Algebraic and geometric aspects of modern theory of integrable systems

Lecture 10

1 Hamiltonian actions and moment maps

A symplectic action of a Lie algebra $\mathfrak{g}$ on (M,ω): An action $\rho : \mathfrak{g} \to \Gamma(TM)$ such that $\mathcal{L}_{\rho(v)}\omega = 0$ for any $v \in \mathfrak{g}$. Here $\mathcal{L}$ is the Lie derivative, the Cartan formula for it gives:

$$\mathcal{L}_{\rho(v)}\omega = i_{\rho(v)}d\omega + di_{\rho(v)}\omega = di_{\rho(v)}\omega.$$

A weakly hamiltonian action of a Lie algebra $\mathfrak{g}$ on (M,ω): An action $\rho : \mathfrak{g} \to \Gamma(TM)$ such that there exists a linear map $\mathcal{J} : \mathfrak{g} \to \mathcal{E}(M)$ and the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathcal{J}} & \mathcal{E}(M) \\ & {\scriptstyle\rho}\searrow & \downarrow{\scriptstyle \eta(\cdot)} \\ & & \Gamma(TM) \end{array},$$

i.e. $\rho(v) = \eta(\mathcal{J}(v))$ for any $v \in \mathfrak{g}$.

Remark If $\mathcal{J}$ is finite-dimensional, we can weaken the requirement: the map $\mathcal{J}$ a priori need not be linear (i.e. we only require that any vector field $\rho(v)$ is hamiltonian). If $\mathcal{J}$ is any map with the property $\rho(\cdot) = \eta(\mathcal{J}(\cdot))$, we can make it linear: let $e_1, \ldots, e_k$ be a basis of $\mathfrak{g}$, put $\mathcal{J}'(e_i) := \mathcal{J}(e_i), i = 1, \ldots, k$, and extend this by linearity. The new map $\mathcal{J}'$ satisfies $\rho(v) = \eta(\mathcal{J}'(v))$ and is linear.

Any weakly hamiltonian action is symplectic: $d_{i_{\eta(f)}}\omega = ddf = 0$. Conversely, any symplectic action is *locally* weakly hamiltonian: $di_{\rho(v)}\omega = 0$ implies by the Poincaré lemma that $i_{\rho(v)}\omega = df$ for some function f, hence $\rho(v) = \eta(f)$.

A moment map of a weakly hamiltonian action $\rho : \mathfrak{g} \to \Gamma(TM)$: the map $J : M \to \mathfrak{g}^*$ "dual to $\mathcal{J}$", i.e.

$$\mathcal{J}(v)(x) = \langle v, J(x)\rangle, x \in M, v \in \mathfrak{g}.$$

Let $\mathcal{J}' : \mathfrak{g} \to \mathcal{E}(M)$ be another map with the property $\rho(v) = \eta(\mathcal{J}'(v))$. Then $\eta((\mathcal{J}' - \mathcal{J})(v)) = 0$, hence $C := \mathcal{J}' - \mathcal{J}$ takes values in the space of Casimir functions of η (equal to $\mathbb{R}$ if M is connected, which is assumed) and $J' = J + C$, where $C : \mathfrak{g} \to \mathbb{R}$ is a linear map. The corresponding moment map $J' : M \to \mathfrak{g}$ is given by

$$\langle v, J'(x)\rangle = \mathcal{J}'(v)(x) = \mathcal{J}(v)(x) + C(v),$$

i.e. differs from J by a constant addend $C \in \mathfrak{g}^*$.

Remark: The map $\mathcal{J}$ determines the moment map J by the formula above uniquely, but the converse also is true. Thus any smooth map $J : M \to \mathfrak{g}^*$ generates a weakly hamiltonian action of $\mathfrak{g}$ on M such that one of its moment maps coincide with J.

A hamiltonian action of a Lie algebra $\mathfrak{g}$ on (M, ω)**:** A weakly hamiltonian action $\rho : \mathfrak{g} \to \Gamma(TM)$ such that among linear maps $\mathcal{J} : \mathfrak{g} \to \mathcal{E}(M)$ with the property $\rho(\cdot) = \eta(\mathcal{J}(\cdot))$ there exists a homomorphism of Lie algebras $(\mathfrak{g}, [,])$ and $(\mathcal{E}(M), \{,\}_\eta)$.

Remark Note that for any other map $\mathcal{J}' = \mathcal{J} + C$ with $\rho(\cdot) = \eta(\mathcal{J}'(\cdot))$ we have: $\mathcal{J}'[v,w] = \mathcal{J}[v,w] + C([v,w]) = \{\mathcal{J}v, \mathcal{J}w\}_\eta + C([v,w]) = \{\mathcal{J}v + C(v), \mathcal{J}w + C(w)\}_\eta + C([v,w]) = \{\mathcal{J}'v, \mathcal{J}'w\}_\eta + C([v,w])$. thus $\mathcal{J}'$ is a homomorphism if an only if C vanishes on the commutant $[\mathfrak{g}, \mathfrak{g}] = \{[v,w] \mid v, w \in \mathfrak{g}\}$ of the Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ is semisimple (as $\mathfrak{sl}(n, \mathbb{R}), \mathfrak{so}(n, \mathbb{R}), \mathfrak{sp}(n, \mathbb{R})$), we have $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, hence the homomorphic $\mathcal{J}$ is defined uniquely.

