Algebraic and geometric aspects of modern theory of integrable systems

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Contents

1 Sketch of the (introductory part of the) course

1. Symplectic manifolds and hamiltonian equations.

Symplectic manifold: (M, ω), $\omega \in \Gamma(\bigwedge^2 T^*M)$, locally $\omega = \omega_{ij}(x)dx^i \wedge dx^j$, ω nondegenerate (i.e. the matrix ω_{ij} nondegenrate) and closed, $d\omega = 0$.

Prototype: the canonical symplectic form on T^*Q, $\omega = dp_i \wedge dq^i$, here q^i local coordinates on Q, p_i the corresponding momenta.

Hamiltonian differential equation on (M, ω): a differential equation given by a "hamiltonian" vector field $v(H) := \omega_{ij}^{-1}(x)\frac{\partial H}{\partial x^i}$, here $H \in C^\infty(M)$, a hamiltonian function.

Prototype: the "natural" mechanical system on $M = T^*Q$, the hamiltonian function $H(p) = -(1/2m) \parallel p \parallel^2 - U(\pi_M(p)) \in C^\infty(T^*Q)$ is the Legendre transform of the lagrangian function $L = (m/2) \parallel w \parallel^2 - U(\tau_M(w)) \in C^\infty(TQ)$, here $\parallel \cdot \parallel$ is a norm on tangent vectors generated by some Riemannian metric on Q.

2. Completely integrable systems in the Arnold–Liouville sense.

First integral of differential equation given by a vector field v on M: a function $f \in C^\infty(M)$ such that $vf \equiv 0$.

First integrals in involution of a hamiltonian vector field $v(H)$ on (M, ω): functions $f_1, f_2, \ldots$ such that $v(H)f_i \equiv 0$ and $\{f_i, f_j\} \equiv 0$, here $\{f, g\} := v(f)g$ (the Poisson bracket of functions f, g). In particular, $\{H, f_i\} \equiv 0$.

The Arnold–Liouville theorem: Let (M, ω) be symplectic, $\dim M = 2n$. Assume a hamiltonian vector field $v(H)$ admits n functionally independent integrals in involution. Then

1. if the common level sets of these integrals are compact and connected, they are (n-dimensional) tori $\mathbb{T}^n = \{(\varphi_1, \ldots, \varphi_n) \mathrm{mod} 2\pi\}$;
2. the restriction of the initial hamiltonian equation to $\mathbb{T}^n$ gives an almost periodic motion on $\mathbb{T}^n$, i.e. in the "angle coordinates" φ the equation has the form
$$\frac{d\overrightarrow{\varphi}}{dt} = \overrightarrow{a},$$
here $\overrightarrow{a} = (a_1, \ldots, a_n)$ is a constant vector depending only on the level;
3. the initial equation can be integrated in "quadratures", i.e. the solutions can be obtained by means of a finite number of algebraic operations and operations of taking integral.

Integrable contra chaotic systems: A trajectory of "chaotic" system can be dense in a phase space M, a trajectory of "integrable" system lies on tori of dimension $\leqslant (1/2)\dim M$.

3. Poisson manifolds and Lie algebras

Nondegenerate Poisson manifolds: $(M, \eta), \eta \in \Gamma(\bigwedge^2 TM)$, locally $\eta = \eta^{kl}\frac{\partial}{\partial x^k} \wedge \frac{\partial}{\partial x^l}$, such that $\eta^{kl} = \omega_{kl}^{-1}$, here $\omega = \omega_{ij}(x)dx^i \wedge dx^j$ a symplectic form. How to encode the condition $d\omega = 0$ in terms of η? One of possible answers: Jacobi identity for the Poisson bracket $\{f,g\} := \eta^{kl}\frac{\partial f}{\partial x^k}\frac{\partial g}{\partial x^l}$, $\{\{f,g\},h\} + \text{cyclic permutations} = 0$.

General Poisson manifolds: $(M,\eta), \eta \in \Gamma(\bigwedge^2 TM)$ such that the Jacobi identity holds for the corresponding Poisson bracket.

Symplectic leaves and Casimir functions of Poisson manifolds: Given a Poisson manifold (M,η), there exist a splitting $M = \bigcup_{t\in T} M_t$ of the manifold M to submanifolds M_t such that $\eta|_{M_t}$ is nondegenerate, i.e. inverse to some symplectic form. Casimir function is a function $f \in C^\infty(M)$ such that $\eta^{kl}\frac{\partial f}{\partial x^k} \equiv 0$, i.e. $\{f,g\} = 0$ for any $g \in C^\infty(M)$. Another characterization of Casimir functions: functions whose level sets coincide with the symplectic leaves M_t.

Lie algebras: A vector space $\mathfrak{g}$ with a skew-symmetric binary operation $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying Jacobi identity.

Examples of Lie algebras:

1. $\mathfrak{g} := \mathfrak{gl}(n,\mathbb{R})$, real $n \times n$-matrices, $[A,B] := AB - BA$, commutator of matrices;
2. $\mathfrak{g} := \mathfrak{sl}(n,\mathbb{R})$, real $n \times n$-matrices with zero trace;
3. $\mathfrak{g} := \mathfrak{so}(n,\mathbb{R})$, real skew-symmetric $n \times n$-matrices.

Lie-Poisson structures as examples of Poisson manifolds: Given a Lie algebra $\mathfrak{g}$ and a basis $e_1,\ldots,e_n$, let c_{ij}^k be the coresponding structure constants, i.e. $[e_i,e_j] = c_{ij}^k e_k$. Put $\eta_{\mathfrak{g}} := c_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, here $x^i = e_i$ (elements of $\mathfrak{g}$ regarded as linear functions on the dual space $\mathfrak{g}^*$). Then $\eta_{\mathfrak{g}}$ is a Poisson structure on $\mathfrak{g}^*$.

Symplectic leaves of the Lie-Poisson structures: They coincide with the so-called coadjoint orbits on $\mathfrak{g}^*$. For instance, take one of the Lie algebras from the examples above. Then it has a scalar product $(A|B) := Tr(AB)$ by means of which we can identify $\mathfrak{g}$ with $\mathfrak{g}^*$. After this identification the symplectic leaves of $\eta_{\mathfrak{g}}$ become $\{XYX^{-1} \mid X \in G\}$, here $Y \in \mathfrak{g}$ is fixed, G is the set (the group) of 1.) nondegenerate $n\times n$-matrices; 2.) $n\times n$-matrices with determinant one; 3.) orthogonal $n \times n$-matrices (i.e. $XX^T = I$). The corresponding Casimir functions are $Tr(X), Tr(X^2), \ldots$.

3. Poisson and manifolds and reductions

"The Noether principle": If a vector field on $\mathbb{R}^n$ admits a one-parametric group of diffeomorphisms preserving this vector field, the problem of integrating of the corresponding differential equation is reduced to a problem of integrating of another differential equation on $\mathbb{R}^{n-1}$.

Symplectic version of the Noether principle: If a hamiltonian function of a hamiltonian equation is invariant under some one-parametric group of transformations of the phase space M^{2n} which preserve also the symplectic form, then the equation can be reduced to another hamiltonian equation on a phase space of dimension $2n-2$. Looking a little bit more globally one can say that we will reduce our initial system to a hamiltonian system on some Poisson manifold of dimension $2n-1$ (the above mentioned phase space of dimension $2n-2$ is a symplectic leaf).

Example 1, rotation invariant natural mechanical system: Take $Q = \mathbb{R}^n$, the euclidian metric and a rotation invariant potential U. The group of rotations of Q can be extended to a group of diffeomorphisms of T^*Q, preserving the canonical symplectic form (the hamiltonian H will be also preserved by this group).

Example 2, the Euler top: The mechanical system of free rigid body: the configuration space Q is $SO(3) = \{X \in \mathfrak{gl}(3, \mathbb{R}) \mid XX^T = I\}$; the potential is zero, the metric is a "left-invariant" (i.e. invariant with respect to left translations $Y \mapsto XY$ of $SO(3)$) metric on $SO(3)$ depending of the shape of the body. The Noether principle (using the whole 3-parametric group of symmetries of T^*Q and H obtained from extension of the left translations to T^*Q) allows to reduce this system from T^*Q to $\mathfrak{so}(3, \mathbb{R})^*$ with the Lie–Poisson structure $\eta_{\mathfrak{so}(3,\mathbb{R})}$. The problem of finding first integrals in involution is now carried from a (bigger) symplectic manifold to a (smaller) Poisson manifold.

4. What will we do afterwards?

The main ideas which we will try to implement are

1. to find some mechanisms of building big families of functions in involution (with respect to $\eta_{\mathfrak{g}}$) on Lie algebras $\mathfrak{g}$ (on their duals $\mathfrak{g}^*$);

2. to recognize among these functions some "physically reasonable" hamiltonians;

3. to prove that the remaining functions (interpreted as first integrals of the corresponding hamiltonian equation) form a "complete" family, i.e. lead to a completely integrable system.

2 Preliminaries on manifolds

A chart on a topological space M: A pair (U, ψ), here $U \subset M$ is an open set, $\psi : U \to \mathbb{R}^n$ is a homeomorphism onto its image. Two charts $(U_1, \psi_1), (U_2, \psi_2)$ are *compatible* if $\psi_1 \circ \psi_2^{-1}|_{\mathrm{im}(U_1 \cap U_2)} : \mathrm{im}(U_1 \cap U_2) \to \mathbb{R}^n$ is smooth (analytical) mapping. The components of the vector $\psi = (\psi_1, \ldots, \psi_n)$ are called *local coordinates* on M.

An atlas on a topological space M: A collection of pairwise compatible charts $\mathcal{A} := \{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ such that $M = \bigcup_{\alpha \in A} U_\alpha$. Two atlases are *equivalent or compatible* if ...

A manifold: A topological space endowed with a class of equivalent atlases.

Example: The sphere S^2 with two stereographic projections (from the north and south poles).

A vector bundle $E \to M$ over a manifold M: A surjective map $\pi : E \to M$, here E is a topological space, such that here is a structure of a vector space on each fiber $E_x := \pi^{-1}(x), x \in M$, and there is an atlas $\mathcal{A} := \{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ on M and homeomorphisms $\Psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^m$ with the properties:

1. the following diagram is commutative

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\Psi_\alpha} & U_\alpha \times \mathbb{R}^m \\ \downarrow \pi & & \downarrow \pi_1 \\ U_\alpha & = & U_\alpha \end{array} ;$$

2. the map $\widetilde{\Psi}_{\alpha,x} := \Psi_\alpha|_{E_x}$ is a linear isomorphism of the vector spaces E_x and $\mathbb{R}^m$;

3. the collection $\{(\pi^{-1}(U_\alpha), \Psi_\alpha)\}_{\alpha \in A}$ is an atlas on E, in particular $\Psi_\alpha \circ \Psi_\beta^{-1}(x, y) = (x, \widetilde{\Psi}_{\alpha,x} \circ \widetilde{\Psi}_{\beta,x}^{-1}(y)), x \in U_\alpha \cap U_\beta, y \in \mathbb{R}^m$, and the functions $\widetilde{\Psi}_{\alpha\beta,x} := \widetilde{\Psi}_{\alpha,x} \circ \widetilde{\Psi}_{\beta,x}^{-1}$ are linear isomorphisms of $\mathbb{R}^m$ which smoothly depend on $x \in M$.

The functions $\widetilde{\Psi}_{\alpha\beta,x}$ are called *transition functions* of the vector bundle. Given the base M and the collection of transition functions, one can reconstruct the initial vector bundle (up to an isomorphism).

A section of a vector bundle $E \to M$: A mapping $s : M \to E$ such that $\pi(s(x)) = x$ for any $x \in M$. The space of sections will be denoted by $\Gamma(E)$.

Example 1, the tangent bundle $TM \xrightarrow{\tau_M} M$: Let M be a manifold with an atlas $\mathcal{A} := \{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$. Put $\widetilde{\Psi}_{\alpha\beta,x} := \frac{\partial \psi_{\alpha\beta}(\varphi_\beta(x))}{\partial \varphi_\beta}$, here $\psi_{\alpha\beta} := \psi_\alpha \circ \psi_\beta^{-1} : \mathbb{R}^n \to \mathbb{R}^n$. Below we give an explicit description of TM.

A tangent vector at x to M: A *curve* in M is a mapping $c : \mathbb{R} \to M$. Two curves c_1, c_2 such that $c_1(0) = c_2(0) = x$ are *equivalent* at x if the derivatives of the functions $f(c_1(t))$ and $f(c_2(t))$ coincide at 0 for any $f \in \mathcal{E}(M)$ ($\mathcal{E}(M)$ is $C^\infty(M)$ or the space of analytic functions on M depending on the category). Note that c_1, c_2 are equivalent at x if and only if $\frac{d}{dt}|_{t=0}(\psi^i \circ c_1)(t) = \frac{d}{dt}|_{t=0}(\psi^i \circ c_2)(t), i = 1, \ldots, n$, for some (consequently for any) chart (U, ψ) with $x \in U$.

A class $v = [c]_x$ of equivalence of curves at x is called a *tangent vector* at x. We say that v is *tangent* to c (and to any other representative of the class) at x. A tangent vector in local coordinates $(\psi^1, \dots, \psi^n)$ is represented by the n-tuple $(\frac{d}{dt}|_{t=0}(\psi^1 \circ c)(t), \dots, \frac{d}{dt}|_{t=0}(\psi^n \circ c)(t))$, here c is any representative of the class. Since we can add such n-tuples and multiply them by scalars, the set of tangent vectors inherits a structure of vector space (which is independent of the choice of local coordinates). Given two local coordinate systems ψ_α, ψ_β the corresponding n-tuples are related by

$$\frac{d}{dt}|_{t=0}(\psi^i_\alpha \circ c)(t) = \frac{\partial \psi^i_{\alpha\beta}(\varphi_\beta(x))}{\partial \varphi^j_\beta} \frac{d}{dt}|_{t=0}(\psi^j_\beta \circ c)(t).$$

Tangent vectors as differentiations: A *differentiation* of the ring $\mathcal{E}(M)$ at x is a linear mapping $l : \mathcal{E}(M) \to \mathbb{R}$ such that $l(fg) = l(f)g(x) + f(x)l(g), f, g \in \mathcal{E}(M)$. Given a tangent vector v at x which is represented by a curve c, we construct a differentiation $\tilde{v}$ by $\tilde{v}(f) := \frac{d}{dt}|_{t=0}(f \circ c)(t)$. It does not depend on the choice of representative.

Let $\psi = (\psi^1, \dots, \psi^n) : U \to \mathbb{R}^n$ be local coordinates on M such that $\psi(x) = 0$. Then $c := \psi^{-1}(L^i)$, where L^i is the i-th coordinate line in $\mathbb{R}^n$, gives (a local) curve with $c(0) = x$. The corresponding vector is denoted $\frac{\partial}{\partial \psi^i}$. The vectors (differentiations) $\frac{\partial}{\partial \psi^i}, i = 1, \dots, n$, form a basis of the vector space $T_x M$.

A vector field on M: A section of the tangent bundle TM, i.e. a tangent vector $v(x) \in T_x M$ (smoothly, analytically) depending on $x \in M$. In a local chart (U, ψ) can be expressed as $v(x) = v^i(x)\frac{\partial}{\partial \psi^i}$, here $v^i(x)$ are functions.

Any vector field v is a differentiation of the ring $\mathcal{E}(M)$, i.e. a linear endomorphism of $\mathcal{E}(M)$ such that $v(fg) = v(f)g + fv(g), f, g \in \mathcal{E}(M)$. In local coordinates $(vf)(x) = v^i(x)\frac{\partial f}{\partial \psi^i}(x)$.

The space $\Gamma(TM)$ of vector fields is a vector field over $\mathbb{R}$ and a module over the ring $\mathcal{E}(M)$.

The commutator of vector fields on M: Given two differentiations v_1, v_2 of the ring $\mathcal{E}(M)$, the commutator $[v_1, v_2] := v_1 v_2 - v_2 v_1$ is again a differentiation: $v_1 v_2(fg) = v_1((v_2 f)g + f(v_2)g) = (v_1 v_2 f)g + (v_2 f)(v_1 g) + (v_1 f)(v_2 g) + f(v_1 v_2 g)$, so $[v_1, v_2](fg) = ([v_1, v_2]f)g - f([v_1, v_2]g)$. In local coordinates $[v_1, v_2]^i(x) = v^j_1(x)\frac{\partial v^i_2(x)}{\partial \psi^j} - v^j_2(x)\frac{\partial v^i_1(x)}{\partial \psi^j}$.

A bivector field on M: A section η of the second exterior power of the tangent bundle $\bigwedge^2 TM$. Locally $\eta = \eta^{ij}(x)\frac{\partial}{\partial \psi^i} \wedge \frac{\partial}{\partial \psi^j}$.

Example 2, the cotangent bundle $T^*M \xrightarrow{\pi_M} M$: The bundle dual to TM. The transition functions: $\widetilde{\Psi}^{-1}_{\alpha\beta,x}$. We denote by $d\psi^1, \dots, d\psi^n$ the basis of $T^*_x M$ dual to the basis $\frac{\partial}{\partial \psi^1}, \dots, \frac{\partial}{\partial \psi^n}$.

A covector field on M (differential 1-form): A section γ of the bundle T^*M. Locally $\gamma = \gamma_i(x)d\psi^i$.

A differential 2-form on M: A section ω of the second exterior power of the cotangent bundle $\bigwedge^2 T^*M$. Locally $\omega = \omega_{ij}(x)d\psi^i \wedge d\psi^j$.

A morphism of vector bundles $E_1 \xrightarrow{\pi_1} M, E_2 \xrightarrow{\pi_2} M$ over M: A map $\mu : E_1 \to E_2$ such that the following diagram is commutative

$$\begin{array}{ccc} E_1 & \xrightarrow{\mu} & E_2 \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ M & = & M \end{array}$$

and the induced mappings $\mu_x : E_{1,x} \to E_{2,x}$ are linear for any $x \in M$.

Differential k-forms as morphisms $\bigotimes^k TM \to M \times \mathbb{R}$: any differential k-form σ can be interpreted as such a morphism which is skew-symmetric. In other words, σ is a map form $\Gamma(TM) \times \cdots \times \Gamma(TM) \to \mathcal{E}(M)$ which is multilinear over the ring $\mathcal{E}(M)$ and skew-symmetric.

The exterior derivative $d : \Gamma(\bigwedge^k T^*M) \to \Gamma(\bigwedge^{k+1} T^*M)$: The Cartan formula gives $df(X) = Xf$ for $f \in \mathcal{E}(M)$, $(d\gamma)(X,Y) = X\gamma(Y) - Y\gamma(X) - \gamma([X,Y]), X, Y \in \Gamma(TM)$ for $\gamma \in \Gamma(T^*M)$ and $(d\omega)(X,Y,Z) = \sum_{c.p.X,Y,Z} X\omega(Y,Z) - \omega([X,Y],Z)$ for $\omega \in \Gamma(\bigwedge^2 T^*M)$.

Bivector fields and 2-forms as morphisms: Let $\eta \in \Gamma(\bigwedge^2 TM)$ and $\gamma \in \Gamma(T^*M)$. The *contraction* $\gamma \lrcorner \eta =: \eta(\gamma)$ (in the first index) is a vector field defined by $v = v^j(x)\frac{\partial}{\partial \psi^j}, v^j(x) := \gamma_i(x)\eta^{ij}(x)$. Since this operation is pointwise it defines a morphism of bundles $\eta^\sharp : T^*M \to TM$. Note that it is skew-symmetric, i.e. $(\eta^\sharp)^* = -\eta^\sharp$. Conversely, given such a morphism, we can construct a bivector field.

Analogously, a differential 2-form ω defines a skew-symmetric morphism $\omega^\flat : TM \to T^*M$.

3 Ordinary differential equations on manifolds

References: [Arn75]

Ordinary differential equation:

$$\frac{dc}{dt}(x) = v(x), (\text{or } \dot{x} = v(x) \text{ for short})$$

here $v \in \Gamma(TM)$ is given, c is unknown. A *solution* of this equation (or a *trajectory* of v) with an initial condition $x_0 \in M$ is a curve $c : \mathbb{R} \to M$ such that $c(0) = x_0$ and the vector $v(x)$ is tangent to c at any $x \in c(\mathbb{R})$.

