Algebraic and geometric aspects of modern theory of integrable systems

Lecture 11

1 Right and left actions on T^*G. Hamiltonian actions and completely integrable systems

The cotangent lift of a vector field: Put $M := T^*Q$. Let $\zeta \in \Gamma(TQ)$ be a vector field. Then it can be interpreted as a function $H_\chi : T^*Q \to \mathbb{R}, H_\zeta(\alpha) := \langle \alpha, \zeta|_x \rangle, \alpha \in T^*_x Q$. Put $\zeta^\sqcup := \eta(-H_\zeta), \eta := \omega^{-1}$, where ω is the canonical symplectic form on T^*Q. We say that $\zeta^\sqcup$ is the *cotangent lift* of ζ.

In the (q,p)-local coordinates on T^*Q we have $H_\zeta(q,p) = p_i \zeta^i(q)$ for $\zeta = \zeta^i(q)\frac{\partial}{\partial q^i}$ (because $H_\zeta(\alpha) = \alpha_i \zeta^i(q)$ for $\alpha = \alpha_i dq^i$) and $\zeta^\sqcup = \frac{\partial H_\zeta}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H_\zeta}{\partial q^i}\frac{\partial}{\partial p_i} = \zeta^i(q)\frac{\partial}{\partial q^i} - p_j \frac{\partial \zeta^j}{\partial q^i}\frac{\partial}{\partial p_i}$. Note that $H_\zeta = \lambda(\zeta)$, where $\lambda = pdq$ is the canonical Liouville 1-form on M.

FACT. *The map $\zeta \mapsto \zeta^\sqcup : \Gamma(TQ) \to \Gamma(TM)$ is a homomorphism of Lie algebras.*

Proof We will prove that the map $\zeta \mapsto -H_\zeta : (\Gamma(TQ), [,]) \to (\mathcal{E}(M), \{,\}_\eta)$ is a homomorphism of Lie algebras. Indeed, $\{-H_\zeta, -H_\xi\}_\eta = -\frac{\partial H_\zeta}{\partial p_i}\frac{\partial H_\xi}{\partial q^i} + \frac{\partial H_\xi}{\partial p_i}\frac{\partial H_\zeta}{\partial q^i} = -\zeta^i(q) p_j \frac{\partial \xi^j}{\partial q^i} + \xi^i(q) p_j \frac{\partial \zeta^j}{\partial q^i} = -H_{[\zeta,\xi]}$. □

Thus we get a (hamiltonian) right action $\zeta \mapsto \zeta^\sqcup$ of the Lie algebra $\Gamma(TQ)$ on M.

The cotangent lift of a right action $\rho : \mathfrak{g} \to \Gamma(TQ)$: this is a hamiltonian action $\rho^\sqcup : \mathfrak{g} \to \Gamma(TM)$ given by $\rho^\sqcup(v) := (\rho(v))^\sqcup$. The corresponding map $\mathcal{J} : \mathfrak{g} \to \mathcal{E}(M)$ is given by $v \mapsto -H_{\rho(v)}$ and the corresponding moment map $J : M \to \mathfrak{g}^*$ is given by $\langle v, J(x) \rangle = \mathcal{J}(v)(x) = -H_{\rho(v)}(x) = -\lambda(\rho(v))(x), v \in \mathfrak{g}, x \in M$.

Left and right invariant vector fields on a Lie group G: Let G be a Lie group, $\mathfrak{g} = T_e G$ its Lie algebra. Given $g \in G$ put $L_g : G \to G, L_g g' := gg', R_g : G \to G, R_g g' := g'g$. Given $v \in \mathfrak{g}$ put

$$v_l(g) := (L_g)_* v, v_r(g) := (R_g)_* v.$$

The vector field v_l is left invariant, i.e. for any $g' \in G$ we have $(L_{g'})_* v_l(g) = v_l(g'g)$. Indeed, $(L_{g'})_* v_l(g) = (L_{g'})_*(L_g)_* v = (L_{g'g})_* v = v_l(g'g)$. Analogously v_r is right invariant.

FACT. *1. The maps $v \mapsto v_l : \mathfrak{g} \to \Gamma(TG), v \mapsto v_r : \mathfrak{g} \to \Gamma(TG)$ are a homomorphism and an antihomomorphism of Lie algebras, respectively.*

2. $[v_l, w_r] = 0$ for any $v, w \in \mathfrak{g}$.

Remark: Item 2 is an infinitesimal emanation of the fact that L_g and $R_{g'}$ commute for any $g, g' \in G$.

