''}(g)f = 0$. $\square$

It is not clear from this fact whether $\{f, g\}_{\eta^t} = 0$ if f, g are Casimir functions of the *same* bivector $\eta^{t'}$. We will discuss this question in the next lecture.