A solution always exists *locally* and is unique: in local coordinates $(\psi^1, \ldots, \psi^n)$ we have $v = v^i(x)\frac{\partial}{\partial \psi^i}$ and the initial equation is equivalent to the system of ODE

$$\frac{dc^i(t)}{dt} = \psi^i(c^1(t), \ldots, c^n(t)), i = 1, \ldots, n$$

with the initial condition $c^i(0) = x_0^i, i = 1, \ldots, n$, and we can use the corresponding existence-uniqueness theorem.

Globally, if $\operatorname{supp} v := \overline{\{x \in M \mid v(x) \neq 0\}}$ is compact (eg. M is compact itself) one can extend any local solution to a global (in time and space) solution. [1]

Example 1: "nonextendability in time": $M :=]0, 1[, \dot{x} = 1$.

Example 2: "nonextendability in space": $M := \mathbb{R}, \dot{x} = x^2$.

Example 3: "Winding line on a torus": $M := \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, the vector field $v_{a,b} := a\frac{\partial}{\partial x^1} + b\frac{\partial}{\partial x^2}$, where $a, b \in]0, \infty[$ are fixed, can be projected onto the vector field $\tilde{v}_{a,b}$ on $\mathbb{T}^2$. Its trajectories are the projections $t \to P(x^1 + at, x^2 + bt)$ of the lines $t \to (x^1 + at, x^2 + bt)$.

Rational case: b/a is a rational number, $b = m\lambda, a = n\lambda$ for some $\lambda \in \mathbb{R}$. Then for $t := 1/\lambda$ we have $(x^1 + at, x^2 + bt) = (x^1 + m, x^2 + n)$ and $P(x^1 + at, x^2 + bt) = P(x^1, x^2)$ (the trajectory is closed, i.e. periodic).

Irrational case: b/a is an irrational number (any trajectory is dense in M).

4 Submanifolds, foliations and distributions

A submanifold S of M of codimension r: A subset $N \subset M$ such that there exists an atlas $\mathcal{A} := \{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ on M with $N \cap U_\alpha = \{x \in U_\alpha \mid \psi^1(x) = 0, \ldots \psi^r(x) = 0\}$ for those $\alpha \in A$ for which $N \cap U_\alpha \neq \emptyset$.

Smooth maps and submanifolds: A smooth map $F : M_1 \to M_2$ is called an *immersion* if $T_m F : T_m M_1 \to T_{F(m)} M_2$ is injective for any $m \in M_1$. The image of an injective immersion is called

[1] see V. I. Arnold, *Równania różniczkowe zwyczajne*, PWN, 1975, Theorem 1,§35.

an *immersed submanifold*. An injective immersion F is an *embedding* if F is a homeomorphism onto $F(M_1)$, where $F(M_1)$ is endowed with the topology induced from M_2.

Remarks: 1. The image $N := F(M_1)$ of an embedding is a submanifold in M_2 and, vice versa, given a submanifold $N \subset M$, the inclusion $N \hookrightarrow M$ is an embedding. 2. If $N \subset M$ is an immersed submanifold, then for any $x \in N$ there exists an open neighbourhood U of x in M such that the connected component of $N \cap U$ containing x is a submanifold in U. Vice versa, ...

Example of an immersed submanifold, which is not a submanifold: "The irrational torus winding" $\mathbb{R} \to \mathbb{T}^2$.

A foliation $\mathcal{F}$ of codimension r on M : A collection $\mathcal{F} = \{F_\beta\}_{\beta\in B}$ of path-connected sets on M such that there exists an atlas $\mathcal{A} := \{(U_\alpha, \psi_\alpha)\}_{\alpha\in A}$ on M with the following properties:

1. $M = \bigcup_{\beta\in B} F_\beta$;
2. $F_\beta \cap F_\gamma = \emptyset$ for any $\beta, \gamma, \beta \neq \gamma$;
3. for any $\alpha \in A$ and any $(c_1, \dots, c_r) \in \psi_\alpha(U_\alpha)$ the set $\{x \in U_\alpha \mid \psi^1(x) = c_1, \dots \psi^r(x) = c_r\}$ coincides with one of the path-connected components of the set $U_\alpha \cap \mathcal{F}_\beta$ if it is nonempty.

By the remark above the sets F_β are immersed submanifolds.

A distribution $\mathcal{D}$ on M of codimension r: A subbundle of the tangent bundle TM with the r-codimensional fiber, or in other words a collection of subspaces $D_x \subset T_xM$ smoothly (analytically) depending on $x \in M$. Such a distribution is locally spanned by $n - r$ linearly independent (at each point) vector fields.

Example: the distribution tangent to a foliation: $\mathcal{D} = T\mathcal{F} := \{v \in TM \mid v \text{ is tangent to } \mathcal{F}\}$.

Integrable distribution: A distribution which is tangent to some foliation.

Involutive distribution: A distribution $\mathcal{D}$ such that for any two vector fields $X, Y \in \Gamma(TM)$ which are tangent to $\mathcal{D}$ (i.e. $X(x), Y(x) \in D_x$ for any $x \in M$) their commutator $[X, Y]$ is also tangent to $\mathcal{D}$ (equivalently, locally there exist $v_1, \dots, v_m, v_i \in \Gamma(TM)$, and functions f_{ij}^k such that $\mathrm{Span}\{v_1, \dots, v_m\} = \mathcal{D}$ and $[v_i, v_j] = f_{ij}^k v_k$; *Exercise: prove the equivalence*).

The Frobenius theorem: A distribution $\mathcal{D}$ is integrable if and only if it is involutive.

Example of nonintegrable distribution: $X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, Y = \frac{\partial}{\partial y}$.

A generalized distribution $\mathcal{D}$ on M of codimension r: A collection of subspaces $D_x \subset T_xM$ locally spanned by $n - r$ vector fields linearly independent at least at one point (but not necessarily linearly independent at other points).

A generalized foliation $\mathcal{F}$ on M: ...

Example of a generalized foliation which is not a foliation: The trajectories of a vector field $x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}$.

The generalized Frobenius theorem (Nagano 1966): An analytic generalized distribution $\mathcal{D}$ is integrable if and only if it is *involutive*, i.e. for any two vector fields $X, Y \in \Gamma(TM)$ which are tangent to $\mathcal{D}$ (i.e. $X(x), Y(x) \in D_x$ for any $x \in M$) their commutator $[X, Y]$ is also tangent to $\mathcal{D}$.

An example of smooth involutive nonintegrable distribution: Let $\varphi(x)$ be a smooth function on $\mathbb{R}$ such that $\varphi(x) \equiv 0$ for $x \leqslant 0$ and $\varphi(x) > 0$ for $x > 0$. Take $X = \frac{\partial}{\partial x}, Y = \varphi\frac{\partial}{\partial y}$ on $\mathbb{R}^2$. Then $[X, Y] := \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial y}$ can be expressed as a linear combination of X, Y. But it is nonintegrable: look at its "leaves".

5 Symplectic and nondegenerate Poisson manifolds

A symplectic form on M: A differential 2-form (2-form for short) ω on M such that

1. ω is nondegenerate, i.e. $\omega^\flat$ is an isomorphism of bundles, or, equivalently, $\omega_{ij}(x)$ is a nondegenerate matrix for any x in some (consequently in any) local coordinate system;

2. $d\omega = 0$.

A nondegenerate Poisson structure on M: A bivector field (bivector for short) η such that $\eta^\sharp : T^*M \to TM$ is inverse to $\omega^\flat : TM \to T^*M$ for some symplectic form ω.

The Poisson bracket on $\mathcal{E}(M)$: Given a bivector field $\eta : T^*M \to TM$ (not necessarily Poisson), put $\{f, g\} := \eta(df)g, f, g \in \mathcal{E}(M)$. (From now on we will often skip $\sharp$ and $\flat$ indices.) Then $\{,\}$ is a bilinear skew-symmetric operation on $\mathcal{E}(M)$. We say that $\eta(f) := \eta(df)$ is a *hamiltonian* vector field corresponding to the function f.

Proposition. Let η be a nondegenerate bivector. Then it is Poisson if and only if $\{,\}$ satisfies the Jacobi identity, $\sum_{c.p.\, f,g,h}\{\{f, g\}, h\} = 0$. $\square$

Proof Put $\omega := \eta^{-1}$, i.e. $\omega(\eta(\alpha), v) = \alpha(v)$ for any vector field v and 1-form α. Then $\eta(f)\omega(\eta(g), \eta(h)) = \eta(f)(dg(\eta(h))) = \eta(f)(\eta(h)g) = \eta(f)\{h, g\} = \{f, \{h, g\}\} = -\{f, \{g, h\}\}$ and $\omega([\eta(f), \eta(g)], \eta(h)) = -\omega(\eta(h), [\eta(f), \eta(g)]) = -dh([\eta(f), \eta(g)]) = -[\eta(f), \eta(g)]h = -\eta(f)\eta(g)h + \eta(g)\eta(f)h = -\eta(f)\{g, h\} + \eta(g)\{f, h\} = -\{f, \{g, h\}\} + \{g, \{f, h\}\}$. Thus $d\omega(\eta(f), \eta(g), \eta(h)) = \sum_{c.p.\, f,g,h} \eta(f)\omega(\eta(g), \eta(h)) - \omega([\eta(f), \eta(g)], \eta(h)) = -\sum_{c.p.\, f,g,h}\{g, \{f, h\}\}$. So, if $d\omega = 0$, then $\{,\}$ satisfies the Jacobi identity.

Conversely, if the JI holds, $d\omega$ vanishes on all hamiltonian vector fields. To finish the proof it remains to note that the hamiltonian vector fields span T_xM at any $x \in M$. Indeed, it is enough to take $\eta(x^i)$, where (x^i) are local coordinates.

Example: the canonical symplectic structure on the cotangent bundle T^*Q: Let $\pi_Q : T^*Q \to Q$ be a cotangent bundle to a manifold Q. There is a canonical differential 1-form $\lambda \in \Gamma(T^*M), M := T^*Q$ determined uniquely by the following condition: for any $\alpha \in \Gamma(T^*Q)$, the following equality holds $\alpha^*\lambda = \alpha$, here α in the LHS is regarded as a map $\alpha : Q \to T^*Q$. We call λ the *Liouville* 1-form. If $(U, q^1, \ldots, q^n)$ is a local chart on Q, the 1-forms $dq^1, \ldots, dq^n$ form a basis of the vector space $T_x^*Q, x \in U$, and define the chart $(\pi_Q^{-1}(U), q^1, \ldots, q^n, p_1, \ldots, p_n)$. In these coordinates $\lambda = p_i dq^i$. Indeed, $\alpha : (q^1, \ldots, q^n) \mapsto (q^1, \ldots, q^n, \alpha_1(q), \ldots, \alpha_n(q))$, where $\alpha = \alpha_i(q)dq^i$. Thus $\alpha^*\lambda = \alpha_i(q)dq^i = \alpha$.

The canonical symplectic form ω on M is given by $\omega := d\lambda$, or, locally, $\omega = dp_i \wedge dq^i$.

Hamiltonian differential equation on a symplectic manifold (M, ω)**:** The ODE related to a hamiltonian vector field $\eta(f), f \in \mathcal{E}(M)$, here $\eta = \omega^{-1}$. In the context of the example above (in the canonical coordinates (q, p)): $\eta = -\frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}, \eta(H) = \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i}$, the corresponding equations read:

$$\dot{q}^i = \frac{\partial H(q,p)}{\partial q^i}, \dot{p}_i = -\frac{\partial H(q,p)}{\partial p_i}.$$

6 Poisson structures, their characteristic distributions, symplectic leaves and Casimir functions

References: [dSW99]

A Poisson structure on M**:** A bivector $\eta : T^*M \to TM$ (not necessarily nondegenerate) such that the corresponding bracket $\{,\}$ on $\mathcal{E}(M)$ satisfies the Jacobi identity (JI for short).

Digression on Lie algebras: A *Lie algebra* is a vector space $\mathfrak{g}$ endowed with a bilinear skew-symmetric operation $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the JI:

1. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \ \forall x, y, z \in V$, or, equivalently,
2. $\mathrm{ad}_x[y, z] = [\mathrm{ad}_x y, z] + [y, \mathrm{ad}_x z] \ \forall x, y, z \in V$, where $\mathrm{ad}_x y := [x, y]$, or, equivalently,
3. $\mathrm{ad}_{[x,y]} = [\mathrm{ad}_x, \mathrm{ad}_y] \ \forall x, y \in V$, where the bracket in the RHS denotes the commutator of the operators.

The second condition means that ad_x is a differentiation of the bracket $[,]$. The third one has the following interpretation. A pair $(V, [,])$, where V is a vector space and $[,] : V \times V \to V$ is a bilinear operation, is called an *algebra*. Given algebras $(V_1, [,]_1)$ and $(V_2, [,]_2)$, we say that a linear map $L : V_1 \to V_2$ is a *homomorphism* of algebras, if $L[x, y]_1 = [Lx, Ly]_2 \ \forall x, y \in V_1$.

So the third condition means that the map $x \mapsto \mathrm{ad}_x : V \to \mathrm{End}(V)$ a *homomorphism* of algebras $(V, [,])$ and $(\mathrm{End}(V), [,])$. Note that the last algebra is in fact a Lie algebra. A homomorphism of Lie algebras $(\mathfrak{g}, [,]) \to (\mathrm{End}(V), [,])$ is called a *representation* of the Lie algebra $(\mathfrak{g}, [,])$ in the vector space V (so $x \mapsto \mathrm{ad}_x$ is a representation of $\mathfrak{g}$ in $\mathfrak{g}$).

Consider the Lie algebra $(\mathcal{E}(M), \{,\})$ on a Poisson manifold. The corresponding ad_f-operator, $f \in \mathcal{E}(M)$, coincides with $\eta(f) : \mathcal{E}(M) \to \mathcal{E}(M)$.

The characteristic (generalized) distribution of a Poisson structure $\eta : T^*M \to TM$**:** $\mathcal{D}_\eta := \mathrm{im}\, \eta$ (locally generated by the hamiltonian vector fields $\eta(x^1), \ldots, \eta(x^n)$, where $(x^1, \ldots, x^n)$ are some local coordinates).

By the third condition above the map $f \mapsto \eta(f), (\mathcal{E}(M), \{,\}) \to (\Gamma(TM), [,])$ is a homomorphism of Lie algebras, here $[,]$ is the commutator of vector fields. This implies involutivity of $\mathcal{D}_\eta$: $[\eta(x^i), \eta(x^j)] = \eta(\{x^i, x^j\}) = \eta(\eta^{ij}(x))$, where $\eta = \eta^{ij}(x)\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$. On the other hand, $\eta(f) = \eta^{ij}(x)\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j} = \frac{\partial f}{\partial x^k}\eta(x^k)$ for any f. In particular, $[\eta(x^i), \eta(x^j)]$ is a linear combination (with smooth coefficients) of $\eta(x^1), \ldots, \eta(x^n)$.

Theorem: The characteristic distribution $\mathcal{D}_\eta$ is integrable (we call the corresponding foliation *characteristic* or *symplectic*).

Proof In analytic category this follows from the involutivity of $\mathcal{D}$ by the generalized Frobenius theorem. In the smooth case this is also true, but the proof is more complicated, so we skip it. □

Digression on linear algebra of bivectors: Let V be a vector space and e a bivector on V. Then e can be treated as: 1) an element $e \in \bigwedge^2 V$; 2) a linear skew-symmetric map $e^\sharp : V^* \to V$; 3) a bilinear form $\tilde{e}$ on V^*.

PROPOSITION. Let $W := \operatorname{im} e^\sharp \subset V$. Then there exists a correctly defined bivector $e|_W \in \bigwedge^2 W$, called the *restriction* of e to W. Moreover, the restriction $e|_W$ is nondegenerate, i.e. $e|_W^\sharp : W^* \to W$ is an isomorphism.

Proof I. A theorem from linear algebra says that there exists a basis $v_1, \ldots, v_n$ of V such that $e = v_1 \wedge v_2 + \cdots + v_{2k-1} \wedge v_{2k}$ (the number $2k$ is equal to $\dim W$ and is called the *rank* of e). It is easy to see that $v_1, \ldots, v_{2k}$ span W. □

Proof II. e is skew-symmetric, i.e. $(e^\sharp)^* = -e^\sharp$. This implies $\ker e^\sharp = (\operatorname{im} e^\sharp)^\perp$, where $(\cdot)^\perp$ stands for the annihilator of $(\cdot)$. So the natural isomorphism $\hat{e} : V^*/\ker e^\sharp \to \operatorname{im} e^\sharp = W$ induced by $e^\sharp$ can regarded as a map from $W^* \cong V^*/(W^\perp)$ to $W \subset V$. The map $\hat{e}$ being skew-symmetric induces the element of $\bigwedge^2 W$, which we denote by $e|_W$. □

Proof III. Let ω be a skew-symmetric bilinear form on a vector space L. Put $\ker \omega := \{x \in L \mid \omega(x, y) = 0\ \forall y \in L\}$. The form is called *nondegenerate* if $\ker \omega = \{0\}$.

Any ω induces a nondegenerate skew-symmetric bilinear form on the vector space $L/\ker \omega$.

Treating e as a skew-symmetric bilinear form $\tilde{e}$ on V^* we have $\ker \tilde{e} = \ker e^\sharp$. The restriction $e|_W$ treated as a skew-symmetric bilinear form on $W^* \cong V^*/\ker \tilde{e}$ is the above mentioned nondegenerate form induced from $\tilde{e}$. □

Symplectic leaves of a Poisson structure η on M: The leaves of the characteristic foliation $\mathcal{D}_\eta$. Since $D_{\eta,x} = \operatorname{im} \eta_x^\sharp$ for any $x \in M$, the bivector η admits a restriction $\eta|_S$ to any symplectic leaf $S \subset M$, which is a nondegenerate bivector on S. Moreover, since any hamiltonian vector field $\eta(f)$ is tangent to S at points of S, the value $\{f, g\}(x) = (\eta(f)g)(x), x \in S$, depends only of $g|_S$ and by the skew-symmetry the same is true with respect to f. In other words, $\{f|_S, g|_S\}_{\eta|_S} = (\{f, g\}_\eta)|_S$ for any $f, g \in \mathcal{E}(M)$ and the operation $\{,\}_{\eta|_S}$ satisfies the JI, hence $\eta|_S$ is a nondegenerate Poisson structure on S. This explains the term "symplectic leaf" ($(\eta|_S)^{-1}$ is a symplectic form).

Example 1: Let $M := \mathbb{R}^2, \eta = x^1 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}$. On the open set $U := \{x^1 \neq 0\}$ the form $(\eta|_U)^{-1} = -(1/x^1)dx^1 \wedge dx^2$ is symplectic. Thus the JI holds for $\{,\}_\eta$ on U and by continuity it holds also on the whole M. The symplectic leaves are U and all the points on the line $\{x^1 = 0\}$.

Example 2: Let $M := \mathbb{R}^3, \eta = \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}$. On each plane $P_c := \{x^3 = c\}$ the form $(\eta|_{P_c})^{-1} = -dx^1 \wedge dx^2$ is symplectic. The JI holds for $\{,\}_\eta$ on P_c for any $c \in \mathbb{R}$. Since P_c sweep the whole space M as c runs through $\mathbb{R}$, the JI holds for $\{,\}_\eta$ globally. The symplectic leaves are the planes P_c.

Example 3: Let $M := \mathbb{R}^3, \eta = x^1 \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} + x^2 \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}$ (we will prove that this is a Poisson bivector later). The symplectic leaves are ...

Example 4: Let $M = \mathbb{T}^2 \times \mathbb{R}$, let y be a coordinate on the second component. Put $\eta = \tilde{v}_{a,b} \wedge \frac{\partial}{\partial y}$, where $\tilde{v}_{a,b}$ is the generator of winding line. η is Poisson because locally it looks like the bivector from Example 2. If b/a is irrational, the symplectic leaves (which are two-dimensional) are dense in M.