FACT. *A map $\mathcal{J} : \mathfrak{g} \to \mathcal{E}(M)$ is a homomorphism if and only if the corresponding moment map $J : M \to \mathfrak{g}^*$ is Poisson, here $\mathfrak{g}^*$ is endowed with the Lie-Poisson structure $\eta_\mathfrak{g}$.*

Proof Let $e_1, \ldots, e_n$ be a basis of $\mathfrak{g}$ and let $y_1, \ldots, y_n$ be the the elements of this basis regarded as linear functions on $\mathfrak{g}^*$. With these notation we have in view of f the definition of the moment map the following equalities: $\mathcal{J}e_i = J^* y_i, i = 1, \ldots, n$.

Denote by c_{ij}^k the corresponding structure constants: $[e_i, e_j] = c_{ij}^k e_k$. Assume $\mathcal{J}$ is a homomorphism, i.e. $\{\mathcal{J}e_i, \mathcal{J}e_j\}_\eta = c_{ij}^k \mathcal{J}e_k$. This can be rewritten as $\{J^* y_i, J^* y_j\}_\eta = c_{ij}^k J^* y_k = J^* \{y_i, y_j\}_{\eta_\mathfrak{g}}$, which means the Poisson property of the moment map. Inverting the considerations we get also another implication. □

Remark: Similarly to the case of weakly hamiltonian actions any smooth Poisson map $J : M \to \mathfrak{g}^*$ generates a hamiltonian action of g on M such that one of its moment maps coincide with J.

Hamiltonian actions and projectability: Let $\rho : \mathfrak{g} \to \Gamma(TM)$ be a hamiltonian action such that its orbits form a foliation $\mathcal{F}$ and the factor space $M/\mathcal{F}$ is good. Let $p : M \to M' := M/\mathcal{F}$ be the natural projection. Then $\eta := \omega^{-1}$ is projectable with respect to p. Indeed, $T\mathcal{F} = \langle \eta(\mathcal{J}e_1), \ldots, \eta(\mathcal{J}e_n) \rangle$ and the dual foliation is given by $\{\mathcal{J}e_1 = c_1, \ldots, \mathcal{J}e_n = c_n\}$, i.e. coincides with the fibers of the moment map. As a result we get a dual pair of Poisson maps

$$\begin{array}{ccc} & (M, \eta) & \\ {\scriptstyle p} \swarrow & & \searrow {\scriptstyle J} \\ (M', p_* \eta) & & (\mathfrak{g}^*, \eta_\mathfrak{g}). \end{array}$$

Example 1: Let $H : M \to \mathbb{R}$ be any function with the nonvanishing differential. Then we have $\rho : \mathbb{R} \to \Gamma(TM), 1 \mapsto \eta(H), \mathcal{J} : \mathbb{R} \to \mathcal{E}(M), 1 \mapsto H, J = H, T\mathcal{F} = \langle \eta(H) \rangle$

$$\begin{array}{ccc} & (M, \eta) & \\ {\scriptstyle p} \swarrow & & \searrow {\scriptstyle H=J} \\ (M/\mathcal{F}, p_* \eta) & & (\mathbb{R} = \mathbb{R}^*, 0). \end{array}$$

In particular, if $M := T^*\mathbb{R}^2 \setminus \{0\}, \omega = dp \wedge dq, H(q,p) = q_1^2 + q_2^2 + (p^1)^2 + (p^2)^2$, we get the Hopf fibrations over the symplectic leaves of $p_* \eta$.

Example 2: Let $M \subset \mathfrak{g}^*$ be a coadjoint orbit endowed with the canonical symplectic form $\omega := (\eta_{\mathfrak{g}}|_M)^{-1}$. Then the coadjoint action $\rho : \mathfrak{g} \to \Gamma(T\mathfrak{g}^*), v \mapsto \widetilde{ad_v^*}$ is hamiltonian. Indeed, $\widetilde{ad_v^*} = \eta_{\mathfrak{g}}(v')$ (see Lecture **??**) where v' denotes the linear function on $\mathfrak{g}^*$ defined by an element $v \in \mathfrak{g}$. Thus $\mathcal{J} : \mathfrak{g} \to \mathcal{E}(M)$ is given by $v \mapsto v'|_M$ and $J : M \to \mathfrak{g}^*$ coincides with the inclusion $M \hookrightarrow \mathfrak{g}^*$.

