Algebraic and geometric aspects of modern theory of integrable systems

Lecture 1

1. Sketch of the (introductory part of the) course

1. Symplectic manifolds and hamiltonian equations.
Symplectic manifold: (M, ω), $\omega \in \Gamma(\bigwedge^2 T^*M)$, locally $\omega = \omega_{ij}(x) dx^i \wedge dx^j$, ω nondegenerate (i.e. the matrix ω_{ij} nondegenrate) and closed, $d\omega = 0$.
Prototype: the canonical symplectic form on T^*Q, $\omega = dp_i \wedge dq^i$, here q^i local coordinates on Q, p_i the corresponding momenta.
Hamiltonian differential equation on (M, ω): a differential equation given by a "hamiltonian" vector field $v(H) := \omega_{ij}^{-1}(x) \frac{\partial H}{\partial x^i}$, here $H \in C^\infty(M)$, a hamiltonian function.
Prototype: the "natural" mechanical system on $M = T^*Q$, the hamiltonian function $H(p) = -(1/2m) \parallel p \parallel^2 - U(\pi_M(p)) \in C^\infty(T^*Q)$ is the Legendre transform of the lagrangian function $L = (m/2) \parallel w \parallel^2 - U(\tau_M(w)) \in C^\infty(TQ)$, here $\parallel \cdot \parallel$ is a norm on tangent vectors generated by some Riemannian metric on Q.

2. Completely integrable systems in the Arnold–Liouville sense.
First integral of differential equation given by a vector field v on M: a function $f \in C^\infty(M)$ such that $vf \equiv 0$.
First integrals in involution of a hamiltonian vector field $v(H)$ on (M, ω): functions $f_1, f_2, \ldots$ such that $v(H)f_i \equiv 0$ and $\{f_i, f_j\} \equiv 0$, here $\{f, g\} := v(f)g$ (the Poisson bracket of functions f, g). In particular, $\{H, f_i\} \equiv 0$.
The Arnold–Liouville theorem: Let (M, ω) be symplectic, $\dim M = 2n$. Assume a hamiltonian vector field $v(H)$ admits n functionally independent integrals in involution. Then

1. if the common level sets of these integrals are compact and connected, they are (n-dimensional) tori $T^n = \{(\varphi_1, \ldots, \varphi_n) \mathrm{mod} 2\pi\}$;
2. the restriction of the initial hamiltonian equation to T^n gives an almost periodic motion on T^n, i.e. in the "angle coordinates" φ the equation has the form
$$\frac{d\overrightarrow{\varphi}}{dt} = \overrightarrow{a},$$
here $\overrightarrow{a} = (a_1, \ldots, a_n)$ is a constant vector depending only on the level;
3. the initial equation can be integrated in "quadratures", i.e. the solutions can be obtained by means of a finite number of algebraic operations and operations of taking integral.

Integrable contra chaotic systems: A trajectory of "chaotic" system can be dense in a phase space M, a trajectory of "integrable" system lies on tori of dimension $\leqslant (1/2)\dim M$.

3. Poisson manifolds and Lie algebras

Nondegenerate Poisson manifolds: $(M, \vartheta), \vartheta \in \Gamma(\bigwedge^2 TM)$, locally $\vartheta = \vartheta^{kl}\frac{\partial}{\partial x^k} \wedge \frac{\partial}{\partial x^l}$, such that $\vartheta^{kl} = \omega^{-1}_{kl}$, here $\omega = \omega_{ij}(x)dx^i \wedge dx^j$ a symplectic form. How to encode the condition $d\omega = 0$ in terms of ϑ? One of possible answers: Jacobi identity for the Poisson bracket $\{f, g\} := \vartheta^{kl}\frac{\partial f}{\partial x^k}\frac{\partial g}{\partial x^l}$, $\{\{f, g\}, h\} + \text{cyclic permutations} = 0$.

General Poisson manifolds: $(M, \vartheta), \vartheta \in \Gamma(\bigwedge^2 TM)$ such that the Jacobi identity holds for the corresponding Poisson bracket.

Symplectic leaves and Casimir functions of Poisson manifolds: Given a Poisson manifold (M, ϑ), there exist a splitting $M = \bigcup_{t\in T} M_t$ of the manifold M to submanifolds M_t such that $\vartheta|_{M_t}$ is nondegenerate, i.e. inverse to some symplectic form. Casimir function is a function $f \in C^\infty(M)$ such that $\vartheta^{kl}\frac{\partial f}{\partial x^k} \equiv 0$, i.e. $\{f, g\} = 0$ for any $g \in C^\infty(M)$. Another characterization of Casimir functions: functions whose level sets coincide with the symplectic leaves M_t.