Casimir functions of a Poisson structure η on M: Let $U \subset M$ be an open set. We say that $f \in \mathcal{E}(U)$ is a *Casimir function* if $\eta(f) \equiv 0$ on U. In particular, since $\{f, g\} = \eta(f)g$ on U the Casimir functions constitute the centre of the Lie algebra $(\mathcal{E}(U), \{,\}|_U)$. The space of Casimir functions over U will be denoted by $\mathcal{C}_\eta(U)$.

Proposition. The Casimir functions are constant on the leaves of the symplectic foliation.

Proof We have $\eta(f)g = -\eta(g)f = 0$ for any $f \in \mathcal{C}_\eta(U), g \in \mathcal{E}(U)$. So, since $\eta(g)$ span the characteristic distribution, f is constant along its leaves. $\square$

Example 1': $\mathcal{C}_\eta(M) = \mathbb{R}$, the space of constant functions.

Example 2': $\mathcal{C}_\eta(M) = \mathcal{F}un(x^3)$, the space of functions functionally generated by x^3.

Example 3': $\mathcal{C}_\eta(M) = \mathcal{F}un((x^1)^2 + (x^2)^2 + (x^3)^2)$. Hence the symplectic leaves are the concentric spheres and the point $\{(0,0,0)\}$.

Example 4': If b/a is irrational $\mathcal{C}_\eta(M) = \mathbb{R}$. However, for sufficiently small U the space $\mathcal{C}_\eta(U)$ will be functionally generated by one nonconstant function. So "local Casimirs" are not obtained as the restriction of the "global Casimirs".

7 Lie–Poisson structures

References: [dSW99]

Definition I: Let $(\mathfrak{g}, [,])$ be a finite-dimensional Lie algebra, $\mathfrak{g}^*$ its dual space (space of linear functionals on $\mathfrak{g}$). Given $f, g \in \mathcal{E}(\mathfrak{g}^*)$ define $\{f,g\}_{\mathfrak{g}}(x) := \langle x, [df|_x, dg|_x]\rangle, x \in \mathfrak{g}^*$. Here we identify $T_x^*\mathfrak{g}^*$ with $\mathfrak{g}$, $\langle , \rangle$ stands for the canonical pairing between vectors and covectors.

Digression: Let M be a manifold, $\{,\} : \mathcal{E}(M) \times \mathcal{E}(M) \to \mathcal{E}(M)$ a bilinear operation being a differentiation with respect to each argument: $\{fg,h\} = f\{g,h\} + g\{f,h\}, \{f,gh\} = \{f,g\}h + \{f,h\}g$. Then it can be shown that there is a tensor $\eta \in \Gamma(\bigotimes^2 TM)$ such that $\{f,g\} = \eta(df,dg)$. Let us show this in the case when $\{,\}$ is skew-symmetric.

Indeed, since $\{f,\cdot\}, \{g,\cdot\}$ are differentiations they are vector fields, say X_f, X_g. Let $f \in \mathcal{E}(M)$ be such that $df|_x = 0$ for some $x \in M$. Then $\langle X_f|_x, dg|_x\rangle = (X_f g)(x) = \{f,g\}(x) = -\{g,f\}(x) = -(X_g f)(x) = -\langle X_g|_x, df|_x\rangle = 0$. Here $g \in \mathcal{E}(M)$ is arbitrary, hence $X_f|_x = 0$. Thus the map $df|_x \to X_f|_x$ depends only on the value of df at x, i.e. is given by a morphism $T^*M \to TM$.

Definition II: Let $(\mathfrak{g}, [,])$ be a finite-dimensional Lie algebra, $e_1, \dots, e_n \in \mathfrak{g}$ its basis, $x_1 = e_1, \dots, x_n = e_n$ these vectors regarded as linear functions on $\mathfrak{g}^*$ (in particular $x_1, \dots, x_n$ are linear coordinates on $\mathfrak{g}^*$). Let $[e_i, e_j] = c_{ij}^k e_k$ (c_{ij}^k are called the *structure constants* corresponding to the basis $e_1, \dots, e_n$). Put $\eta_{\mathfrak{g}} := c_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$.

Proposition. The bivector corresponding to the bracket $\{,\}_{\mathfrak{g}}$ coincides with $\eta_{\mathfrak{g}}$.

Proof Let $\eta = \eta^{ij}(x)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ be the bivector corresponding to $\{,\}_{\mathfrak{g}}$. Take $f := x_i, g := x_j$, then $\{f,g\}(x) = \eta^{ij}(x)$. On the other hand, by Definition I, $\{f,g\}(x) = \langle x, [x_i, x_j]\rangle = c_{ij}^k x_k$. $\square$

Exercise: 1) Let $\eta \in \Gamma(\bigwedge^2 TM)$, in local coordinates $\eta = \eta^{ij}(x)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$. Show that the JI for $\{,\}, \{f,g\} = \eta^{ij}(x)\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$ holds if and only if the expression

$$[\eta,\eta]_S^{ijk} := \sum_{c.p.\, i,j,k} \eta^{ir}(x)\frac{\partial}{\partial x^r}\eta^{jk}(x)$$

vanishes for all $i,j,k \in \{1,\dots,n\}$. 2) Show that, given $\eta, \zeta \in \Gamma(\bigwedge^2 TM), \eta = \eta^{ij}(x)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \zeta = \zeta^{ij}(x)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, the expression

$$[\eta,\zeta]_S^{ijk} := \frac{1}{2}\sum_{c.p.\, i,j,k} \eta^{ir}(x)\frac{\partial}{\partial x^r}\zeta^{jk}(x) + \zeta^{ir}(x)\frac{\partial}{\partial x^r}\eta^{jk}(x)$$

is a local representation of a trivector on M (called the *Schouten bracket* of η and ζ). 3) If $\eta = v_1 \wedge v_2, \zeta = w_1 \wedge w_2, v_i, w_i \in \Gamma(TM)$, then

$$[\eta,\zeta]_S \sim [v_1,w_1] \wedge v_2 \wedge w_2 + v_1 \wedge [v_2,w_1] \wedge w_2 + v_2 \wedge [v_1,w_2] \wedge w_1 + v_1 \wedge w_1 \wedge [v_2,w_2].$$

Here $\sim$ means equality up to a constant.

Proof of the Jacobi identity for the Lie–Poisson structure: $[\eta_{\mathfrak{g}}, \eta_{\mathfrak{g}}]_S^{ijk} = \sum_{c.p.\, i,j,k} c^l_{ir} x_l c^r_{jk}$. The last expression vanishes for all i, j, k if and only if $\sum_{c.p.\, i,j,k} c^l_{ir} c^r_{jk} = 0$ for all l, i, j, k, which is equivalent to the JI for $[,]$. $\square$

8 Actions of Lie algebras and symplectic foliations of Lie–Poisson structures

References: [dSW99]

An action of a Lie algebra $\mathfrak{g}$ on a manifold: A homomorphism of Lie algebras $\rho : (\mathfrak{g}, [,]) \to (\Gamma(TM), [,])$ (in the target space $[,]$ stands for the commutator of vector fields) is called a *(right) action* of $\mathfrak{g}$ on M (a left action corresponds to an antihomomorphism, i.e. a map $\rho : (\mathfrak{g}, [,]) \to (\Gamma(TM), [,])$ such that $\rho([v, w]) = -[\rho(v), \rho(w)], v, w \in \mathfrak{g}$).

Orbits of an action $\rho : (\mathfrak{g}, [,]) \to (\Gamma(TM), [,])$: Put $D_x := \{\rho(v)|_x \mid v \in \mathfrak{g}\}, x \in M$.

Proposition. Let $\mathfrak{g}$ be finite-dimensional. Then the generalized distribution $\mathcal{D} := \{D_x\}_{x \in M}$ is integrable.

Proof The distribution $\mathcal{D}$ is involutive: $[\rho(v), \rho(w)] = \rho([v, w])$. Thus in the analytic category the proof follows from the generalized Frobenius theorem. We skip the proof in the smooth case (roughly it consists in integrating the action of the Lie algebra to a local action of the corresponding Lie group). $\square$

The leaves of the corresponding generalized foliation are called the *orbits* of the action ρ. If the Lie algebra $\mathfrak{g}$ is finite-dimensional, the action can be "integrated" to a local action of a Lie group G such that $\mathfrak{g}$ is its Lie algebra. Then the orbits of the Lie algebra action and of the Lie group action coincide.

Linear representations and actions: Let V be a vector space and $A \in \mathrm{End}(V)$ a linear operator. It induces a uniquely defined vector field $\tilde{A}$ on V given by $x \mapsto (x, Ax) : V \to V \times V \cong TV$. If $e_1, \ldots, e_n$ is a basis of V, $x^1, \ldots, x^n$ the dual basis of V^* (i.e. the coordinates on V) and $Ae_i = A_{ji} e_j$, we have $\tilde{A} = A_{ji} x^i \frac{\partial}{\partial x^j}$.

Exercise: The map $A \mapsto \tilde{A} : \mathrm{End}(V) \to \Gamma(TV)$ is an antihomorphism of Lie algebras, i.e. a left action of the Lie algebra $\mathrm{End}(V)$ on V.

Let $L : (\mathfrak{g}, [,]) \to (\mathrm{End}(V), [,])$ be a representation of a Lie algebra $\mathfrak{g}$ in a vector space V. Then the map $\tilde{L} : \mathfrak{g} \to \Gamma(TM), \tilde{L}(x) := \widetilde{L(x)}$ is a left action of $\mathfrak{g}$ on the manifold V. Note, that the dual representation $L^* : \mathfrak{g} \to \mathrm{End}(V^*)$ given by $L^*(v) := (L(v))^*$ is an antihomomorphism, hence the map $\tilde{L^*} : \mathfrak{g} \to \Gamma(TM), \tilde{L^*}(v) := \widetilde{(L(v))^*}$ is a right action of $\mathfrak{g}$ on V.

The adjoint and coadjoint actions: Let $(\mathfrak{g}, [,])$ be a Lie algebra. The homomorphism $v \mapsto \mathrm{ad}_v : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, where $\mathrm{ad}_v w := [v, w]$, gives the *adjoint* representation (of $\mathfrak{g}$ on $\mathfrak{g}$). The corresponding (left) action $x \mapsto \widetilde{\mathrm{ad}_x} : \mathfrak{g} \to \Gamma(T\mathfrak{g})$ is also called *adjoint*. The homomorphism $v \mapsto \mathrm{ad}_v^* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$, where ad_v^* is the transposed operator to ad_v, and the corresponding (right) action $x \mapsto \widetilde{\mathrm{ad}_v^*} : \mathfrak{g} \to \Gamma(T\mathfrak{g}^*)$ are called the *coadjoint* (anti)representation and action, respectively.

The symplectic leaves of the Lie-Poisson structure $\eta_{\mathfrak{g}}$ on $\mathfrak{g}^*$ coincide with the orbits of the coadjoint action : We claim that $\widetilde{\mathrm{ad}_v^*} = \eta_{\mathfrak{g}}(v')$, where v' denotes the linear function on $\mathfrak{g}^*$ defined by an element $v \in \mathfrak{g}$. Indeed, let $v = v^j e_j$, then $v' = v^j x_j$. Here $x_1, \dots, x_n$ are the elements $e_1, \dots, e_n$ regarded as linear functions on $\mathfrak{g}^*$. Then $\mathrm{ad}_v e_i = v^j c_{ji}^k e_k, \mathrm{ad}_v^* x^i = v^j c_{jk}^i x^k$, hence $\widetilde{\mathrm{ad}_v^*} = v^j c_{jk}^i x_i \frac{\partial}{\partial x_k}$. The last expression obviously coincides with $\eta_{\mathfrak{g}}(v')$. $\square$

An invariant symmetric bilinear form on $(\mathfrak{g}, [,])$: A symmetric bilinear form $(,) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ satisfying the equality $(\mathrm{ad}_x y, z) = -(y, \mathrm{ad}_x z)$ for any $x, y, z \in \mathfrak{g}$.

PROPOSITION. Let $(,)$ be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$. Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by means of the map $v \mapsto (v, \cdot)$. Then the adjoint orbits become coadjoint ones under this identification.

Proof Indeed, if $A : \mathfrak{g} \to \mathfrak{g}$ is a linear operator the transposed operator $A^* : \mathfrak{g}^* \to \mathfrak{g}^*$ becomes the adjoint one under this identification: $(A^* y, z) = (y, Az)$ for any $y, z \in \mathfrak{g}$. Thus ad_x^* becomes $-\mathrm{ad}_x$. $\square$

Notations (for the Lie algebras): $\mathfrak{gl}(n, \mathbb{R}) := \{n \times n -$ matrices with real entries$\}, \mathfrak{sl}(n, \mathbb{R}) := \{x \in \mathfrak{gl}(n, \mathbb{R}) \mid \mathrm{Tr}(x) = 0\}, \mathfrak{so}(n, \mathbb{R}) := \{x \in \mathfrak{gl}(n, \mathbb{R}) \mid x = -x^T\}, \mathfrak{sp}(n, \mathbb{R}) := \{x \in \mathfrak{gl}(2n, \mathbb{R}) \mid xJ + Jx^T = 0\}$, here $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, I_n being the identity $n \times n$-matrix. It is easy to see that $x \in \mathfrak{sp}(n, \mathbb{R})$ if and only if $x = \begin{bmatrix} a & b \\ c & -a^T \end{bmatrix}$, here $a, b, c, \in \mathfrak{gl}(n, \mathbb{R}), b = b^T, c = c^T$.

The sets above are Lie algebras with respect to the commutator of matrices.

Notations (for the Lie Groups): $GL(n, \mathbb{R}) := \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid \det X \neq 0\}, SL(n, \mathbb{R}) := \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid \det X = 1\}, SO(n, \mathbb{R}) := \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid XX^T = I_n\}, SP(n, \mathbb{R}) := \{X \in \mathfrak{gl}(2n, \mathbb{R}) \mid XJX^T = J\}$. All these sets are groups with respect to the matrix multiplication. It is easy to see that if $x \in \mathfrak{g}$, where $\mathfrak{g}$ is one of the Lie algebras above, then $\exp(x) \in G$, where G is the corresponding Lie group. Also $\mathfrak{g} = T_I G$.

The Lie algebras from Examples 1-5, below, have an invariant nondegenerate symmetric form $(x, y) = \mathrm{Tr}(xy)$ by means of which we can make an identification $\mathfrak{g} \cong \mathfrak{g}^*$. The coadjoint orbits are identified with the adjoint ones, which can be described as the orbits of the corresponding Lie group with respect to the conjugation of matrices: $\{XxX^{-1} \mid X \in G\}, x \in \mathfrak{g}$.

Example 1: $\mathfrak{g} := \mathfrak{gl}(n, \mathbb{R}), \mathcal{C}_{\eta_{\mathfrak{g}}}(\mathfrak{g}) = \mathcal{F}un(\mathrm{Tr}(x), \mathrm{Tr}(x^2), \dots, \mathrm{Tr}(x^n))$.

Example 2: $\mathfrak{g} := \mathfrak{sl}(n, \mathbb{R}), \mathcal{C}_{\eta_{\mathfrak{g}}}(\mathfrak{g}) = \mathcal{F}un(\mathrm{Tr}(x^2), \dots, \mathrm{Tr}(x^n))$. In particular, for $n = 2$ we have a basis $e_1 := e_{11} - e_{22}, e_2 := e_{12}, e_2 := e_{21}$ and the commutation relations $[e_1, e_2] = 2e_2, [e_1, e_3] = -2e_3, [e_2, e_3] = e_1$. Hence $\eta_{\mathfrak{g}} = x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + 2x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} - 2x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3}$. The Casimir function

$\mathrm{Tr}(x^2)$ reads as $x_1^2/2 + 2x_2x_3$. The symplectic leaves are the 1-sheet hyperboloids, sheets of 2-sheet hyperboloids, two sheets of the cone (without zero) and the point 0.

Example 3: $\mathfrak{g} := \mathfrak{so}(2n, \mathbb{R}), \mathcal{C}_{\eta_{\mathfrak{g}}}(\mathfrak{g}) = \mathcal{F}un(\mathrm{Tr}(x^2), \mathrm{Tr}(x^4) \ldots, \mathrm{Tr}(x^{2n-2}), \mathrm{Pf}(x))$.

Example 4: $\mathfrak{g} := \mathfrak{so}(2n+1, \mathbb{R}), \mathcal{C}_{\eta_{\mathfrak{g}}}(\mathfrak{g}) = \mathcal{F}un(\mathrm{Tr}(x^2), \mathrm{Tr}(x^4) \ldots, \mathrm{Tr}(x^{2n}))$.

Example 5: $\mathfrak{g} := \mathfrak{sp}(n, \mathbb{R}), \mathcal{C}_{\eta_{\mathfrak{g}}}(\mathfrak{g}) = \mathcal{F}un(\mathrm{Tr}(x^2), \mathrm{Tr}(x^4) \ldots, \mathrm{Tr}(x^{2n}))$.

Example 6 (the Heisenberg algebra): $\mathfrak{g} := \mathbb{R}^3, [e_1, e_2] = e_3$, here e_1, e_2, e_3 is the standard basis of $\mathbb{R}^3$. We have $\eta_{\mathfrak{g}} = x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}, \mathcal{C}_{\eta_{\mathfrak{g}}}(\mathfrak{g}^*) = \mathcal{F}un(x_3)$, so the coadjoint orbits consist of the planes $\{x_3 = c\}, c \neq 0$ and of the points of the plane $\{x_3 = 0\}$. The adjoint orbits are generated by the vector fields $c_{ij}^k x^i \frac{\partial}{\partial x^k}$, where $\{x^i\}$ is the basis dual to $\{x_i\}$, i. e. by $x^1 \frac{\partial}{\partial x^3}, x^2 \frac{\partial}{\partial x^3}$, so they are the lines parallel to the x_3-axis and the points of this axis.

9 Symplectic and Poisson reduction

References: [AG90]

Digression on linear algebra of skew-symmetric bilinear forms: Let V be a vector space, $\omega \in \bigwedge^2 V^*, W \subset V$ a subspace. We put $W^{\perp\omega} := \{v \in V \mid \omega(v, W) = 0\}$ and $\ker\omega := V^{\perp\omega} = \{v \in V \mid \omega(v, w) = 0 \ \forall w \in V\}$. We say that W is *isotropic* (*coisotropic*) if $W \subset W^{\perp\omega}$ (respectively $W \supset W^{\perp\omega}$). In case when ω is nondegenerate, or, in other words, symplectic, we call W *lagrangian*, if it is maximal isotropic (i.e. W is isotropic and for any isotropic $W' \supset W$ we have $W' = W$). Equivalently, W is lagrangian if it is minimal coisotropic.

Examples: Let $e_1, \ldots, e_{2n}$ be a basis of V, $e^1, \ldots, e^{2n}$ be the dual basis of V^*, $\omega = e^1 \wedge e^{n+1} + \cdots e^n \wedge e^{2n}$. Then $W_l := \langle e_1, \ldots, e_l\rangle$ is isotropic for any $l \leqslant n$, $W_l^{\perp\omega} = \langle e_1, \ldots e_n, e_{n+l+1}, \ldots e_{2n}\rangle$ is coisotropic, W_n is lagrangian.

A coisotropic submanifold of a symplectic manifold (M, ω)**:** A submanifold $N \subset M$ such that T_xN is a coisotropic subspace of the sympectic vector space (T_xM, ω_x) for any $x \in M$.

Proposition. Let $f_1, \ldots, f_k \in \mathcal{E}(M)$ be such that $N = \{x \in M \mid f_1(x) = 0, \ldots f_k(x) = 0\}$. Then N is coisotropic if and only if $\{f_i, f_j\}|_N \equiv 0, i, j = 1, \ldots, k$.