Example 3: Let $\rho : \mathfrak{g} \to \Gamma(TM)$ be a hamiltonian action with a moment map $J : M \to \mathfrak{g}^*$ and let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. Then $\rho|_{\mathfrak{h}} : \mathfrak{h} \to \Gamma(TM)$ is a hamiltonian action and its moment map $J_{\mathfrak{h}} : M \to \mathfrak{h}^*$ is given by $i^* \circ J$, where $i^* : \mathfrak{g}^* \to \mathfrak{h}^* = \mathfrak{g}/\mathfrak{h}^{\perp}$ is the projection dual to the inclusion $i : \mathfrak{h} \hookrightarrow \mathfrak{g}$.

Remark about relations between weakly hamiltonian and hamiltonian actions: Let $\rho : \mathfrak{g} \to \Gamma(TM)$ be a weakly hamiltonian action. Let us examine obstructions for ρ to be a hamiltonian action.

Let $\mathcal{J} : \mathfrak{g} \to \mathcal{E}(M)$ be map with the property $\rho(\cdot) = \eta(\mathcal{J}(\cdot))$. Put $c(v, w) := \{\mathcal{J}v, \mathcal{J}w\}_\eta - \mathcal{J}([v, w])$.

FACT. *1. $c(v, w)$ is a constant function for any $v, w \in \mathfrak{g}$;*

2. c is a 2-cocycle on the Lie algebra $\mathfrak{g}$, i.e. it is a bilinear skew-symmetric function on $\mathfrak{g}$ satisfying $\sum_{c.p.\ v,w,u} c([v, w], u) = 0$ for any $v, w, u \in \mathfrak{g}$.

Proof Item 1. We have $\eta(c(v, w)) = \eta(\{\mathcal{J}v, \mathcal{J}w\}_\eta - \mathcal{J}([v, w])) = [\eta(\mathcal{J}v), \eta(\mathcal{J}w)] - \rho([v, w]) = [\rho(v), \rho(w)] - \rho([v, w]) = 0$, hence $c(v, w)$ is a Casimir function for η.
Item 2. We have $\{\mathcal{J}[v, w], \mathcal{J}u\}_\eta = \eta(\mathcal{J}[v, w])\mathcal{J}u = \rho([v, w])\mathcal{J}u = [\rho(v), \rho(w)]\mathcal{J}u = \rho(v)\rho(w)\mathcal{J}u - \rho(w)\rho(v)\mathcal{J}u = \rho(v)\eta(\mathcal{J}w)\mathcal{J}u - \rho(w)\eta(\mathcal{J}v)\mathcal{J}u = \rho(v)\{\mathcal{J}w, \mathcal{J}u\}_\eta - \rho(w)\{\mathcal{J}v, \mathcal{J}u\}_e = \{\mathcal{J}v, \{\mathcal{J}w, \mathcal{J}u\}_\eta\}_\eta - \{\mathcal{J}w, \{\mathcal{J}v, \mathcal{J}u\}_\eta\}_\eta$.

Hence $\sum_{c.p.\ v,w,u} c([v, w], u) = \sum_{c.p.\ v,w,u} \{\mathcal{J}[v, w], \mathcal{J}u\}_\eta - \mathcal{J}([[v, w], u])) = 0$ due to the Jacobi identity for $[,]$ and $\{,\}_\eta$. □

It is known that for a semisimple $\mathfrak{g}$ any 2-cocycle c is cohomologically trivial, i.e. there exists $C \in \mathfrak{g}^*$ such that $c(v, w) = C([v, w])$.

FACT. *If the cocycle c is trivial, the map $\mathcal{J}' := \mathcal{J} + C : \mathfrak{g} \to \mathcal{E}(M)$ is a homomorphism.*

Proof $\mathcal{J}'([v, w]) = \mathcal{J}([v, w]) + C([v, w]) = \mathcal{J}([v, w]) + \{\mathcal{J}v, \mathcal{J}w\}_\eta - \mathcal{J}([v, w]) = \{\mathcal{J}v, \mathcal{J}w\}_\eta = \{\mathcal{J}'v, \mathcal{J}'w\}_\eta$. □

We conclude that for semisimple $\mathfrak{g}$ any weakly hamiltonian action is hamiltonian.

In general, the cocycle c is nontrivial. Note that c is defined nonuniquely, since so is the map $\mathcal{J}$. Taking $\mathcal{J}' = \mathcal{J} + C$ (see one of the *Remarks* above) we get the formula $c'(v, w) = \{\mathcal{J}'v, \mathcal{J}'w\}_\eta - \mathcal{J}'([v, w]) = \{\mathcal{J}v, \mathcal{J}w\}_\eta - \mathcal{J}([v, w]) - C([v, w]) = c(v, w) - C([v, w])$, i.e. the nontriviality of c does not depend on the choice of $\mathcal{J}$. So there exist weakly hamiltonian actions not being hamiltonian. For such actions the moment maps are not Poisson, but one can modify the Poisson structure on $\mathfrak{g}^*$ (adding a cocycle to $\eta_{\mathfrak{g}}$ and obtaining a Poisson structure with affine coefficients) in such a way that the moment maps will be Poisson.