Example: Let $G := GL(n,\mathbb{R})$ (nondegenerate $n \times n$-matrices with real entries) $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{R}) = T_I G$ (all $n \times n$-matrices with real entries). Since G is an open set in a vector space, we have $TG = G \times \mathfrak{g}$ and any vector field is of the form $X \mapsto (X, V(X))$ i.e. is represented by a matrix valued function $V(X) = \begin{bmatrix} V_{11}(X) & \dots & V_{1n}(X) \\ & \dots & \\ V_{n1}(X) & \dots & V_{nn}(X) \end{bmatrix}$. It is easy to see that if $V \in \mathfrak{g}$, then $V_l(X) = XV, V_r = VX$. In other words, $V_l = X_{ij}V_{jk}\partial_{ik}, V_r = V_{ij}X_{jk}\partial_{ik}$. Thus we have $[V_l, W_l] = (X_{ij}V_{jk}\partial_{ik}X_{i'j'}W_{j'k'})\partial_{i'k'} - \ldots = (X_{ij}V_{jk}\delta_{ii'}\delta_{kj'}W_{j'k'})\partial_{i'k'} - \ldots = (X_{ij}V_{jk}W_{kk'})\partial_{ik'} - (X_{ij}W_{jk}V_{kk'})\partial_{ik'} = X_{ij}[V,W]_{jk'}\partial_{ik'} = [V,W]_l$ and $[V_l, W_r] = (X_{ij}V_{jk}\partial_{ik}W_{i'j'}X_{j'k'})\partial_{i'k'} - (W_{ij}X_{jk}\partial_{ik}X_{i'j'}V_{j'k'})\partial_{i'k'} = (X_{ij}V_{jk}W_{i'j'}\delta_{ij'}\delta_{kk'})\partial_{i'k'} - (W_{ij}X_{jk}\delta_{ii'}\delta_{kj'}V_{j'k'})\partial_{i'k'} = (X_{ij}V_{jk}W_{i'i})\partial_{i'k} - (W_{ij}X_{jk}V_{kk'})\partial_{ik'} = 0$.

Let us define a right action $\rho_l : v \mapsto v_l^{\sqcup} : \mathfrak{g} \to \Gamma(TT^*G)$ of $\mathfrak{g}$ on T^*G and a left action $\rho_r : v \mapsto v_r^{\sqcup} : \mathfrak{g} \to \Gamma(TT^*G)$ of $\mathfrak{g}$ on T^*G. These actions are hamiltonian, the corresponding $\mathcal{J}$-maps are given by $\mathcal{J}_l : v \mapsto -H_{v_l}$ and $\mathcal{J}_r : V \mapsto -H_{v_r}$ and the corresponding moment maps $J_l, J_r : T^*G \to \mathfrak{g}^*$ are $\langle J_l(x), v\rangle = -H_{v_l}(x), \langle J_r(x), v\rangle = -H_{v_r}(x), x \in T^*G, v \in \mathfrak{g}$.

Fact. *The orbits of the action ρ_l coincide with the fibers of the moment map J_r and vice versa.*

Proof We know that the fibers of the moment map J_r are skew-orthogonal with respect to ω to the orbits of the action ρ_r. Let us prove that the orbits of ρ_l are also skew-orthogonal to that of ρ_r.

Indeed, $\omega(\eta(H_{v_l}), \eta(H_{v_r})) = dH_{v_l}(\eta(H_{v_r})) = \eta(H_{v_r})H_{v_l} = \{H_{v_r}, H_{v_l}\}_\eta = -H_{[v_r,v_l]} = 0$. $\square$

Summarizing, we get the following dual pair of Poisson maps:

$$\begin{array}{ccc} & (T^*G, \eta) & \\ {\scriptstyle -J_r} \swarrow & & \searrow {\scriptstyle J_l} \\ (\mathfrak{g}^*, \eta_{\mathfrak{g}}) & & (\mathfrak{g}^*, \eta_{\mathfrak{g}}). \end{array}$$

Complete families of functions in involution: Let (M, η) be a Poisson structure. Let $\operatorname{Sing}\eta$ denote the union of all symplectic leaves of η of nonmaximal dimension.

We say that a set $I \subset \mathcal{E}(M)$ is a family of functions *in involution* if $\{f, g\}_\eta = 0$ for any $f, g \in I$. We say that a family I of functions in involution is *complete* if there exists an open dense set $U \subset M$ such that $\dim \operatorname{Span}\{d_x f \mid f \in I\} = \operatorname{rank}\eta_x + (1/2)\dim(M - \operatorname{rank}\eta_x)$ for any $x \in U \setminus (U \cap \operatorname{Sing}\eta)$ (in other words, the common level sets of functions from I form a lagrangian foliation in any symplectic leaf of η on $U \setminus (U \cap \operatorname{Sing}\eta)$).