Proof Let $\eta := \omega^{-1}$, then $(T_xN)^{\perp\omega_x} = \langle \eta(f_1)|_x, \ldots, \eta(f_k)|_x\rangle$. Indeed, if $w \in T_xN$, we have $\omega_x(w, \eta(f_i)|_x) = -\omega_x(\eta(f_i)|_x, w) = -d_xf_i(w) = 0$. So the inclusion $(T_xN)^{\perp\omega_x} \subset T_xN$ is equivalent to the equality $d_xf_j(\eta(f_i)|_x) = 0, i, j = 1, \ldots, k$. On the other hand, $d_xf_j(\eta(f_i)|_x) = (\eta(f_i)f_j)|_x = \{f_i, f_j\}|_x$. $\square$

A coisotropic foliation of a symplectic manifold (M, ω)**:** A foliation $\mathcal{F}$ on M such that each leaf locally is a coisotropic submanifold.

Proposition. Let $U \subset M$ be an open set such that $\mathcal{F}$ on U is given by $\{x \in U \mid f_1(x) = c_1, \ldots f_k(x) = c_k\}$ for some $f_1, \ldots, f_k \in \mathcal{E}(U)$. Then N is coisotropic if and only if $\{f_i, f_j\} \equiv 0$ on U for any $i, j = 1, \ldots, k$.

Linear version of symplectic reduction: Let (V, ω) be a symplectic vector space, $W \subset V$ a coisotropic subspace (i.e. $W^{\perp\omega} \subset W$). Put $W' := W/W^{\perp\omega}$ and let $p : W \to W'$ be the natural projection. Then there exists a unique symplectic form ω' on W' such that

$$p^*\omega' = \omega|_W,$$

i.e. $\omega(v, w) = \omega'(pv, pw)$ for any $v, w \in W$. To show this we observe that $W^{\perp\omega} = \ker\omega|_W$, so we can put $\omega'(v + W^{\perp\omega}, w + W^{\perp\omega}) := \omega(v, w)$.

Digression on factor manifolds: Let M be a manifold and $\mathcal{K}$ a foliation on M. The relation "$x \sim y \Leftrightarrow$ (x and y belong to the same leaf)" is an equivalence relation on M and we shall denote by $M/\mathcal{K}$ the topological quotient space $M/\sim$. We say that $M/\mathcal{K}$ is *good* if the space $M/\mathcal{K}$ has a structure of a smooth (analytic) manifold whose underlying topology is the quotient one such that the canonical projection $M \to M/\mathcal{K}$ is a smooth (analytic) submersion (recall that a smooth map is a *submersion* if the tangent map i s surjective at each point). If such a smooth (analytic) structure exists it is unique.

Note that for any foliation $\mathcal{K}$ and small enough open sets $U \subset M$ the factor space $U/\mathcal{K}$ is good.

Symplectic reduction on a symplectic manifold (M, ω): Let $N \subset M$ be a coisotropic submanifold. Put $D_x := (T_x N)^{\perp \omega_x} \subset T_x M$.

PROPOSITION. $\mathcal{D} := \{D_x\}_{x \in N}$ is a integrable distribution on N.

Proof Let $f_1, \ldots, f_k \in \mathcal{E}(U)$ be local functions defining N. Then $D_x = \langle \eta(f_1)|_x, \ldots, \eta(f_k)|_x \rangle$ and $[\eta(f_i), \eta(f_j)]|_x = \eta_x(\{f_i, f_j\}|_x) = 0$. $\square$

Put $\mathcal{K}$ for the foliation such that $T\mathcal{K} = \mathcal{D}$ Assume that $N' := N/\mathcal{K}$ is good and put $p : N \to N'$ for the natural projection.

PROPOSITION. There exists a unique symplectic form ω' on N' such that

$$p^* \omega' = \omega|_N.$$

Proof Perform the linear symplectic reduction at each point. $\square$

Example: Let $M := T^*\mathbb{R}^2 \cong \mathbb{R}^4$ and let $\omega = dp \wedge dq$ be the canonical form. Let $N := \{(q, p) \mid H(q, p) = 1\}, H(q, p) = q_1^2 + q_2^2 + (p^1)^2 + (p^2)^2$. Then $T\mathcal{K}$ is generated by $\eta(H) = 2(q_1 \frac{\partial}{\partial p^1} - p^1 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial p^2} - p^2 \frac{\partial}{\partial q_2})$. This vector field has 3 first integrals: $H, f_1 := (q_1^2 + (p^1)^2) - (q_2^2 + (p^2)^2), f_2 := 2(q_1 p^2 - q_2 p^1)$. Put $f_3 := 2(q_1 q_2 + p^1 p^2)$ and consider the map $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$ given by $(q, p) \mapsto (H, f_1, f_2, f_3)$. Restricting the map φ to the sphere $N = S^3$, we get the map $\psi : N \to \mathbb{R}^3$. In fact, because of the relation $f_2^2 + f_3^2 = H^2 - f_1^2$ the image of ψ lies in the 2-dimensional sphere $N' := S^2$ in $\mathbb{R}^3$.

Exercise: 1) If $f' := (f_1', f_2', f_3') \in N'$, the preimage $\psi^{-1}(f)$ is a "great circle" on N contained in the plane

$$\begin{aligned} (1 + f_1')q_2 - f_3' q_1 + f_2' p^1 &= 0 \\ (1 + f_1')p^2 - f_2' q_1 - f_3' p^1 &= 0 \end{aligned}$$

for $f \neq (0, 0, -1)$ and in the plane $\{q_1 = 0, p^1 = 0\}$ for $f = (0, 0, -1)$. 2) This plane is a complex one dimensional subspace of the space $\mathbb{C}^2 \cong \mathbb{R}^4$, where the complex coordinates are given by $q_1 + ip^1, q_2 + ip^2$.

The fibration $\psi : S^3 \to S^2$ is called the *Hopf fibration*. As a result of the symplectic reduction we get a symplectic structure on $S^2 \cong \mathbb{CP}^1$. Analogous construction gives a symplectic structure on $\mathbb{CP}^n$.

A particular case of Poisson reduction (informally): Let $\mathcal{F}$ be a coisotropic foliation on (M, ω) and let $\mathcal{D} := \{D_x\}_{x \in M}, D_x := (T_x \mathcal{F})^{\perp \omega_x}$. Then $\mathcal{D}$ is an integrable distribution, put $\mathcal{K}$ for the foliation such that $T\mathcal{K} = \mathcal{D}$. Assume that $M' := M/\mathcal{K}$ is good.

Now perform the symplectic reduction with respect to each leaf of $\mathcal{F}$. As a result we will get a foliation of M' by a symplectic (immersed) submanifolds. In fact this is a symplectic foliation of some degenerate Poisson structure η' on M'.

Digression on projectability of tensor fields: Let $p : M \to M'$ be a surjective submersion. Put $p_{*,x} : T_x M \to T_{p(x)}M'$ for the tangent map. A vector field $v \in \Gamma(TM)$ is said to be *projectable* with respect to p if there exists a vector field $v' \in \Gamma(TM')$ such that $v'_{p(x)} = p_{*,x} v_x$ for any $x \in M$.

Let $(x^1, \ldots, x^k, y^1, \ldots, y^l)$ be local coordinates on M such that $(x^1, \ldots, x^k)$ are local coordinates on M' and p is given by $p(x, y) = x$. Then v is projectable if and only if $v = u^i(x)\frac{\partial}{\partial x^i} + w^j(x, y)\frac{\partial}{\partial y^j}$ (and if v is so, $v' = u^i(x)\frac{\partial}{\partial x^i}$).

Analogously, one can define the projectability of bivector fields and show that $\eta = \eta^{ij}(x, y)\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + \zeta^{tu}(x, y)\frac{\partial}{\partial x^t} \wedge \frac{\partial}{\partial y^u} + \xi^{rs}(x, y)\frac{\partial}{\partial y^r} \wedge \frac{\partial}{\partial y^s}$ is projectable if and only if $\eta^{ij}(x, y) = \eta^{ij}(x)$ is independent of y.

Poisson reduction formally: Let $p : M \to M'$ be a surjective submersion, $\mathcal{K}$ be the foliation of the fibers of p. Let $\eta \in \Gamma(\bigwedge^2 TM)$ be a nondegenerate Poisson structure and let $\omega := \eta^{-1}$.

PROPOSITION. (Liebermann, Weinstein) The following conditions are equivalent: 1) η is projectable with respect to p; 2) the distribution $\mathcal{D}, \mathcal{D} := \{D_x\}_{x \in M}, D_x := (T_x\mathcal{K})^{\perp \omega_x}$, is integrable; 3) the set of functions $S := p^*(\mathcal{E}(M'))$ constant along $\mathcal{K}$ is a Lie subalgebra with respect to $\{,\}_\eta$.

Moreover, if η is projectable, $\eta' := p_*\eta$ is Poisson and the map p is Poisson, i.e. $p^* : (\mathcal{E}(M'), \{,\}_{\eta'}) \to (\mathcal{E}(M), \{,\}_\eta)$ is a homomorphism of Lie algebras.

Proof Locally the leaves of $\mathcal{K}$ are given by $\{x^1 = c_1, \ldots, x^k = c_k\}$ in the (x, y)-coordinates, so $D_x = \langle \eta(x^1)|_x, \ldots, \eta(x^k)|_x \rangle$ (we do not assume $\{x^i, x^j\}_\eta = 0$). $\mathcal{D}$ is integrable if and only if $[\eta(x^i), \eta(x^j)] = \eta(\{x^i, x^j\})$ is a linear combination of $\eta(x^1), \ldots, \eta(x^k)$. Let us show that this last condition is equivalent to condition 3). Indeed, put $f(x, y) := \{x^i, x^j\}_\eta$ and observe, that $\eta(f) = \frac{\partial f}{\partial x^t}\eta(x^t) + \frac{\partial f}{\partial y^u}\eta(y^u)$. Thus $\eta(f)$ is a linear combination of $\eta(x_1), \ldots, \eta(x_k)$ if and only if the function f does not depend on y, i.e. belongs to S. So we have proven 2) $\Longleftrightarrow$ 3).

Since $\eta^{ij}(x, y) = f(x, y)$ (see the previous subsection), we see that η is projectable if and only if f does not depend on y, hence 1) $\Longleftrightarrow$ 3).

Finally, if η is projectable, the Poisson bracket corresponding to η' is the restriction of $\{,\}_\eta$ to S, hence satisfies the JI. $\square$

Dual pairs of foliations (Poisson maps): Note that in the construction above we get a foliation $\mathcal{F}$ such that $T\mathcal{F} = (T\mathcal{K})^{\perp\omega}$. Since taking the skew-orthogonal complement of a subspace twice gives the initial subspace, we also have $T\mathcal{K} = (T\mathcal{F})^{\perp\omega}$. In such a situation we say that the foliations $\mathcal{K}, \mathcal{F}$ (and the natural projections $M \to M/\mathcal{K}, M \to M/\mathcal{F}$) form a *dual pair*. In the particular case discussed in the context of symplectic reduction $\mathcal{F}$ was a coisotropic foliation and $\mathcal{K}$ an isotropic one (since $T\mathcal{K} = (T\mathcal{F})^{\perp\omega} \subset T\mathcal{F}$).

10 Hamiltonian reduction and the Arnold-Liouville theorem

References: [Arn81]

Reduction of a hamiltonian system on (M,ω): Let $v = \eta(H), \eta := \omega^{-1}, H \in \mathcal{E}(M)$. Assume that $p : M \to M'$ is surjective submersion such that η is projectable with respect to p and H is constant along the fibers of p. Then v is also projectable with respect to p. Indeed, $p_* v = \eta'(H')$, where $\eta' := p_*\eta, H' \in \mathcal{E}(M')$ is the unique function such that $H = p^* H'$. The hamiltonian system on (M', η') given by the hamiltonian vector field $v' := p_* v$ is called the *reduction* of the initial hamiltonian system with respect to p.

First integrals and reduction: Assume that $g \in \mathcal{E}(M)$ is a first integral of the hamiltonian vector field $v = \eta(H)$, i.e. $vg = 0$. The last can be rewritten as $\eta(H)g = 0$, or, equivalently, $\{H, g\}_\eta = 0$. Consider the foliation $\mathcal{F} := \{g = const\}$ of the level sets of the function g (we assume that $dg \neq 0$ everywhere) and the dual 1-dimensional foliation $\mathcal{K}$ generated by $\eta(g)$. Then H is constant along the leaves of $\mathcal{K}$ (because $\eta(g)H = -\eta(H)g = 0$) and the system can be reduced with respect to the projection $M \to M/\mathcal{K}$ (at least locally, since locally the factor space $M/\mathcal{K}$ is good).

Conclusion: any first integral allows to reduce a bihamiltonian system on $(M, \omega), \dim M = 2n$, to a new hamiltonian system which is defined on a symplectic manifold of dimension $2n - 2$.

More generally: k first integrals $g_1, \ldots, g_k$ in involution (i.e. such that $\{g_i, g_j\} = 0$) allow to reduce the number of independent variables by $2k$.

Even more generally: Assume there exists $S \subset \mathcal{E}(M)$, a Lie subalgebra with respect to $\{,\}_\eta$ consisting of the first integrals of a hamiltonian system. Then it can be reduced to a hamiltonian system on a smaller symplectic manifold. The last is a symplectic leaf of the reduced Poisson structure obtained by the reduction of η with respect to the action of the Lie algebra $\eta(S) \subset \mathit{\Gamma}(TM)$. The dimension of this manifold depends on the structure of the Lie algebra $(S, \{,\}_\eta|_S)$.

The Arnold–Liouville theorem: Let (M,ω) be symplectic, $\dim M = 2n$. Assume a hamiltonian vector field $v(H)$ admits n functionally independent integrals $g_1 = H, g_2, \ldots, g_n$ in involution. Then

1. if the common level sets $M_c := \{x \in M \mid g_i = c_i, i = 1, \ldots, n\}$ of these integrals are compact and connected, they are diffeomorphic to (n-dimensional) tori $\mathbb{T}^n = \{(\varphi_1, \ldots, \varphi_n) \mathrm{mod} 2\pi\}$;

2. the restriction of the initial hamiltonian equation to $\mathbb{T}^n$ gives an almost periodic motion on $\mathbb{T}^n$, i.e. in the "angle coordinates" φ the equation has the form

$$\frac{d\overrightarrow{\varphi}}{dt} = \overrightarrow{a},$$

here $\overrightarrow{a} = (a_1, \ldots, a_n)$ is a constant vector depending only on the level;

3. the initial equation can be integrated in "quadratures", i.e. the solutions can be obtained by means of a finite number of algebraic operations and operations of taking integral.

Proof. The functional independence of $g_1, \ldots, g_n$ means linear independence of the differentials $dg_1, \ldots, dg_n$ at each point of M_c. By the implicit function theorem M_c is a submanifold of M.

LEMMA. 1 *The vector fields $\eta(g_1), \ldots, \eta(g_n)$ are commuting, tangent to M_c and linearly independent at each point of M_c.*

Proof The linear independence follows from that of $dg_1, \ldots, dg_n$ and from nondegeneracy of η. The vector fields are tangent to M_c because $\eta(g_i)g_j = \{g_i, g_j\} = 0$. The equality $\eta(\{g_i, g_j\}) = [\eta(g_i), \eta(g_j)]$ shows the commuting property. □

LEMMA. 2 *Let N be a compact connected n-dimensional manifold which has n linearly independent (at each point) commuting vector fields $v_1, \ldots, v_n$. Then N is diffeomorphic to n-dimensional torus.*

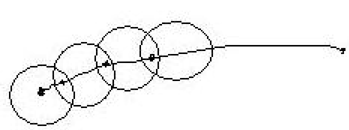
Sketch of proof Let $G_i^t, i = 1, \ldots, n$, be the corresponding 1-parametric groups of diffeomorphisms of N. In other words, $\frac{d}{dt}|_{t=0} G_i^t x = v_i|_x$ for any $x \in N$ and $G_i^{t_1+t_2} = G_i^{t_1} \circ G_i^{t_2} = G_i^{t_2} \circ G_i^{t_1}$ for any $t_1, t_2 \in \mathbb{R}$ (and $G_i^0 = \mathrm{Id}, G_i^{-t} = (G_i^t)^{-1}$). Note that G_i^t exist since by compactness of N the vector fields v_i are complete.

Due to the commuting property of vector fields the diffeomorphisms also commute: $G_i^t \circ G_j^{t'} = G_j^{t'} \circ G_i^t$. Thus the n-parametric family of diffeomorphisms $G^{\mathbf{t}} : N \to N, G^{\mathbf{t}} := G_1^{t_1} \cdots G_n^{t_n}, \mathbf{t} := (t_1, \ldots, t_n)$, has the property $G^{\mathbf{t}+\mathbf{t}'} = G^{\mathbf{t}} \circ G^{\mathbf{t}'} = G^{\mathbf{t}'} \circ G^{\mathbf{t}}$ (and $G^{\mathbf{0}} = \mathrm{Id}, G^{-\mathbf{t}} = (G^{\mathbf{t}})^{-1}$). In other words, we get an action of the commutative group $\mathbb{R}^n$ on N.

LEMMA. 3 *This action is transitive, i.e. for any two points $x_0, x_1 \in N$ there exists $\mathbf{t}$ such that $G^{\mathbf{t}} x_0 = x_1$.*

Proof Fix x_0 and consider the map $\psi : \mathbb{R}^n \to N, \psi(\mathbf{t}) := G^{\mathbf{t}} x_0$. This map is a local diffeomorphism: there exists an open set $U, \mathbf{0} \subset U \subset \mathbb{R}^n$ such that $\psi|_U$ is a diffeomorphism onto $V := \psi(U)$. Indeed, the derivative $\psi'(\mathbf{0})$ sends the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ to $\{v_1|_{x_0}, \ldots, v_n|_{x_0}\}$. The last vectors are independent, hence $\psi'(\mathbf{0})$ is nondegenerate and we can use the inverse function theorem.

Now connect x_0 with x_1 by a curve and cover this curve by a finite number of sets similar to V.



Choose a point y_i in each of the pairwise intersections of these sets and put $y_0 := x_0, y_m := x_1$. It is clear that there exist $\mathbf{t}_i$ such that $G^{\mathbf{t}_i} y_i = y_{i+1}$. Finally, put $\mathbf{t} := \mathbf{t}_0 + \cdots + \mathbf{t}_{m-1}$. $\square$

LEMMA. 4 *The stabilizer* $G_{x_0} := \{\mathbf{t} \in \mathbb{R}^n \mid G^{\mathbf{t}} x_0 = x_0\} \subset \mathbb{R}^n$ *of the point* $x_0 \in N$ *with respect to this action is a discrete additive subgroup of* $\mathbb{R}^n$*, independent of the choice of* x_0.

Proof Given any action of a group G on a set X, one proves immediately that the stabilizers are subgroups and the stabilizers of points lying on one orbit are conjugate. Here N consists of one orbit and the group is commutative. Thus G_{x_0} is a subgroup, the same for any point.

To prove that it is discrete, observe that the set U can not contain any point of G_{x_0} different from $\mathbf{0}$. $\square$

LEMMA. 5 *For any discrete subgroup* $\Lambda \subset \mathbb{R}^n$ *there exist linearly independent vectors* $l_1, \dots, l_k \in \mathbb{R}^n, k \leqslant n$, *such that* $\Lambda = \{\sum_{i=1}^k z_i l_i \mid z_i \in \mathbb{Z}\}$.

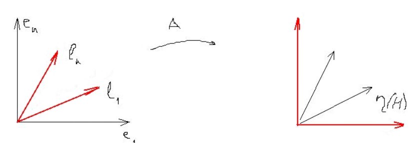
For the proof see the book: V. I. Arnold "Metody matematyczne mechaniki klasycznej" (PWN, 1981), Chapter 49.