Lie algebras: A vector space $\mathfrak{g}$ with a skew-symmetric binary operation $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying Jacobi identity.

Examples of Lie algebras:

1. $\mathfrak{g} := \mathfrak{gl}(n, \mathbb{R})$, real $n \times n$-matrices, $[A, B] := AB - BA$, commutator of matrices;
2. $\mathfrak{g} := \mathfrak{sl}(n, \mathbb{R})$, real $n \times n$-matrices with zero trace;
3. $\mathfrak{g} := \mathfrak{so}(n, \mathbb{R})$, real skew-symmetric $n \times n$-matrices.

Lie-Poisson structures as examples of Poisson manifolds: Given a Lie algebra $\mathfrak{g}$ and a basis $e_1, \ldots, e_n$, let c^k_{ij} be the coresponding structure constants, i.e. $[e_i, e_j] = c^k_{ij}e_k$. Put $\vartheta_{\mathfrak{g}} := c^k_{ij}x_k\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}$, here $x^i = e_i$ (elements of $\mathfrak{g}$ regarded as linear functions on the dual space $\mathfrak{g}^*$). Then $\vartheta_{\mathfrak{g}}$ is a Poisson structure on $\mathfrak{g}^*$.

Symplectic leaves of the Lie-Poisson structures: They coincide with the so-called coadjoint orbits on $\mathfrak{g}^*$. For instance, take one of the Lie algebras from the examples above. Then it has a scalar product $(A|B) := Tr(AB)$ by means of which we can identify $\mathfrak{g}$ with $\mathfrak{g}^*$. After this identification the symplectic leaves of $\vartheta_{\mathfrak{g}}$ become $\{XYX^{-1} \mid X \in G\}$, here $Y \in \mathfrak{g}$ is fixed, G is the set (the group) of 1.) nondegenerate $n\times n$-matrices; 2.) $n\times n$-matrices with determinant one; 3.) orthogonal $n \times n$-matrices (i.e. $XX^T = I$). The corresponding Casimir functions are $Tr(X), Tr(X^2), \ldots$.

3. Poisson and manifolds and reductions

"The Noether principle": If a vector field on $\mathbb{R}^n$ admits a one-parametric group of diffeomorphisms preserving this vector field, the problem of integrating of the corresponding differential equation is reduced to a problem of integrating of another differential equation on $\mathbb{R}^{n-1}$.

Symplectic version of the Noether principle: If a hamiltonian function of a hamiltonian equation is invariant under some one-parametric group of transformations of the phase space M^{2n} which preserve also the symplectic form, then the equation can be reduced to another hamiltonian equation on a phase space of dimension $2n - 2$. Looking a little bit more globally one can say that we will reduce our initial system to a hamiltonian system on some Poisson manifold of dimension $2n-1$ (the above mentioned phase space of dimension $2n - 2$ is a symplectic leaf).

Example 1, rotation invariant natural mechanical system: Take $Q = \mathbb{R}^n$, the euclidian metric and a rotation invariant potential U. The group of rotations of Q can be extended to a group of diffeomorphisms of T^*Q, preserving the canonical symplectic form (the hamiltonian H will be also preserved by this group).

Example 2, the Euler top: The mechanical system of free rigid body: the configuration space Q is $SO(3) = \{X \in \mathfrak{gl}(3,\mathbb{R}) \mid XX^T = I\}$; the potential is zero, the metric is a "left-invariant" (i.e. invariant with respect to left translations $Y \mapsto XY$ of $SO(3)$) metric on $SO(3)$ depending of the shape of the body. The Noether principle (using the whole 3-parametric group of symmetries of T^*Q and H obtained from extension of the left translations to T^*Q) allows to reduce this system from T^*Q to $\mathfrak{so}(3,\mathbb{R})^*$ with the Lie–Poisson structure $\vartheta_{\mathfrak{so}(3,\mathbb{R})}$. The problem of finding first integrals in involution is now carried from a (bigger) symplectic manifold to a (smaller) Poisson manifold.

4. What will we do afterwards?

The main ideas which we will try to implement are

1. to find some mechanisms of building big families of functions in involution (with respect to $\vartheta_{\mathfrak{g}}$) on Lie algebras $\mathfrak{g}$ (on their duals $\mathfrak{g}^*$);

2. to recognize among these functions some "physically reasonable" hamiltonians;

3. to prove that the remaining functions (interpreted as first integrals of the corresponding hamiltonian equation) form a "complete" family, i.e. lead to a completely integrable system.