Example 1. Let η be nondegenerate. Then I is complete if and only if the common level sets form a lagrangian foliation on an open dense subset in M.

Example 2. Let M be 3-dimensional and $\operatorname{rank}\eta_x = 2$ on an open dense subset $U \subset M$. Assume f is a Casimir function for η on U and g is any function whose differential is linearly independent of that of f on U. Then f, g functionally generate a complete set of functions in involution.

For instance, let $M = \mathfrak{g} = \mathfrak{so}(3,\mathbb{R}) = \mathbb{R}^3, \eta = \eta_{\mathfrak{g}}$. Then $f = x_1^2 + x_2^2 + x_3^2$ and we can take any independent g, say $g = x_1$. The corresponding lagrangian foliation consists of the circles obtained by the intersections of concentric spheres and parallel planes $\{x_1 = \mathit{const}\}$. We can take $U = \mathbb{R}^3 \setminus \{x_2 = 0, x_3 = 0\}$.

Let (M,η) be a nondegenerate Poisson structure and let $p' : M \to M', p'' : M \to M''$ be a dual pair of surjective Poisson maps. Put $\eta' := p'_*\eta, \eta'' := p''_*\eta$.

$$\begin{array}{ccc} & (M,\eta) & \\ {}^{p'}\swarrow & & \searrow^{p''} \\ (M',\eta') & & (M'',\eta''). \end{array}$$

Fact. *Assume $I' \subset \mathcal{E}(M'), I'' \subset \mathcal{E}(M'')$ are complete families of functions in involution for η', η'' respectively. Put $((p')^*I') = \{((p')^*f) \mid f \in I'\}$ and $((p'')^*I'') = \{((p'')^*g) \mid g \in I''\}$. Then the set $I := ((p')^*I') + ((p'')^*I'') \subset \mathcal{E}(M)$ is a complete family of functions in involution for η.*

Proof Let us first prove that the functions from I are in involution. Indeed, the functions form $(p')^*I'$ are in involution because so are the functions from I' and the map $(p')^*$ is a homomorphism of Poisson brackets. The same argument works for $(p'')^*I''$. Finally, any function f' from $(p')^*I'$ commutes with any function f'' from $(p'')^*I''$ due to the skew-orthogonality of the fibers of p' and p'' (recall that $\eta(f'), \eta(f'')$ are tangent to the fibers of p'', p', respectively): $\{f,g\}_\eta = \eta(f)g = -\omega(\eta(f),\eta(g)) = 0$.

Now let us prove the completeness. Let $\mathcal{F}', \mathcal{F}''$ denote the foliations of fibers of p', p'' respectively. Notice that $D := T\mathcal{F}' + T\mathcal{F}''$ is an integrable generalized distribution. Indeed, let (x,y) be local coordinates on M such that the foliation $\mathcal{F}'$ is given by $\{x^1 = c_1, \dots, x^k = c_k\}$. Then $D = \langle \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^{n-k}}, \eta(x^1), \dots, \eta(x^k)\rangle$, here $n := \dim M$. Since η is projectable along $\mathcal{F}'$, the vector fields $\eta(x^1), \dots, \eta(x^k)$ form an involutive generalized distribution (see the Liebermann–Weinstein criterion of projectability). Since the coefficients of these vector fields depend only on x they commute with $\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^{n-k}}$. Obviously the generalized foliation $\mathcal{F}$ tangent to D is the pull-back (with respect to p'') of the symplectic foliation of η'' (whose characteristic distribution is spanned by $\eta(x^1), \dots, \eta(x^k)$). Due to the symmetry of the objects with prime and double prime we deduce that $\mathcal{F}$ is also the pull-back with respect to p' of the symplectic foliation of η'. We conclude that $\operatorname{corank} \eta'_{p'(z)} = \operatorname{corank} \eta''_{p''(z)}$ for any $z \in M$ (here by definition the corank of a bivector η on a manifold M at a point $z \in M$ is the difference $\dim M - \operatorname{rank} \eta_z$).