Now we are ready to finish the proof of Lemma 2. Any orbit O of a (smooth) action of a Lie group G on a manifold is diffeomorphic to the factor manifold G/G_{x_0}, where $x_0 \in O$ is any element. In our case $O = N$ is diffeomorphic $\mathbb{R}^n/G_{x_0} = \mathbb{T}^k \times \mathbb{R}^{n-k} = \{(\varphi_1, \dots, \varphi_k; y_1, \dots, y_{n-k})\}, \varphi_i \mathrm{mod} 2\pi$. By compactness of N we conclude that $k = n$ and $N \cong \mathbb{T}^n$. $\square$

So we have proven the first item of the A–L theorem. To show item 2 fix c and observe that the diffeomorphism $\mathbb{T}^n \to N := M_c$ can be included to the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^n \\ \downarrow p & & \downarrow \psi \\ \mathbb{T}^n & \longrightarrow & N \end{array} .$$

Here p is the natural projection and A is the linear isomorphism mapping the vectors $2\pi e_1, \dots,$ $2\pi e_n$, where $e_1, \dots, e_n$ is the standard base in $\mathbb{R}^n$, to $l_1, \dots, l_n$.



Obviously, $\eta(H) = v_1 = \psi_*(E_1)$, where E_1 is the constant vector field on $\mathbb{R}^n$ equal to e_1 at $\mathbf{0}$. Thus in the φ-coordinates on N the hamiltonian vector field $\eta(H)$ has the form $\eta(H) = (a_1, \dots, a_n)$ for some $a_i \in \mathbb{R}$. $\square$

In order to prove item 3 of the A–L theorem we will build special coordinates on M, the "action-angle" coordinates.

11 The "action-angle" coordinates

References: [Arn81]

Digression: the Darboux theorem: Let ω be a symplectic form on $M, \dim M = 2n$. Then in a neighbourhood of any point there exist local coordinates (q^i, p_i) such that

$$\omega = dp_i \wedge dq^i.$$

The coordinates (q^i, p_i) are called the *Darboux coordinates* (another name: the *canonical coordinates*) for ω. Note that the Darboux coordinates are not unique. Given such coordinates (q, p), any local symplectomorphism F, i.e. a local diffeomorphism preserving ω, will produce new Darboux coordinates (F^*q, F^*p).

Description of the "action-angle" coordinates: In the context of the Arnold–Liouville theorem, we will build specific Darboux coordinates on M, the "action-angle" coordinates. Let $N := M_c$ be a fixed common level set of the functions $g_1, \ldots, g_n$.

The "angles": Note that, although we have defined the angles $\varphi_1, \ldots, \varphi_n$ on a single level set, in fact these functions are defined also on the neighbour levels and depend smoothly on the level. Indeed, we can repeat the construction of the map $\psi : \mathbb{R}^n \to N$ on neighbour levels. As a result we will get a n-parameter family of maps $\psi_c : \mathbb{R}^n \to M_c$ to which there corresponds a n-parameter family of linear maps $A_c : \mathbb{R}^n \to \mathbb{R}^n$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{A_c} & \mathbb{R}^n \\ \downarrow p & & \downarrow \psi_c \\ \mathbb{T}^n & \longrightarrow & M_c \end{array} .$$

The maps ψ_c and A_c smoothly depend on c, consequently so do the angles on M_c.

The (g, φ)-coordinates: We claim that in a neighbourhood of N the functions $g_1, \ldots, g_n$ together with the angles $\varphi_1, \ldots, \varphi_n$ form a system of local coordinates. Indeed, the functions $g_1, \ldots, g_n$ are functionally independent by the assumption. They are also independent of the angles, because they are constant on the vector fields $\eta(g_1), \ldots, \eta(g_n)$ which are linear combinations of $\frac{\partial}{\partial\varphi_1}, \ldots, \frac{\partial}{\partial\varphi_n}$.

The "action-angle" coordinates (I, φ): These are coordinates such that: 1) the functions $(I^1, \ldots, I^n)$ depend only on g; 2) $\omega = dI^i \wedge d\varphi_i$. In particular, (I, φ) are the Darboux coordinates on (M, ω).

The initial differential equation in different coordinate systems: Recall that in the φ-coordinates on N the hamiltonian vector field $\eta(H)|_N$ has the form $\eta(H) = (a_1, \ldots, a_n)$ for some $a_i \in \mathbb{R}$. The corresponding differential equation is of the form:

$$\frac{d\overrightarrow{\varphi}}{dt} = \overrightarrow{a}(c),$$

and its solutions are

$$\overrightarrow{\varphi}(t) = \overrightarrow{\varphi}(0) + t\overrightarrow{a}(c).$$

Thus in the (g, φ)-coordinates the flow of $\eta(H)$ is given by the equation

$$\frac{d\overrightarrow{g}}{dt} = 0, \frac{d\overrightarrow{\varphi}}{dt} = \overrightarrow{a}(g).$$

Analogously, in the (I, φ)-coordinates the initial equation is of the form

$$\frac{d\overrightarrow{I}}{dt} = 0, \frac{d\overrightarrow{\varphi}}{dt} = \overrightarrow{a}(I).$$

However, due to the fact that (I, φ)-coordinates are canonical, we get $\overrightarrow{a}(I) = -\frac{\partial H}{\partial \overrightarrow{I}}, \frac{\partial H}{\partial \overrightarrow{\varphi}} = 0$. Thus, knowing the "action-angle" coordinates, we can easily calculate the vector of "frequences" $\overrightarrow{a}$.

Finally the solutions of this equation are given by

$$\overrightarrow{I} = const, \overrightarrow{\varphi}(t) = \overrightarrow{\varphi}(0) - t\frac{\partial H}{\partial \overrightarrow{I}}.$$

Construction of the "action-angle" coordinates (a sketch): *Geometrically* we can explain this construction as follows.

Let $(M, \omega), \dim M = 2n$, be a symplectic manifold and $g : M \to B$ a lagrangian fibration, i.e. a surjective submersion all fibers of which are lagrangian submanifolds in M. Let $c \in B$. Let us explain how the construction of the action of $\mathbb{R}^n$ on $M_c := g^{-1}(c)$ described in the previous section can be done simultaneously for all points from some neighbourhood U of c.

Namely, let $\alpha \in T_c^*B$ and let $f \in \mathcal{E}(B)$ be such that $d_c f = \alpha$. It is easy to see that the vector field $\eta(g^*f), \eta := \omega^{-1}$, is tangent to M_c (because $\eta(g^*f)g^*h = 0$ for any $f, h \in \mathcal{E}(B)$) and its restriction $\eta(g^*f)|_B$ is independent of the choice of f (if f' is another function with $d_c f' = \alpha$, we have $(g^*f - g^*f')|_B \equiv 0$ and $\eta(g^*f - g^*f')|_B \equiv 0$). Thus we get a linear mapping $\alpha \mapsto v(\alpha) := \eta(g^*f)|_B : T_c^*B \to \Gamma(TB)$, i.e. an action of the abelian (commutative) Lie algebra T_c^*B on M_c. Integrating this action (i.e. passing from vector fields to their flows) we get an action of the abelian group T_c^*B on M_c. Recall that fixing a point $x_c \in M_c$ we obtain a lattice $\Lambda_{x_c} \subset T_c^*B$, the stabilizer of x_c with respect to this action.

Now allow c to move over U. Repeating this construction for all points in U, we will have to choose $x_c \in M_c$, i.e. a section of g. Let us do this smoothly. As a result our lattice Λ_{x_c} will depend smoothly on c and we will get n one-forms $l_1, \ldots, l_n \in \Gamma(T^*B)$, the generators of this lattice.

It turns out that: 1) the section $c \mapsto x_c$ can be so chosen that its image will be a lagrangian submanifold in M; 2) if it will be chosen in such a way, the corresponding one-forms $l_1, \ldots, l_n$ will be closed, i.e. locally $l_i = dI_i$ for some functions I_i.

These last are the action coordinates we are looking for.

Analytically one can calculate the action coordinates as follows.

PROPOSITION. If $U \subset B$ is small enough the symplectic form ω is exact on $g^{-1}(U)$.

Proof We will use the De Rham theorem: $H^k_{DR}(M, \mathbb{R}) \cong (H_k(M, \mathbb{R}))^*$. Here $H^k_{DR}(M, \mathbb{R})$ stands for the space of the k-th De Rham cohomology, i.e. the factor space of closed modulo exact smooth k-forms. $H_k(M, \mathbb{R})$ is the space of the so-called singular k-th homology. It is known that it is isomorphic to its "smooth variant", which can be described as follows. Let $C_k(M, \mathbb{R})$ denote the space of finite formal linear combinations $a^i f_i$, where $a^i \in \mathbb{R}$ and f_i are smooth k-simplices in M, i.e. smooth maps from open neighbourhoods of k-dimensional simplices in $\mathbb{R}^k$ to M. The boundary operator $\partial_k : C_k(M,\mathbb{R}) \to C_{k-1}(M,\mathbb{R})$ satisfies the identity $\partial_{k-1} \circ \partial_k = 0$, so one can set $H_k(M,\mathbb{R}) := \ker \partial_k / \operatorname{im} \partial_{k+1}$.

Given $\alpha \in \Gamma(\bigwedge^k T^*M), a^i f_i \in C_k(M,\mathbb{R})$, put $\langle \alpha, a^i f_i \rangle := a^i \int_{\operatorname{im}(f_i)} \alpha$. The De Rham theorem says that in fact this pairing 1) induces a pairing between $H^k_{DR}(M,\mathbb{R})$ and $H_k(M,\mathbb{R})$ (this follows from the Stokes formula); 2) the induced pairing is nondegenerate.

In particular, it follows from the De Rham theorem that if the integral of a closed k-form over all the smooth k-cycles (i.e. the elements of $\ker \partial_k$) is zero, then this form is exact.

Now any 2-cycle f in $g^{-1}(U)$ is homotopically equivalent to some cocycle $\tilde{f}$ in $M_c, c \in U$. Thus $\int_{\operatorname{im} f} \omega = \int_{\operatorname{im} \tilde{f}} \omega = 0$. The last equality holds due to the fact that the restriction of ω to M_c is zero. $\square$

Let λ be the corresponding potential, $d\lambda = \omega$. Let $\gamma_{1,c}, \ldots, \gamma_{n,c}$ be the smooth closed curves on $M_c \cong \mathbb{T}^N$ representing the basis of $H_1(M_c,\mathbb{R}) \cong \mathbb{R}^n$. Put

$$I_i(c) := (1/2\pi) \int_{\gamma_{i,c}} \lambda.$$

Proposition. 1. This does not depend on the choice of the representatives.

2. This does not depend on the choice of the potential.

Proof follows from the Stokes formula.

Example (harmonic oscillator I): Let $M = \mathbb{R}^2, H = (1/2)(p^2+q^2), \omega = dp \wedge dq$. Then in the polar coordinates $q = r\cos\varphi, p = r\sin\varphi$ we have $dp \wedge dq = -\sin\varphi dr \wedge r\sin\varphi d\varphi + \cos\varphi r d\varphi \wedge \cos\varphi dr = -rdr \wedge d\varphi = d(-r^2/2) \wedge d\varphi$. Hence $I = -H$.

Example (harmonic oscillator II): Let $M = \mathbb{R}^2, H = (1/2)(a^2p^2 + b^2q^2), \omega = dp \wedge dq$. The hamiltonian vector field is $\eta(H) = -a^2p\frac{\partial}{\partial q} + b^2q\frac{\partial}{\partial p}$, here $\eta = \omega^{-1} = -\frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}$. The level sets $M_c = \{(q,p) \mid H(q,p) = c\}$ are ellipses $\{(q,p) \mid q^2/(2c/b^2) + q^2/(2c/a^2) = 1\}$ with the semiaxes $\sqrt{2c}/b, \sqrt{2c}/a$. Note that the standard parametrization of the ellipse, $\varphi \mapsto (\sqrt{2c}/b\cos\varphi, \sqrt{2c}/a\sin\varphi)$ is not a trajectory of $\eta(H)$

The receipt gives $I(c) = \frac{1}{2\pi}\int_{M_c} pdq = \frac{1}{2\pi}\int_{\overline{M}_c} \omega = -\frac{c}{ab}$, which up to $-\frac{1}{2\pi}$ is the area of the figure $\overline{M}_c := \{(q,p) \mid q^2/(2c/b^2) + q^2/(2c/a^2) \leqslant 1\}$ bounded by the ellipse. From this we conclude that $H = -abI$ and that the solution of the hamiltonian system

$$\dot{q} = -a^2p, \dot{p} = b^2q$$

is given by $H = c, \varphi(t) = \varphi(0) - t\frac{\partial H}{\partial I} = \varphi(0) + tab$ or, in other words, by

$$t \mapsto ((\sqrt{2c}/b)\cos(t_0 + tab), (\sqrt{2c}/a)\sin(t_0 + tab)).$$

12 Hamiltonian actions and moment maps

References: [dSW99]

A symplectic action of a Lie algebra $\mathfrak{g}$ on (M,ω)**:** An action $\rho : \mathfrak{g} \to \Gamma(TM)$ such that $\mathcal{L}_{\rho(v)}\omega = 0$ for any $v \in \mathfrak{g}$. Here $\mathcal{L}$ is the Lie derivative, the Cartan formula for it gives:

$$\mathcal{L}_{\rho(v)}\omega = i_{\rho(v)}d\omega + di_{\rho(v)}\omega = di_{\rho(v)}\omega.$$

A weakly hamiltonian action of a Lie algebra $\mathfrak{g}$ on (M,ω)**:** An action $\rho : \mathfrak{g} \to \Gamma(TM)$ such that there exists a linear map $\mathcal{J} : \mathfrak{g} \to \mathcal{E}(M)$ and the following diagram is commutative:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathcal{J}"] \arrow[dr, "\rho"'] & \mathcal{E}(M) \arrow[d, "\eta(\cdot)"] \\ & \Gamma(TM) \end{tikzcd} ,$$

i.e. $\rho(v) = \eta(\mathcal{J}(v))$ for any $v \in \mathfrak{g}$.

Remark If $\mathcal{J}$ is finite-dimensional, we can weaken the requirement: the map $\mathcal{J}$ a priori need not be linear (i.e. we only require that any vector field $\rho(v)$ is hamiltonian). If $\mathcal{J}$ is any map with the property $\rho(\cdot) = \eta(\mathcal{J}(\cdot))$, we can make it linear: let $e_1, \ldots, e_k$ be a basis of $\mathfrak{g}$, put $\mathcal{J}'(e_i) := \mathcal{J}(e_i), i = 1, \ldots, k$, and extend this by linearity. The new map $\mathcal{J}'$ satisfies $\rho(v) = \eta(\mathcal{J}'(v))$ and is linear.

Any weakly hamiltonian action is symplectic: $d_{i_{\eta(f)}}\omega = ddf = 0$. Conversely, any symplectic action is *locally* weakly hamiltonian: $di_{\rho(v)}\omega = 0$ implies by the Poincaré lemma that $i_{\rho(v)}\omega = df$ for some function f, hence $\rho(v) = \eta(f)$.

A moment map of a weakly hamiltonian action $\rho : \mathfrak{g} \to \Gamma(TM)$**:** the map $J : M \to \mathfrak{g}^*$ "dual to $\mathcal{J}$", i.e.

$$\mathcal{J}(v)(x) = \langle v, J(x)\rangle, x \in M, v \in \mathfrak{g}.$$

Let $\mathcal{J}' : \mathfrak{g} \to \mathcal{E}(M)$ be another map with the property $\rho(v) = \eta(\mathcal{J}'(v))$. Then $\eta((\mathcal{J}' - \mathcal{J})(v)) = 0$, hence $C := \mathcal{J}' - \mathcal{J}$ takes values in the space of Casimir functions of η (equal to $\mathbb{R}$ if M is connected, which is assumed) and $J' = J + C$, where $C : \mathfrak{g} \to \mathbb{R}$ is a linear map. The corresponding moment map $J' : M \to \mathfrak{g}$ is given by

$$\langle v, J'(x)\rangle = \mathcal{J}'(v)(x) = \mathcal{J}(v)(x) + C(v),$$

i.e. differs from J by a constant addend $C \in \mathfrak{g}^*$.

Remark: The map $\mathcal{J}$ determines the moment map J by the formula above uniquely, but the converse also is true. Thus any smooth map $J : M \to \mathfrak{g}^*$ generates a linear map $\mathcal{J} : \mathfrak{g} \to \mathcal{E}(M)$ and, consequently, a linear map $\rho : \mathfrak{g} \to \Gamma(TM)$ ($\rho := \eta \circ \mathcal{J}$). In order that this map is an action, we need to make some additional assumptions on J (see the end this lecture).

A hamiltonian action of a Lie algebra $\mathfrak{g}$ on (M,ω)**:** A weakly hamiltonian action $\rho : \mathfrak{g} \to \Gamma(TM)$ such that among linear maps $\mathcal{J} : \mathfrak{g} \to \mathcal{E}(M)$ with the property $\rho(\cdot) = \eta(\mathcal{J}(\cdot))$ there exists a homomorphism of Lie algebras $(\mathfrak{g}, [,])$ and $(\mathcal{E}(M), \{,\}_\eta)$.

Remark Note that for any other map $\mathcal{J}' = \mathcal{J} + C$ with $\rho(\cdot) = \eta(\mathcal{J}'(\cdot))$ we have: $\mathcal{J}'[v,w] = \mathcal{J}[v,w] + C([v,w]) = \{\mathcal{J}v, \mathcal{J}w\}_\eta + C([v,w]) = \{\mathcal{J}v + C(v), \mathcal{J}w + C(w)\}_\eta + C([v,w]) = \{\mathcal{J}'v, \mathcal{J}'w\}_\eta + C([v,w])$. thus $\mathcal{J}'$ is a homomorphism if an only if C vanishes on the commutant $[\mathfrak{g},\mathfrak{g}] = \{[v,w] \mid v,w \in \mathfrak{g}\}$ of the Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ is semisimple (as $\mathfrak{sl}(n,\mathbb{R}), \mathfrak{so}(n,\mathbb{R}), \mathfrak{sp}(n,\mathbb{R})$), we have $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$, hence the homomorphic $\mathcal{J}$ is defined uniquely.

Proposition. *A map $\mathcal{J} : \mathfrak{g} \to \mathcal{E}(M)$ is a homomorphism if and only if the corresponding moment map $J : M \to \mathfrak{g}^*$ is Poisson, here $\mathfrak{g}^*$ is endowed with the Lie-Poisson structure $\eta_\mathfrak{g}$.*

Proof Let $e_1, \ldots, e_n$ be a basis of $\mathfrak{g}$ and let $y_1, \ldots, y_n$ be the the elements of this basis regarded as linear functions on $\mathfrak{g}^*$. With these notation we have in view of f the definition of the moment map the following equalities: $\mathcal{J}e_i = J^* y_i, i = 1, \ldots, n$.

Denote by c_{ij}^k the corresponding structure constants: $[e_i, e_j] = c_{ij}^k e_k$. Assume $\mathcal{J}$ is a homomorphism, i.e. $\{\mathcal{J}e_i, \mathcal{J}e_j\}_\eta = c_{ij}^k \mathcal{J}e_k$. This can be rewritten as $\{J^* y_i, J^* y_j\}_\eta = c_{ij}^k J^* y_k = J^*\{y_i, y_j\}_{\eta_\mathfrak{g}}$, which means the Poisson property of the moment map. Inverting the considerations we get also another implication. $\square$

Remark: Similarly to the case of weakly hamiltonian actions any smooth Poisson map $J : M \to \mathfrak{g}^*$ generates a hamiltonian action of g on M such that one of its moment maps coincide with J.

Hamiltonian actions and projectability: Let $\rho : \mathfrak{g} \to \Gamma(TM)$ be a hamiltonian action such that its orbits form a foliation $\mathcal{F}$ and the factor space $M/\mathcal{F}$ is good. Let $p : M \to M' := M/\mathcal{F}$ be the natural projection. Then $\eta := \omega^{-1}$ is projectable with respect to p. Indeed, $T\mathcal{F} = \langle \eta(\mathcal{J}e_1), \ldots, \eta(\mathcal{J}e_n) \rangle$ and the dual foliation is given by $\{\mathcal{J}e_1 = c_1, \ldots, \mathcal{J}e_n = c_n\}$, i.e. coincides with the fibers of the moment map. As a result we get a dual pair of Poisson maps

$$
\begin{tikzcd}
 & (M,\eta) \arrow[dl, "p"'] \arrow[dr, "J"] & \\
(M', p_*\eta) & & (\mathfrak{g}^*, \eta_\mathfrak{g}).
\end{tikzcd}
$$

Example 1: Let $H : M \to \mathbb{R}$ be any function with the nonvanishing differential. Then we have $\rho : \mathbb{R} \to \Gamma(TM), 1 \mapsto \eta(H), \mathcal{J} : \mathbb{R} \to \mathcal{E}(M), 1 \mapsto H, J = H, T\mathcal{F} = \langle \eta(H) \rangle$

$$
\begin{tikzcd}
 & (M,\eta) \arrow[dl, "p"'] \arrow[dr, "H=J"] & \\
(M/\mathcal{F}, p_*\eta) & & (\mathbb{R} = \mathbb{R}^*, 0).
\end{tikzcd}
$$

In particular, if $M := T^*\mathbb{R}^2 \setminus \{0\}, \omega = dp \wedge dq, H(q,p) = q_1^2 + q_2^2 + (p^1)^2 + (p^2)^2$, we get the Hopf fibrations over the symplectic leaves of $p_*\eta$.

Example 2: Let $M \subset \mathfrak{g}^*$ be a coadjoint orbit endowed with the canonical symplectic form $\omega := (\eta_\mathfrak{g}|_M)^{-1}$. Then the coadjoint action $\rho : \mathfrak{g} \to \Gamma(T\mathfrak{g}^*), v \mapsto \widetilde{ad_v^*}$ is hamiltonian. Indeed, $\widetilde{ad_v^*} = \eta_\mathfrak{g}(v')$ (see Lecture 8) where v' denotes the linear function on $\mathfrak{g}^*$ defined by an element $v \in \mathfrak{g}$. Thus $\mathcal{J} : \mathfrak{g} \to \mathcal{E}(M)$ is given by $v \mapsto v'|_M$ and $J : M \to \mathfrak{g}^*$ coincides with the inclusion $M \hookrightarrow \mathfrak{g}^*$.

Example 3: Let $\rho : \mathfrak{g} \to \Gamma(TM)$ be a hamiltonian action with a moment map $J : M \to \mathfrak{g}^*$ and let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. Then $\rho|_{\mathfrak{h}} : \mathfrak{h} \to \Gamma(TM)$ is a hamiltonian action and its moment map $J_{\mathfrak{h}} : M \to \mathfrak{h}^*$ is given by $i^* \circ J$, where $i^* : \mathfrak{g}^* \to \mathfrak{h}^* = \mathfrak{g}/\mathfrak{h}^{\perp}$ is the projection dual to the inclusion $i : \mathfrak{h} \hookrightarrow \mathfrak{g}$.

Remark about relations between weakly hamiltonian and hamiltonian actions: Let $\rho : \mathfrak{g} \to \Gamma(TM)$ be a weakly hamiltonian action. Let us examine obstructions for ρ to be a hamiltonian action.

Let $\mathcal{J} : \mathfrak{g} \to \mathcal{E}(M)$ be map with the property $\rho(\cdot) = \eta(\mathcal{J}(\cdot))$. Put $c(v, w) := \{\mathcal{J}v, \mathcal{J}w\}_\eta - \mathcal{J}([v, w])$.

Proposition. *1. $c(v, w)$ is a constant function for any $v, w \in \mathfrak{g}$;*

2. c is a 2-cocycle on the Lie algebra $\mathfrak{g}$, i.e. it is a bilinear skew-symmetric function on $\mathfrak{g}$ satisfying $\sum_{c.p.\ v,w,u} c([v, w], u) = 0$ for any $v, w, u \in \mathfrak{g}$.

Proof Item 1. We have $\eta(c(v, w)) = \eta(\{\mathcal{J}v, \mathcal{J}w\}_\eta - \mathcal{J}([v, w])) = [\eta(\mathcal{J}v), \eta(\mathcal{J}w)] - \rho([v, w]) = [\rho(v), \rho(w)] - \rho([v, w]) = 0$, hence $c(v, w)$ is a Casimir function for η.

Item 2. We have $\{\mathcal{J}[v, w], \mathcal{J}u\}_\eta = \eta(\mathcal{J}[v, w])\mathcal{J}u = \rho([v, w])\mathcal{J}u = [\rho(v), \rho(w)]\mathcal{J}u = \rho(v)\rho(w)\mathcal{J}u - \rho(w)\rho(v)\mathcal{J}u = \rho(v)\eta(\mathcal{J}w)\mathcal{J}u - \rho(w)\eta(\mathcal{J}v)\mathcal{J}u = \rho(v)\{\mathcal{J}w, \mathcal{J}u\}_\eta - \rho(w)\{\mathcal{J}v, \mathcal{J}u\}_e = \{\mathcal{J}v, \{\mathcal{J}w, \mathcal{J}u\}_\eta\}_\eta - \{\mathcal{J}w, \{\mathcal{J}v, \mathcal{J}u\}_\eta\}_\eta$.

Hence $\sum_{c.p.\ v,w,u} c([v, w], u) = \sum_{c.p.\ v,w,u}\{\mathcal{J}[v, w], \mathcal{J}u\}_\eta - \mathcal{J}([[v, w], u])) = 0$ due to the Jacobi identity for $[,]$ and $\{,\}_\eta$. $\square$

It is known that for a semisimple $\mathfrak{g}$ any 2-cocycle c is cohomologically trivial, i.e. there exists $C \in \mathfrak{g}^*$ such that $c(v, w) = C([v, w])$.

Proposition. *If the cocycle c is trivial, the map $\mathcal{J}' := \mathcal{J} + C : \mathfrak{g} \to \mathcal{E}(M)$ is a homomorphism.*

Proof $\mathcal{J}'([v, w]) = \mathcal{J}([v, w]) + C([v, w]) = \mathcal{J}([v, w]) + \{\mathcal{J}v, \mathcal{J}w\}_\eta - \mathcal{J}([v, w]) = \{\mathcal{J}v, \mathcal{J}w\}_\eta = \{\mathcal{J}'v, \mathcal{J}'w\}_\eta$. $\square$

We conclude that for semisimple $\mathfrak{g}$ any weakly hamiltonian action is hamiltonian.

In general, the cocycle c is nontrivial. Note that c is defined nonuniquely, since so is the map $\mathcal{J}$. Taking $\mathcal{J}' = \mathcal{J} + C$ (see *Remark* above) we get the formula $c'(v, w) = \{\mathcal{J}'v, \mathcal{J}'w\}_\eta - \mathcal{J}'([v, w]) = \{\mathcal{J}v, \mathcal{J}w\}_\eta - \mathcal{J}([v, w]) - C([v, w]) = c(v, w) - C([v, w])$, i.e. the nontriviality of c does not depend on the choice of $\mathcal{J}$. So there exist weakly hamiltonian actions not being hamiltonian. For such actions the moment map is not Poisson, but one can modify the Poisson structure on $\mathfrak{g}^*$ (adding a cocycle to $\eta_{\mathfrak{g}}$ and obtaining a Poisson structure with affine coefficients) in such a way that the moment map will be Poisson.

13 Right and left actions on T^*G. Hamiltonian actions and completely integrable systems

The cotangent lift of a vector field: Put $M := T^*Q$. Let $\zeta \in \Gamma(TQ)$ be a vector field. Then it can be interpreted as a function $H_\zeta : T^*Q \to \mathbb{R}, H_\zeta(\alpha) := \langle \alpha, \zeta|_x \rangle, \alpha \in T^*_x Q$. Put $\zeta^\sqcup := \eta(-H_\zeta), \eta := \omega^{-1}$, where ω is the canonical symplectic form on T^*Q. We say that $\zeta^\sqcup$ is the *cotangent lift* of ζ.

In the (q,p)-local coordinates on T^*Q we have $H_\zeta(q,p) = p_i \zeta^i(q)$ for $\zeta = \zeta^i(q)\frac{\partial}{\partial q^i}$ (because $H_\zeta(\alpha) = \alpha_i \zeta^i(q)$ for $\alpha = \alpha_i dq^i$) and $\zeta^\sqcup = \frac{\partial H_\zeta}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H_\zeta}{\partial q^i}\frac{\partial}{\partial p_i} = \zeta^i(q)\frac{\partial}{\partial q^i} - p_j \frac{\partial \zeta^j}{\partial q^i}\frac{\partial}{\partial p_i}$. Note that $H_\zeta = \lambda(\zeta)$, where $\lambda = pdq$ is the canonical Liouville 1-form on M.

PROPOSITION. *The map $\zeta \mapsto \zeta^\sqcup : \Gamma(TQ) \to \Gamma(TM)$ is a homomorphism of Lie algebras.*

Proof We will prove that the map $\zeta \mapsto -H_\zeta : (\Gamma(TQ), [,]) \to (\mathcal{E}(M), \{,\}_\eta)$ is a homomorphism of Lie algebras. Indeed, $\{-H_\zeta, -H_\xi\}_\eta = -\frac{\partial H_\zeta}{\partial p_i}\frac{\partial H_\xi}{\partial q^i} + \frac{\partial H_\xi}{\partial p_i}\frac{\partial H_\zeta}{\partial q^i} = -\zeta^i(q)p_j\frac{\partial \xi^j}{\partial q^i} + \xi^i(q)p_j\frac{\partial \zeta^j}{\partial q^i} = -H_{[\zeta,\xi]}$. $\square$

Thus we get a (hamiltonian) right action $\zeta \mapsto \zeta^\sqcup$ of the Lie algebra $\Gamma(TQ)$ on M.

The cotangent lift of a right action $\rho : \mathfrak{g} \to \Gamma(TQ)$: this is a hamiltonian action $\rho^\sqcup : \mathfrak{g} \to \Gamma(TM)$ given by $\rho^\sqcup(v) := (\rho(v))^\sqcup$. The corresponding map $\mathcal{J} : \mathfrak{g} \to \mathcal{E}(M)$ is given by $v \mapsto -H_{\rho(v)}$ and the corresponding moment map $J : M \to \mathfrak{g}^*$ is given by $\langle v, J(x)\rangle = \mathcal{J}(v)(x) = -H_{\rho(v)}(x) = -\lambda(\rho(v))(x), v \in \mathfrak{g}, x \in M$.

Left and right invariant vector fields on a Lie group G: Let G be a Lie group, $\mathfrak{g} = T_eG$ its Lie algebra. Given $g \in G$ put $L_g : G \to G, L_g g' := gg', R_g : G \to G, R_g g' := g'g$. Given $v \in \mathfrak{g}$ put

$$v_l(g) := (L_g)_* v, v_r(g) := (R_g)_* v.$$

The vector field v_l is left invariant, i.e. for any $g' \in G$ we have $(L_{g'})_* v_l(g) = v_l(g'g)$. Indeed, $(L_{g'})_* v_l(g) = (L_{g'})_*(L_g)_* v = (L_{g'g})_* v = v_l(g'g)$. Analogously v_r is right invariant.

PROPOSITION. *1. The maps $v \mapsto v_l : \mathfrak{g} \to \Gamma(TG), v \mapsto v_r : \mathfrak{g} \to \Gamma(TG)$ are a homomorphism and an antihomomorphism of Lie algebras, respectively.*

2. $[v_l, w_r] = 0$ for any $v, w \in \mathfrak{g}$.

Remark: Item 2 is an infinitesimal emanation of the fact that L_g and $R_{g'}$ commute for any $g, g' \in G$.

Example: Let $G := GL(n, \mathbb{R})$ (nondegenerate $n \times n$-matrices with real entries) $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) = T_IG$ (all $n \times n$-matrices with real entries). Since G is an open set in a vector space, we have $TG = G \times \mathfrak{g}$ and any vector field is of the form $X \mapsto (X, V(X))$ i.e. is represented by a matrix valued function $V(X) = \begin{bmatrix} V_{11}(X) & \dots & V_{1n}(X) \\ & \dots & \\ V_{n1}(X) & \dots & V_{nn}(X) \end{bmatrix}$. It is easy to see that if $V \in \mathfrak{g}$, then $V_l(X) = XV, V_r = VX$. In other words, $V_l = X_{ij}V_{jk}\partial_{ik}, V_r = V_{ij}X_{jk}\partial_{ik}$. Thus we have $[V_l, W_l] = (X_{ij}V_{jk}\partial_{ik}X_{i'j'}W_{j'k'})\partial_{i'k'} - \ldots = (X_{ij}V_{jk}\delta_{ii'}\delta_{kj'}W_{j'k'})\partial_{i'k'} - \ldots = (X_{ij}V_{jk}W_{kk'})\partial_{ik'} - (X_{ij}W_{jk}V_{kk'})\partial_{ik'} =$

$X_{ij}[V,W]_{jk'}\partial_{ik'} = [V,W]_l$ and $[V_l, W_r] = (X_{ij}V_{jk}\partial_{ik}W_{i'j'}X_{j'k'})\partial_{i'k'} - (W_{ij}X_{jk}\partial_{ik}X_{i'j'}V_{j'k'})\partial_{i'k'} = (X_{ij}V_{jk}W_{i'j'}\delta_{ij'}\delta_{kk'})\partial_{i'k'} - (W_{ij}X_{jk}\delta_{ii'}\delta_{kj'}V_{j'k'})\partial_{i'k'} = (X_{ij}V_{jk}W_{i'i})\partial_{i'k} - (W_{ij}X_{jk}V_{kk'})\partial_{ik'} = 0$.

Let us define a right action $\rho_l : v \mapsto v_l^{\sqcup} : \mathfrak{g} \to \Gamma(TT^*G)$ of $\mathfrak{g}$ on T^*G and a left action $\rho_r : v \mapsto v_r^{\sqcup} : \mathfrak{g} \to \Gamma(TT^*G)$ of $\mathfrak{g}$ on T^*G. These actions are hamiltonian, the corresponding $\mathcal{J}$-maps are given by $\mathcal{J}_l : v \mapsto -H_{v_l}$ and $\mathcal{J}_r : V \mapsto -H_{v_r}$ and the corresponding moment maps $J_l, J_r : T^*G \to \mathfrak{g}^*$ are $\langle J_l(x), v\rangle = -H_{v_l}(x), \langle J_r(x), v\rangle = -H_{v_r}(x), x \in T^*G, v \in \mathfrak{g}$.

PROPOSITION. *The orbits of the action ρ_l coincide with the fibers of the moment map J_r and vice versa.*

Proof We know that the fibers of the moment map J_r are skew-orthogonal with respect to ω to the orbits of the action ρ_r. Let us prove that the orbits of ρ_l are also skew-orthogonal to that of ρ_r.

Indeed, $\omega(\eta(H_{v_l}), \eta(H_{v_r})) = dH_{v_l}(\eta(H_{v_r})) = \eta(H_{v_r})H_{v_l} = \{H_{v_r}, H_{v_l}\}_\eta = -H_{[v_r,v_l]} = 0$. $\square$

Summarizing, we get the following dual pair of Poisson maps:

$$\begin{tikzcd}[column sep=small] & (T^*G,\eta) \arrow[dl, "-J_r"'] \arrow[dr, "J_l"] & \\ (\mathfrak{g}^*, \eta_{\mathfrak{g}}) & & (\mathfrak{g}^*, \eta_{\mathfrak{g}}). \end{tikzcd}$$

Complete families of functions in involution: Let (M,η) be a Poisson structure. Let $\operatorname{Sing}\eta$ denote the union of all symplectic leaves of η of nonmaximal dimension.

We say that a set $I \subset \mathcal{E}(M)$ is a family of functions *in involution* if $\{f,g\}_\eta = 0$ for any $f, g \in I$. We say that a family I of functions in involution is *complete* if there exists an open dense set $U \subset M$ such that $\dim \operatorname{Span}\{d_x f \mid f \in I\} = (1/2)\operatorname{rank}\eta_x + \dim M - \operatorname{rank}\eta_x = \dim M - (1/2)\operatorname{rank}\eta_x$ for any $x \in U \setminus (U \cap \operatorname{Sing}\eta)$ (in other words, the common level sets of functions from I form a lagrangian foliation in any symplectic leaf of η on $U \setminus (U \cap \operatorname{Sing}\eta)$).

Example 1. Let η be nondegenerate. Then I is complete if and only if the common level sets form a lagrangian foliation on an open dense subset in M.

Example 2. Let M be 3-dimensional and $\operatorname{rank}\eta_x = 2$ on an open dense subset $U \subset M$. Assume f is a Casimir function for η on U and g is any function whose differential is linearly independent of that of f on U. Then f, g functionally generate a complete set of functions in involution.

For instance, let $M = \mathfrak{g} = \mathfrak{so}(3,\mathbb{R}) = \mathbb{R}^3, \eta = \eta_{\mathfrak{g}}$. Then $f = x_1^2 + x_2^2 + x_3^2$ and we can take any independent g, say $g = x_1$. The corresponding lagrangian foliation consists of the circles obtained by the intersections of concentric spheres and parallel planes $\{x_1 = const\}$. We can take $U = \mathbb{R}^3 \setminus \{x_2 = 0, x_3 = 0\}$.

Let (M,η) be a nondegenerate Poisson structure and let $p' : M \to M', p'' : M \to M''$ be a dual pair of surjective Poisson maps. Put $\eta' := p'_*\eta, \eta'' := p''_*\eta$.

$$\begin{tikzcd}[column sep=small] & (M,\eta) \arrow[dl, "p'"'] \arrow[dr, "p''"] & \\ (M', \eta') & & (M'', \eta''). \end{tikzcd}$$

PROPOSITION. *Assume $I' \subset \mathcal{E}(M'), I'' \subset \mathcal{E}(M'')$ are complete families of functions in involution for η', η'' respectively. Put $((p')^*I') = \{((p')^*f) \mid f \in I'\}$ and $((p'')^*I'') = \{((p'')^*g) \mid g \in I''\}$. Then the set $I := ((p')^*I') + ((p'')^*I'') \subset \mathcal{E}(M)$ is a complete family of functions in involution for η.*

Proof Let us first prove that the functions from I are in involution. Indeed, the functions form $(p')^*I'$ are in involution because so are the functions from I' and the map $(p')^*$ is a homomorphism of Poisson brackets. The same argument works for $(p'')^*I''$. Finally, any function f' from $(p')^*I'$ commutes with any function f'' from $(p'')^*I''$ due to the skew-orthogonality of the fibers of p' and p'' (recall that $\eta(f'), \eta(f'')$ are tangent to the fibers of p'', p', respectively): $\{f, g\}_\eta = \eta(f)g = -\omega(\eta(f), \eta(g)) = 0$.

Now let us prove the completeness. Let $\mathcal{F}', \mathcal{F}''$ denote the foliations of fibers of p', p'' respectively. Notice that $D := T\mathcal{F}' + T\mathcal{F}''$ is an integrable generalized distribution. Indeed, let (x, y) be local coordinates on M such that the foliation $\mathcal{F}'$ is given by $\{x^1 = c_1, \ldots, x^k = c_k\}$. Then $D = \langle \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^{n-k}}, \eta(x^1), \ldots, \eta(x^k) \rangle$, here $n := \dim M$. Since η is projectable along $\mathcal{F}'$, the vector fields $\eta(x^1), \ldots, \eta(x^k)$ form an involutive generalized distribution (see the Liebermann–Weinstein criterion of projectability). Since the coefficients of these vector fields depend only on x they commute with $\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^{n-k}}$. Obviously the generalized foliation $\mathcal{F}$ tangent to D is the pull-back (with respect to p'') of the symplectic foliation of η'' (whose characteristic distribution is spanned by $\eta(x^1), \ldots, \eta(x^k)$). Due to the symmetry of the objects with prime and double prime we deduce that $\mathcal{F}$ is also the pull-back with respect to p' of the symplectic foliation of η'. We conclude that $\operatorname{corank} \eta'_{p'(z)} = \operatorname{corank} \eta''_{p''(z)}$ for any $z \in M$ (here by definition the corank of a bivector η on a manifold M at a point $z \in M$ is the difference $\dim M - \operatorname{rank} \eta_z$).