Let U', U'' stand for the corresponding open dense sets in M', M'' appearing in the definition of the completeness of I', I''. Put $V := (p')^{-1}(U' \setminus (U' \cap \operatorname{Sing} \eta')) \cap (p'')^{-1}(U'' \setminus (U'' \cap \operatorname{Sing} \eta'')), V' := p'(V), V'' := p''(V)$. The above considerations show that and that $(p')^*\mathcal{C}_{\eta'}(V') = (p'')^*\mathcal{C}_{\eta''}(V'') =: Z$ (recall that $\mathcal{C}_\eta(U)$ denotes the space of the Casimir functions of a bivector η over an open set U).

Let us choose a functional basis $\{f_1, \dots, f_{s'}\}$ of I' such that $f_1|_{V'}, \dots, f_{r'}|_{V'}$ is a functional basis of $\mathcal{C}_{\eta'}(V')$ and any functional basis $\{g_1, \dots, g_{s''}\}$ of I''. Then the functions $(p')^*f_{r'+1}, \dots, (p')^*f_{s'}, (p'')^*g_1, \dots, (p'')^*g_{s''}$ are functionally independent on V since

$$\{(p')^*f|_V \mid f \in \mathcal{E}(V')\} \cap \{(p')^*g|_V \mid g \in \mathcal{E}(V'')\} = Z.$$

Now, we have

$$\begin{aligned} s' - r' &= \frac{1}{2}\operatorname{rank} \eta'_{p'(z)} = \frac{1}{2}(\dim T_z\mathcal{F}'' - \dim T_z\mathcal{F}'' \cap T_z\mathcal{F}'), \\ s'' &= \frac{1}{2}\operatorname{rank} \eta''_{p''(z)} + \operatorname{corank} \eta''_{p''(z)} = \frac{1}{2}(\dim T_z\mathcal{F}' - \dim T_z\mathcal{F}'' \cap T_z\mathcal{F}') + \dim T_z\mathcal{F}'' \cap T_z\mathcal{F}', \end{aligned}$$

and, finally

$$s' - r' + s'' = \frac{1}{2}(\dim T_z\mathcal{F}'' + \dim T_z\mathcal{F}') = \frac{1}{2}\dim M.$$

Here z is any point of V. $\square$

Example: the Euler–Manakov top (n-dimensional free rigid body): Let $G = SO(n, \mathbb{R}), M = T^*G$. Let $b(v, w)$ be a positively defined scalar product on $\mathfrak{so}(n, \mathbb{R})^* \cong \mathfrak{so}(n, \mathbb{R}) =: \mathfrak{g}$. Then there exists an operator $A : \mathfrak{so}(n, \mathbb{R}) \to \mathfrak{so}(n, \mathbb{R})$, which is symmetric with respect to the standard scalar product $(v, w) := -\operatorname{Tr}(vw)$, i.e. $(Av, w) = (v, Aw)$, such that $b(v, w) = (Av, w), v, w \in \mathfrak{g}$. Let $b_l : T^*Q \times T^*Q \to \mathbb{R}$ denote the left invariant extension of the scalar product b to a (contravariant) metric on Q and let $B : T^*Q \to \mathbb{R}$ denote the corresponding quadratic form.

The Euler–Manakov top is the hamiltonian system with the hamiltonian function $H := B : M \to \mathbb{R}$ in case when the operator A is given by $A := L^{-1}, Lv := Dv + vD$, where $D := \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, a diagonal matrix with the eigenvalues $\lambda_1, \ldots, \lambda_n$. The eigenvalues λ_i coincide with the "moments of inertia" $\int_V x_i^2 \sigma(x) dx$, where V is the region in $\mathbb{R}^n$ occupied by the body and $\sigma(x)$ is the density function.

Consider the classical Euler case, $n = 3$. The hamiltonian function is left invariant. This means that it belongs to the family $(p')^* I'$ in the notations of the fact above, where $p' = -J_r$. Consider the set I'' functionally generated by the Casimir function f on $\mathfrak{so}^*(3, \mathbb{R})$ and any other independent function g. The functions $H, (p'')^* f, (p'')^* g$, where $p'' := J_l$ are independent first integrals in involution. Thus we have proven the complete integrability of the Euler top (because the dimension of the phase space M is 6).

In the general case ($n > 3$) we need more functions in involution for integrating the system. In the next sections we will construct complete families of functions in involution on $\mathfrak{g}^*$ for any semisimple $\mathfrak{g}$ (these families will play a role of I''). We will also construct complete families of functions in involution on $\mathfrak{so}(n, \mathbb{R})^*$ playing the role of I' and containing the reduced hamiltonian $b(v, v) = -\operatorname{Tr}((Av)v) = \sum_{i<j} (\lambda_i + \lambda_j)^{-1} v_{ij}^2$.