Let U', U'' stand for the corresponding open dense sets in M', M'' appearing in the definition of the completeness of I', I''. Put $V := (p')^{-1}(U' \setminus (U' \cap \operatorname{Sing} \eta')) \cap (p'')^{-1}(U'' \setminus (U'' \cap \operatorname{Sing} \eta'')), V' := p'(V), V'' := p''(V)$. The above considerations show that and that $(p')^*\mathcal{C}_{\eta'}(V') = (p'')^*\mathcal{C}_{\eta''}(V'') =: Z$ (recall that $\mathcal{C}_\eta(U)$ denotes the space of the Casimir functions of a bivector η over an open set U).

Let us choose a functional basis $\{f_1, \ldots, f_{s'}\}$ of I' such that $f_1|_{V'}, \ldots, f_{r'}|_{V'}$ is a functional basis of $\mathcal{C}_{\eta'}(V')$ and any functional basis $\{g_1, \ldots, g_{s''}\}$ of I''. Then the functions $(p')^*f_{r'+1}, \ldots, (p')^*f_{s'}, (p'')^*g_1, \ldots, (p'')^*g_{s''}$ are functionally independent on V since

$$\{(p')^*f|_V \mid f \in \mathcal{E}(V')\} \cap \{(p')^*g|_V \mid g \in \mathcal{E}(V'')\} = Z.$$

Now, we have

$$\begin{aligned} s' - r' &= \frac{1}{2}\operatorname{rank} \eta'_{p'(z)} = \frac{1}{2}(\dim T_z\mathcal{F}'' - \dim T_z\mathcal{F}'' \cap T_z\mathcal{F}'), \\ s'' &= \frac{1}{2}\operatorname{rank} \eta''_{p''(z)} + \operatorname{corank} \eta''_{p''(z)} = \frac{1}{2}(\dim T_z\mathcal{F}' - \dim T_z\mathcal{F}'' \cap T_z\mathcal{F}') + \dim T_z\mathcal{F}'' \cap T_z\mathcal{F}', \end{aligned}$$

and, finally

$$s' - r' + s'' = \frac{1}{2}(\dim T_z\mathcal{F}'' + \dim T_z\mathcal{F}') = \frac{1}{2}\dim M.$$

Here z is any point of V. $\square$

Example: the Euler–Manakov top (n-dimensional free rigid body): Let $G = SO(n, \mathbb{R}), M = T^*G$. Let $b(v, w)$ be a positively defined scalar product on $\mathfrak{so}(n, \mathbb{R})^* \cong \mathfrak{so}(n, \mathbb{R}) =: \mathfrak{g}$. Then there exists an operator $A : \mathfrak{so}(n, \mathbb{R}) \to \mathfrak{so}(n, \mathbb{R})$, which is symmetric with respect to the standard scalar product $(v, w) := -\operatorname{Tr}(vw)$, i.e. $(Av, w) = (v, Aw)$, such that $b(v, w) = (Av, w), v, w \in \mathfrak{g}$. Let $b_l : T^*G \times_G T^*G \to \mathbb{R}$ denote the left invariant extension of the scalar product b to a (contravariant) metric on G and let $B : T^*G \to \mathbb{R}$ denote the corresponding quadratic form.

The Euler–Manakov top is the hamiltonian system with the hamiltonian function $H := B : M \to \mathbb{R}$ in case when the operator A is given by $A := L^{-1}, Lv := Dv + vD$, where $D := \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, a diagonal matrix with the eigenvalues $\lambda_1, \dots, \lambda_n$. The eigenvalues λ_i coincide with the "moments of inertia" $\int_V x_i^2 \sigma(x) dx$, where V is the region in $\mathbb{R}^n$ occupied by the body and $\sigma(x)$ is the density function.

Consider the classical Euler case, $n = 3$. The hamiltonian function is left invariant. This means that it belongs to the family $(p')^* I'$ in the notations of the fact above, where $p' = -J_r$. Consider the set I'' functionally generated by the Casimir function f on $\mathfrak{so}^*(3, \mathbb{R})$ and any other independent function g. The functions $H, (p'')^* f, (p'')^* g$, where $p'' := J_l$ are independent first integrals in involution. Thus we have proven the complete integrability of the Euler top (because the dimension of the phase space M is 6).

In the general case ($n > 3$) we need more functions in involution for integrating the system. In the next sections we will construct complete families of functions in involution on $\mathfrak{g}^*$ for any semisimple $\mathfrak{g}$ (these families will play a role of I''). We will also construct complete families of functions in involution on $\mathfrak{so}(n, \mathbb{R})^*$ playing the role of I' and containing the reduced hamiltonian $b(v, v) = -\operatorname{Tr}((Av)v) = \sum_{i<j} (\lambda_i + \lambda_j)^{-1} v_{ij}^2$.

14 Poisson pencils and families of functions in involution

References: [Mag78]

A Poisson pencil on M: Let a pair (η_1, η_2) of linearly independent bivectors on a manifold M be given. Assume $\eta^t := t_1\eta_1 + t_2\eta_2$ is a Poisson structure for any $t = (t_1, t_2) \in \mathbb{R}^2$. We say that the Poisson structures η_1, η_2 are *compatible* (or form a *bihamiltonian structure* or a *Poisson pair*) and that the whole family $\Theta := \{\eta^t\}_{t\in\mathbb{R}^2}$ is a *Poisson pencil.*

Exercise: Show that the following conditions are equivalent:

1. η^t is Poisson, i.e. $[\eta^t, \eta^t]_S = 0$, for any $t \in \mathbb{R}^2$ (here $[,]_S$ is the Schouten bracket);
2. $[\eta^t, \eta^t]_S = 0$ for any three pairwise nonproportional values of $t \in \mathbb{R}^2$;
3. $[\eta_1, \eta_1]_S = 0, [\eta_1, \eta_2]_S = 0, [\eta_2, \eta_2]_S = 0$.

Example 1: Let η_1, η_2 be bivectors on $\mathbb{R}^n$ with constant coefficients. Then they form a Poisson pair (recall that, given a bivector $\eta = \eta^{ij}(x)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, we have $[\eta, \eta]_S^{ijk} := \sum_{c.p.\, i,j,k} \eta^{ir}(x)\frac{\partial}{\partial x^r}\eta^{jk}(x)$).

Example 2: Let $\mathfrak{g}$ be a Lie algebra and $\eta_{\mathfrak{g}}$ the Lie–Poisson structure on $\mathfrak{g}^*$. Let $c : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be a 2-cocycle on $\mathfrak{g}$, i.e. c is skew-symmetric and $\sum_{c.p.\ v,w,u} c([v, w], u) = 0$ for any $v, w, u \in \mathfrak{g}$. Then $c \in (\mathfrak{g} \wedge \mathfrak{g})^* \cong \mathfrak{g}^* \wedge \mathfrak{g}^*$ can be regarded as a bivector on $\mathfrak{g}^*$ with constant coefficients. It turns out that (η_1, η_2), where $\eta_1 := \eta_{\mathfrak{g}}, \eta_2 := c$, is a Poisson pair.

Indeed, it is easy to see that the bracket $[(v, \alpha), (w, \beta)]' := ([v, w], c(v, w))$ defines a Lie algebra structure on $\mathfrak{g}' := \mathfrak{g} \times \mathbb{R}$ (*Exercise:* check this). The $\mathbb{R}$-component lies in the centre of $\mathfrak{g}'$, we say that $\mathfrak{g}'$ is a *central extension* of $\mathfrak{g}$. The affine subspaces $\mathfrak{g}^*_{x_0} := \mathfrak{g}^* \times x_0 \subset (\mathfrak{g}')^* = \mathfrak{g}^* \times \mathbb{R}$ are Poisson submanifolds of the Poisson manifold $((\mathfrak{g}')^*, \eta_{\mathfrak{g}'})$. The restriction $\eta_{\mathfrak{g}'}|_{\mathfrak{g}^*_{x_0}}$ coincides with $\eta_1 + x_0\eta_2$, i.e. the last bivector is Poisson at least for three different values of x_0. We conclude that (η_1, η_2) is a Poisson pair.

In coordinates this looks as follows. Let $e_1, \ldots, e_n$ be a basis of $\mathfrak{g}$ and $[e_i, e_j] = c^k_{ij}e_k, c(e_i, e_j) = c_{ij}, i, j, k = 1, \ldots, n$, for some constants $c^k_{ij}, c_{ij} \in \mathbb{R}$. Put $\eta'_0 := (0, 1), \eta'_i := (\eta_i, 0) \in \mathfrak{g}', i = 1, \ldots, n$, and let $x'_0, \ldots, x'_n$ denote the same elements regarded as coordinates on $(\mathfrak{g}')^*$. Then $\eta_{\mathfrak{g}'} = (c^k_{ij}x'_k + x'_0c_{ij})\frac{\partial}{\partial x'_i} \wedge \frac{\partial}{\partial x'_j}$ and $\eta^t = (t_1c^k_{ij}x_k + t_2c_{ij})\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$. Here $x_1, \ldots, x_n$ are coordinates on $\mathfrak{g}^*$ corresponding to $e_1, \ldots, e_n$.

Example 3: In a particular case when the cocycle c is trivial, i.e. $c(v, w) = a([v, w])$ for some $a \in \mathfrak{g}^*$ we get a Poisson pencil $\{\eta^t\}, \eta^t := (t_1c^k_{ij}x_k + t_2c^k_{ij}a_k)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, here $a_1, \ldots, a_n$ are coordinates of a in the dual basis $e^1, \ldots, e^n$ of $\mathfrak{g}^*$. In the corresponding Poisson pair (η_1, η_2) the first bivector is the Lie-Poisson one, $\eta_{\mathfrak{g}}$, and the second one is $\eta_{\mathfrak{g}}(a)$, the Lie-Poisson bivector "frozen" at a.

Example 4: Let $\mathfrak{g} : \mathfrak{gl}(n, \mathbb{R})$ and $A \in \mathfrak{g}$. Put $[x, y]_A := xAy - yAx$. It is easy to see that $[,]_A$ is a Lie bracket on $\mathfrak{g}$ for any A (*Exercise*: check this). In particular, for a fixed $A \in \mathfrak{g}$ the bracket $[,]^t := t_1[,] + t_2[,]_A = [,]_{t_1I+t_2A}$ is a Lie bracket for any $t \in \mathbb{R}^2$ (any family of Lie brackets linearly spanned by two fixed brackets will be called a *Lie pencil*). Denote $\mathfrak{g}^t := (\mathfrak{g}, [,]^t)$. The Lie–Poisson structures $\eta_{\mathfrak{g}^t}$ form a Poisson pencil on $\mathfrak{g}^*$.

We get a generalization of this example taking $\mathfrak{g} := \mathfrak{so}(n, \mathbb{R})$ and A a symmetric $n \times n$-matrix.

I mechanism of constructing functions in involution (the Magri–Lenard scheme): Let (η_1, η_2) be a pair of Poisson structures (not necessarily compatible). Assume we can found a sequence of functions $H_0, H_1, \ldots \in \mathcal{E}(M)$ satisfying

$$\begin{array}{rcl} \eta_1(H_0) & = & \eta_2(H_1) \\ \eta_1(H_1) & = & \eta_2(H_2) \\ & \vdots & . \end{array} \tag{1}$$

Proposition. *For any indices i, j the following equality holds:*

$$\{H_i, H_j\}_{\eta_1} = \{H_{i+1}, H_{j-1}\}_{\eta_1}.$$

Proof $\eta_1(H_i)H_j = \eta_2(H_{i+1})H_j = -\eta_2(H_j)H_{i+1} = -\eta_1(H_{j-1})H_{i+1} = \eta_1(H_{i+1})H_{j-1}$ $\square$

Now assume $i < j$. If $j - i = 2k$, we can apply the proposition k times and get $\{H_i, H_j\}_{\eta_1} = \{H_{i+k}, H_{j-k}\}_{\eta_1} = \{H_{i+k}, H_{i+k}\}_{\eta_1} = 0$. If $j - i = 2k + 1$, we get $\{H_i, H_j\}_{\eta_1} = \{H_{i+k}, H_{j-k}\}_{\eta_1} = \{H_{i+k}, H_{i+k+1}\}_{\eta_1} = \eta_1(H_{i+k})H_{i+k+1} = \eta_2(H_{i+k+1})H_{i+k+1} = 0$. Hence the sequence $H_0, H_1, \ldots$ is a family of first integrals in involution for any of vector fields $v_i := \eta_1(H_i), i = 0, 1, \ldots$ Note that all these vector fields are "bihamiltonian", i.e. hamiltonian with respect to both the Poisson structures η_1, η_2.

In general it is hard to find the sequences of functions $H_0, H_1, \ldots$ with the required properties. However, if we assume additionally that (η_1, η_2) is a Poisson pair, there are some cases, when such sequences naturally appear. For instance, assume that all the bivectors $\eta^t := t_1\eta_1 + t_2\eta_2$ of the corresponding Poisson pencil are degenerate. Let $\eta^\lambda := \lambda\eta_1 + \eta_2, \lambda := t_1/t_2$, and let f^λ be a Casimir function of η^λ. It turns out that f^λ depends smoothly, let $f^\lambda = f_0 + \lambda f_1 + \lambda^2 f_2 + \cdots$ be the corresponding Tailor expansion. Then we deduce from the equality $\eta^\lambda(f^\lambda) = 0$ that $0 = \eta_2(f_0), \eta_1(f_0) + \eta_2(f_1), \eta_1(f_1) + \eta_2(f_2), \ldots$ (coefficients of different powers of λ). Thus we can put $H_0 := f_0, H_1 := -f_1, H_2 := f_2, \ldots$ Note that such a Magri–Lenard chain starts from a Casimir function of η_2. If $g^\lambda = g_0 + \lambda g_1 + \cdots$ is another Casimir function of η^λ, we get another sequence of functions in involution. A question arises, is it true that $\{f_i, g_j\}_{\eta_k} = 0$? Another important question concerns the *completeness* of the obtained family of functions.

II mechanism of constructing functions in involution (based on the Casimir functions of a Poisson pencil): Let $\{\eta^t\}_{t \in \mathbb{R}^2}$ be a Poisson pencil on M. Denote by $\mathcal{C}^t(M)$ the space of Casimir functions of η^t.

Proposition. *Let $t', t'' \in \mathbb{R}^2$ be linearly independent and let $f \in \mathcal{C}^{t'}(M), g \in \mathcal{C}^{t''}(M)$. Then*

$$\{f, g\}_{\eta^t} = 0$$

for any $t \in \mathbb{R}^2$.

Proof Indeed for any $t \in \mathbb{R}^2$ there exist $c', c'' \in \mathbb{R}$ such that $t = c't' + c''t''$. Then $\{f, g\}_{\eta^t} = \eta^t(f)g = (c'\eta^{t'} + c''\eta^{t''})(f)g = c''\eta^{t''}(f)g = -c''\eta^{t''}(g)f = 0$. $\square$

It is not clear from this fact whether $\{f, g\}_{\eta^t} = 0$ if f, g are Casimir functions of the *same* bivector $\eta^{t'}$. We will discuss this question in the next lecture.

15 Linear algebra of pairs of bivectors and completeness of families of functions in involution

References: [GZ89, Bol91]

The Jordan–Kronecker decomposition of a pair of bivectors: A bivector b on a vector space V is an element of $\bigwedge^2 V$. We will view a bivector b sometimes as a skew-symmetric map $V^* \to V$ (then its value at $x \in V^*$ will be denoted by $b(x)$) and sometimes as a skew-symmetric bilinear form on V^* (then its value at $x, y \in V^*$ will be denoted by $b(x, y)$). In particular, $b(x, y) = \langle b(x), y \rangle$.

THEOREM. *(Gelfand–Zakharevich, 1989) Given a finite-dimensional vector space V over $\mathbb{C}$ and a pair of bivectors $(b^{(1)}, b^{(2)})$, $b^{(i)} : \bigwedge^2 V^* \to \mathbb{C}$, there exists a direct decomposition $V^* = \oplus_{m=1}^k V_m^*$ such that $b^{(i)}(V_l^*, V_m^*) = 0$ for $i = 1, 2, l \neq m$, and the triples $(V_m^*, b_m^{(1)}, b_m^{(2)})$, where $b_m^{(i)} := b^{(i)}|_{V_m^*}$, are from the following list:*

1. [the Jordan block $\mathbf{j}_{2j_m}(\lambda)$]: $\dim V_m^* = 2j_m$ *and in an appropriate basis of V_m^* the matrices of $b_m^{(1)}, b_m^{(2)}$ are equal to*
$$\begin{bmatrix} \mathbf{0} & I_{j_m} \\ -I_{j_m} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & J_{j_m}(\lambda) \\ -J_{j_m}(\lambda)^T & \mathbf{0} \end{bmatrix}$$
where I_{j_m} is the unity $j_m \times j_m$-matrix and
$$J_{j_m}(\lambda) := \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}$$
is the Jordan $j_m \times j_m$-block with the eigenvalue λ;

2. [the Jordan block $\mathbf{j}_{2j_m}(\infty)$]: $\dim V_m^* = 2j_m$ *and in an appropriate basis of V_m^* the matrices of $b_m^{(1)}, b_m^{(2)}$ are equal to*
$$\begin{bmatrix} \mathbf{0} & J_{j_m}(0) \\ -J_{j_m}(0)^T & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & I_{j_m} \\ -I_{j_m}^T & \mathbf{0} \end{bmatrix};$$

3. [the Kronecker block $\mathbf{k}_{2k_m+1}$]: $\dim V_m^* = 2k_m + 1$ *and in an appropriate basis of V_m^* the matrices of $b_m^{(1)}, b_m^{(2)}$ are equal to*
$$K_{1,k_m} := \begin{bmatrix} \mathbf{0} & B_{1,k_m} \\ -B_{1,k_m}^T & \mathbf{0} \end{bmatrix}, K_{2,k_m} := \begin{bmatrix} \mathbf{0} & B_{2,k_m} \\ -B_{2,k_m}^T & \mathbf{0} \end{bmatrix},$$

where

$$B_{1,k_m} := \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}, B_{2,k_m} := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

($k_m \times (k_m+1)$-matrices).

Kronecker Poisson pencils: Let $\{\eta^t\}_{t\in\mathbb{R}^2}, \eta^t := t_1\eta_1 + t_2\eta_2$, be a Poisson pencil on M. We say that it is *Kronecker at a point* $x \in M$, if the Jordan–Kronecker decomposition of the pair of bivectors $\eta_1|_x, \eta_2|_x$ (regarded as elements of $\bigwedge^2 T_x^{\mathbb{C}}M$, here $T_x^{\mathbb{C}}M$ is the complexified tangent space) does not contain Jordan blocks.

PROPOSITION. *$\{\eta^t\}_{t\in\mathbb{R}^2}$ is Kronecker at x if and only if*

$$\operatorname{rank}(t_1\eta_1|_x + t_2\eta_2|_x) = const, (t_1,t_2) \in \mathbb{C}^2 \setminus \{0\}.$$

Proof It is easy to see that any nontrivial linear combination of matrices K_{1,k_m}, K_{2,k_m} has constant rank equal to $2k_m$. So the rank can "jump" at some $t \neq 0$ if and only if there are Jordan blocks in the decomposition. □

We say that a Poisson pencil Θ on M is *Kronecker* if there exists an open dense set $U \subset M$ such that Θ is Kronecker at any $x \in U$.

Involutivity of Casimir functions for Kronecker Poisson pencils: We have already proven that, if $t', t'' \in \mathbb{R}^2$ are linearly independent, then $\{f,g\}_{\eta^t} = 0$ for any $f \in \mathcal{C}^{t'}(M), g \in \mathcal{C}^{t''}(M), t \in \mathbb{R}^2$. In the same way one can prove that $\eta^t|_x(\alpha,\beta) = 0$ for any $\alpha \in \ker \eta^{t'}|_x, \beta \in \ker \eta^{t''}|_x, t \in \mathbb{R}^2$.

PROPOSITION. *Let $\{\eta^t\}_{t\in\mathbb{R}^2}$ be Kronecker and let $t' \in \mathbb{R}^2, t' \neq 0$. Then $\{f,g\}_{\eta^t} = 0$ for any $f,g \in \mathcal{C}^{t'}(M), t \in \mathbb{R}^2$.*

Proof Fix $x \in U$. Let $t_{(n)} \in \mathbb{R}^2$ be such that $t_{(n)}$ is linearly independent with t' and $t_{(n)} \overset{n\to\infty}{\longrightarrow} t'$. The kernel of the map $\eta^t|_x : T_x^*M \to T_xM$ continuously depend on $t \in \mathbb{R}^2 \setminus \{0\}$ and is of constant dimension. Consequently we can find a sequence of covectors $\alpha_n \in \ker \eta^{t_{(n)}}|_x$ such that $\alpha_n \overset{n\to\infty}{\longrightarrow} d_xg$. We get $\eta^t|_x(d_xf, \alpha_n) = 0$ and by continuity we conclude that $\eta^t|_x(d_xf, d_xg) = 0$. In other words, $\{f,g\}_{\eta^t}(x) = 0$ for any $x \in U$. Since U is dense, using again the continuity argument we get the proof. □

Summarizing, we get the following result.

PROPOSITION. *Let $\Theta = \{\eta^t\}_{t\in\mathbb{R}^2}$ be a Kronecker Poisson pencil and let*

$$\mathcal{C}^{\Theta}(M) := \operatorname{Span}\{\bigcup_{t\in\mathbb{R}^2\setminus\{0\}} \mathcal{C}^t(M)\}.$$

Then $\mathcal{C}^{\Theta}(M)$ is a family of functions in involution with respect to any Poisson bivector η^t.

Remark: It can be shown that in the Kronecker case the family of functions in involution obtained by the Magri-Lenard scheme starting from Casimir functions coincide with the family $\mathcal{C}^{\Theta}(M)$.

Completeness of Casimir functions for Kronecker Poisson pencils: Let (M, η) be a Poisson structure. We say that an open set $W \subset M$ is *correct* for η if the set $W' := W \setminus (W \cap \operatorname{Sing} \eta)$ is nonempty and the common level sets of the functions from $\mathcal{C}^{\eta}(W')$ coincide with the symplectic foliation of η on the set W'. In other words, the set W is correct if the Poisson structure does not have regular symplectic leaves dense in W. Equivalent definition: W is correct if $\{d_x f \mid f \in \mathcal{C}^{\eta}(W)\} = \ker \eta_x$ for any $x \in W'$. Note that in analytic category any sufficiently small open set is correct.

PROPOSITION. *Let $\Theta = \{\eta^t\}_{t \in \mathbb{R}^2}$ be a Kronecker Poisson pencil. Assume $W \subset M$ is an open set that is correct for η^t for a countable set $\{t_{(1)}, t_{(2)}, \ldots\}$ of pairwise linearly independent values of the parameter t and the set $W' := W \setminus \bigcup_{i=1}^{\infty} \operatorname{Sing} \eta^{t_{(i)}}$ is nonempty. Then the set of functions in involution $\mathcal{C}^{\Theta}(W')$ is* complete *with respect to any $\eta^t, t \neq 0$.*

Proof Fix $x \in U \cap W'$. Let us first prove that the set $C_x := \{d_x f \mid f \in \mathcal{C}^{\Theta}(W')\} \subset T_x^* M$ coincides with the set $L_x := \operatorname{Span}\{\bigcup_{t \in \mathbb{R}^2 \setminus \{0\}} \ker \eta_x^t\}$. Indeed, the vector space L_x is finite-dimensional, hence is generated by a finite number of kernels $\ker \eta_x^t = \{d_x f \mid f \in \mathcal{C}^t(W)\}$. Hence $L_x \subset C_x$. The same considerations show that $C_x \subset L_x$.

It is easy to see that the set L_x is of dimension $(1/2)\operatorname{rank} \eta_x^t + \dim M - \operatorname{rank} \eta_x^t$. Assume for a moment that the Jordan–Kronecker decomposition of the pair $\eta_1|_x, \eta_2|_x$ consists of one Kronecker block $\mathbf{k}_{2k_m+1}$. The kernel of the matrix $\lambda K_{1,k_m} + K_{2,k_m}$ is 1-dimensional and is spanned by the vector $[0, \ldots, 0, 1, -\lambda, \ldots, (-\lambda)^{k_m}]$. Taking $k_m + 1$ different values of λ we get $k_m + 1 = (1/2)\operatorname{rank} \eta_x^t + \dim M - \operatorname{rank} \eta_x^t$ linearly independent vectors (recall the Vandermonde determinant) spanning the set L_x. In the case of several Kronecker blocks you repeat these considerations for each block. $\square$

Remark: In fact it is sufficient to require that W is correct for a finite number of η^t. However, this number depends on the number and dimension of the Kronecker blocks, so we make a bit stronger assumption (which in practice is always satisfied).

Example (method of the argument translation): Let $M := \mathfrak{g}^*, \eta_1 := \eta_{\mathfrak{g}}, \eta_2 := \eta_{\mathfrak{g}}(a), S := \operatorname{Sing} \eta_{\mathfrak{g}}$, where $a \in \mathfrak{g}^* \setminus S$. Assume that $\operatorname{codim} S \geqslant 2$ (if $\mathfrak{g}$ is semisimple it is known that $\operatorname{codim} S \geqslant 3$). Note that S is an algebraic set, i.e. it is defined by a finite number of algebraic equations $f_1(x) = 0, \ldots, f_m(x) = 0$ on $\mathfrak{g}^*$. Any algebraic set in a neighbourhood of its generic point is diffeomorphic to a manifold, hence its dimension is correctly defined.

If $e_1, \ldots, e_n$ is a basis of $\mathfrak{g}$ and the corresponding structure constants are defined by $[e_i, e_j] = c_{ij}^k e_k$, the polynomials $f_1, \ldots, f_m$ are the $r \times r$-minors of the matrix $c_{ij}(x) = c_{ij}^k x_k$, where $r = \max_x \operatorname{rank} [c_{ij}(x)]$. Here $x_1 = e_1, \ldots, x_n = e_n$ are the corresponding coordinates on $\mathfrak{g}^*$.

In order to check the condition of Kroneckerity we need to consider the complexification $\mathfrak{g}_{\mathbb{C}}$ of the initial Lie algebra. It can be regarded as a vector space $\operatorname{Span}_{\mathbb{C}}\{e_1, \ldots, e_n\} \cong \mathbb{C}^n$ with the Lie bracket defined by the same structure constants. The set $S_{\mathbb{C}} := \{(z_1, \ldots, z_n) \in \mathfrak{g}_{\mathbb{C}}^* \cong \mathbb{C}^n \mid \operatorname{rank} c_{ij}^k z_k < \max_{z \in \mathbb{C}^n} \operatorname{rank} c_{ij}^k z_k\}$ is a complex algebraic set defined by the equations $f_1(z) = 0, \ldots, f_m(z) = 0$, where $f_1, \ldots, f_m$ are the same polynomials as above. In particular, the set $S_{\mathbb{C}}$ is of complex codimension at least 2.

We know that $t_1\eta_1|_x + t_2\eta_2|_x = c_{ij}^k(t_1x_k + t_2a_k), t_1, t_2 \in \mathbb{C}$. Thus rank $(t_1\eta_1|_x + t_2\eta_2|_x)$ is maximal (over t) and independent of $t \in \mathbb{C}^2 \setminus \{0\}$ if and only if $t_1x + t_2a \in \mathfrak{g}_{\mathbb{C}}^* \setminus S$ if and only if $x \notin \overline{a, S_{\mathbb{C}}}$, where $\overline{a, S_{\mathbb{C}}} := \{z \in \mathfrak{g}_{\mathbb{C}}^* \mid \exists (t_1, t_2) \in \mathbb{C}^2 \setminus \{0\} \colon t_1z + t_2a \in S_{\mathbb{C}}\}$.

Note that the set $S_{\mathbb{C}}$ is homogeneous (stable under rescaling). Passing to the projectivization the set $\overline{a, S_{\mathbb{C}}}$ becomes a cone in $\mathbb{CP}^{n-1}$ over the projectivization of S. This shows that the set $\overline{a, S_{\mathbb{C}}}$ is also algebraic (by the standard arguments from algebraic geometry) and, moreover, $\dim_{\mathbb{C}} \overline{a, S_{\mathbb{C}}} = \dim_{\mathbb{C}} S_{\mathbb{C}} + 1$. In particular $\operatorname{codim}_{\mathbb{C}} \overline{a, S_{\mathbb{C}}} \geqslant 1$ and we can put $U := \mathfrak{g}^* \setminus (\mathfrak{g}^* \cap \overline{a, S_{\mathbb{C}}}) = \mathfrak{g}^* \setminus (\overline{a, S})$. Here $\overline{a, S} := \{x \in \mathfrak{g}^* \mid \exists (t_1, t_2) \in \mathbb{R}^2 \setminus \{0\} \colon t_1x + t_2a \in S\}$ and $\operatorname{codim}_{\mathbb{R}} \overline{a, S} \geqslant 1$. The set U is an open dense set in $\mathfrak{g}^*$ such that $\{\eta^t\}$ is Kronecker at any $x \in U$.

Finally assume that $\mathfrak{g}$ is semisimple. Then $\eta_{\mathfrak{g}}$ has enough global Casimir functions and the whole space $\mathfrak{g}^*$ is a correct set for $\eta_{\mathfrak{g}}$. In particular, the assumptions of the proposition above are satisfied and we get a complete set $\mathcal{C}^{\Theta}(\mathfrak{g}^*)$ of functions in involution (with respect to any η^t). This set is generated by the "translations" $f(x + \lambda a), \lambda \in \mathbb{R}$, of the Casimir functions f of $\eta_{\mathfrak{g}}$.

16 Lie pencils and completely integrable systems

References: [Raï78, Bol91]

Digression on semidirect products of Lie algebras: Let $\rho : \mathfrak{g} \to \mathrm{End}(V)$ be a representation of a Lie algebra $\mathfrak{g}$ on a vector space V.

Exercise: Prove that the formula $[(x,v),(y,w)]' = ([x,y], \rho(x)w - \rho(y)v)$ defines a Lie algebra structure on $\mathfrak{g}' := \mathfrak{g} \times V$.

We put $\mathfrak{g} \times_\rho V := (\mathfrak{g} \times V, [,]')$ and say that $\mathfrak{g} \times_\rho V$ is a *semidirect product* of $\mathfrak{g}$ and V.

Note that the subspaces $\mathfrak{g}_0 := \mathfrak{g} \times \{0\} \subset \mathfrak{g}', \mathfrak{g}_1 := \{0\} \times V \subset \mathfrak{g}'$ satisfy the following commutation relations: $[\mathfrak{g}_0, \mathfrak{g}_0]' \subset \mathfrak{g}_0, [\mathfrak{g}_0, \mathfrak{g}_1]' \subset \mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_1]' = \{0\}$ (in particular $\mathfrak{g}_0$ is an abelian ideal of $\mathfrak{g}'$). And it is easy to see that, given any Lie algebra $\mathfrak{g}' = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with the commutation relations as above, we can put $\rho(x) := \mathrm{ad}'_x|_{\mathfrak{g}_1}, x \in \mathfrak{g}_0$ (here $\mathrm{ad}'_x v := [x,v]'$), and get a representation of a Lie algebra $\mathfrak{g}_0$ on the vector space $\mathfrak{g}_1$ and an isomorphism of $\mathfrak{g}'$ with $\mathfrak{g}_0 \times_\rho \mathfrak{g}_1$ (*Exercise:* prove this).

Given a Lie algebra $\mathfrak{g}$, we call the codimension of a regular coadjoint orbit the *index* of $\mathfrak{g}$. In particular, $\mathrm{ind}\,\mathfrak{g} = \mathrm{corank}\,\eta_{\mathfrak{g}}|_x := \dim \mathfrak{g} - \mathrm{rank}\,\eta_{\mathfrak{g}}|_x$ for generic $x \in \mathfrak{g}^*$.

THEOREM. *(Raïs, 1978)*

$$\mathrm{ind}(\mathfrak{g} \times_\rho V) = \mathrm{ind}\,\mathfrak{g}_v + \mathrm{codim}\, O_\nu.$$

Here $\nu \in V^*$ is a generic element, O_ν is the orbit of this element with respect to the dual (anti) representation $\rho^* : \mathfrak{g} \to \mathrm{End}(V^*), \rho^*(x) := (\rho(x))^*$, and $\mathfrak{g}_\nu := \{x \in \mathfrak{g} \mid \rho^*(x)\nu = 0\}$ is the stabilizer of the element ν with respect to ρ^*.

Example: Let $\mathfrak{g} := \mathfrak{so}(n,\mathbb{R})$ and $\rho : \mathfrak{g} \to \mathrm{End}(\mathbb{R}^n)$ be the standard representation (the skew-symmetric matrices act on vector-columns). Then $\mathfrak{e}(n,\mathbb{R}) := \mathfrak{so}(n,\mathbb{R}) \times_\rho \mathbb{R}^n$ is called the *euclidean* Lie algebra.

The standard euclidean scalar product $(|)$ on $\mathbb{R}^n$ is invariant with respect to ρ, i.e. $(\rho(x)v|w) = -(v|\rho(x)w) = 0$. In particular, we can identify the orbits of ρ and ρ^*. Thus the orbit of ρ^* through an element $\nu \in (\mathbb{R}^n)^* \cong \mathbb{R}^n$ is the sphere $S^{n-1}_{|\nu|}$ of radius $|\nu|$. The stabilizer $\mathfrak{g}_\nu$ is the Lie algebra of rotations "around" ν (i.e. preserving ν) and is isomorphic to the Lie algebra $\mathfrak{so}(n-1,\mathbb{R})$ (of rotations "around" $(1,0,\ldots,0)$). Finally, $\mathrm{ind}\,\mathfrak{e}(n,\mathbb{R}) = \mathrm{ind}\,\mathfrak{so}(n-1,\mathbb{R}) + 1$.

Recall that the ring of Casimir functions of $\eta_{\mathfrak{g}}$ is generated by $\mathrm{Tr}(x^2), \mathrm{Tr}(x^4) \ldots, \mathrm{Tr}(x^{2k})$ for $n = 2k+1$ and by $\mathrm{Tr}(x^2), \mathrm{Tr}(x^4) \ldots, \mathrm{Tr}(x^{2k-2}), \mathrm{Pf}(x)$ for $n = 2k$. Hence $\mathrm{ind}\,\mathfrak{so}(n,\mathbb{R}) = [n/2]$. In particular, $\mathrm{ind}\,\mathfrak{e}(n-1,\mathbb{R}) = [(n-2)/2] + 1 = [n/2] = \mathrm{ind}\,\mathfrak{so}(n,\mathbb{R})$.

Digression on contractions of Lie algebras: Assume $(\mathfrak{g}, [,])$ is a Lie algebra and that there exists a family of Lie brackets $[,]^\lambda$ on $\mathfrak{g}$ continuously depending on the parameter $\lambda \in U \setminus \{\lambda_0\}$, here $U \subset \mathbb{R}^k$ is an open set, $\lambda_0 \in U$ is a fixed element. Assume that $[,] = [,]^\lambda$ for some $\lambda \in U \setminus \{\lambda_0\}$ and that for any $x, y \in \mathfrak{g}$ there exists $\lim_{\lambda \to \lambda_0} [x,y]^\lambda =: [x,y]_0$. Then by the continuity the bracket $[,]_0$ will be a Lie bracket on $\mathfrak{g}$. We will say that $(\mathfrak{g}, [,]_0)$ is a *contraction* of a Lie algebra $(\mathfrak{g}, [,])$.

Example: Let $(\mathfrak{g}, [,])$ be any Lie algebra and let $[,]^\lambda := \lambda[,], \lambda \in \mathbb{R}\setminus\{0\}$. Then $\lim_{\lambda\to 0}[x,y]^\lambda =: [x,y]_0$ exists and gives an abelian Lie bracket on $\mathfrak{g}$.

Lie pencils and complete families of functions in involution: Let $\mathfrak{g} = \mathfrak{so}(n,\mathbb{R}), \mathfrak{g}^t := (\mathfrak{g}, [,]^t)$, where $[,]^t := [,]_{t_1 I + t_2 A}$, $A = \operatorname{diag}(a_1, \dots, a_n)$ is a fixed diagonal matrix with a simple spectrum. The linear map given by $L^t : X \mapsto \sqrt{t_1 I + t_2 A} X \sqrt{t_1 I + t_2 A}$ is an isomorphism of the Lie algebras $\mathfrak{g}^{(1,0)}$ and $\mathfrak{g}^t$ for t nonproportional to $(a_1,-1),\dots,(a_n,-1)$. Indeed, $[L^t X, L^t Y] = \sqrt{t_1 I + t_2 A} X (t_1 I + t_2 A) Y \sqrt{t_1 I + t_2 A} - \sqrt{t_1 I + t_2 A} Y (t_1 I + t_2 A) X \sqrt{t_1 I + t_2 A} = L^t [X,Y]_{t_1 I + t_2 A}$.

We claim that the Lie algebra $(\mathfrak{g}, [,]^t)$ for $t \neq (0,0)$ proportional to one of the vectors $(a_1,-1),\dots,(a_n,-1)$ is isomorphic to $\mathfrak{e}(n-1,\mathbb{R})$ (hence $\mathfrak{e}(n-1,\mathbb{R})$ is a contraction of $\mathfrak{so}(n,\mathbb{R})$). For instance, take $t = (a_1,-1)$. The map $L' : X \mapsto \sqrt{A'} X \sqrt{A'}$, where $A' := \operatorname{diag}(1, 1/\sqrt{a_1 - a_2}, \dots, 1/\sqrt{a_1 - a_n})$, gives the isomorphism of $[,]^{(a_1,-1)}$ with $[,]_B, B := (0,1,\dots,1)$.

Let us prove, that $(\mathfrak{g}, [,]_B)$ is isomorphic to $\mathfrak{e}(n-1,\mathbb{R})$. Put

$$\mathfrak{g}_0 := \left\{ \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & y_{12} & \cdots & y_{1,n-1} \\ 0 & -y_{12} & 0 & \cdots & y_{2,n-1} \\ & & & \cdots & \\ 0 & -y_{1,n-1} & -y_{2,n-1} & \cdots & 0 \end{bmatrix} \mid y_{ij} \in \mathbb{R}, i<j \right\}, \mathfrak{g}_1 := \left\{ \begin{bmatrix} 0 & -y_1 & \cdots & -y_n \\ y_1 & 0 & \cdots & 0 \\ & & \cdots & \\ y_n & 0 & \cdots & 0 \end{bmatrix} \mid y_i \in \mathbb{R} \right\}.$$

Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and it is easy to see that $[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0, [\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_0$. In particular, $\mathfrak{g}_0$ is a Lie subalgebra (isomorphic to $\mathfrak{so}(n-1,\mathbb{R})$). On the other hand we obviously have: 1) $[\mathfrak{g}_0,\mathfrak{g}_0]_B = [\mathfrak{g}_0,\mathfrak{g}_0]$; 2) $[\mathfrak{g}_0,\mathfrak{g}_1]_B = [\mathfrak{g}_0,\mathfrak{g}_1]$; 3) $[\mathfrak{g}_1,\mathfrak{g}_1]_B = \{0\}$. So to finish the proof it remains to notice that the representation $\rho : \mathfrak{g}_0 \to \operatorname{End}(\mathfrak{g}_1), \rho(x) := \operatorname{ad}_x|_{\mathfrak{g}_1}$ is isomorphic to the standard representation of $\mathfrak{so}(n-1,\mathbb{R})$ on $\mathbb{R}^n$ (*Exercise:* check this).

Now we are ready to prove the kroneckerity of the Poisson pencil $\Theta := \{t_1\eta_1 + t_2\eta_2\}_{(t_1,t_2)\in\mathbb{R}^2}$ on $\mathfrak{g}^*$ associated to the Lie pencil $\{(\mathfrak{g}, [,]^t\}_{t\in\mathbb{R}^2}$. Here $\eta_1 := \eta_{\mathfrak{g}}$ is the canonical Lie–Poisson structure on $\mathfrak{so}(n,\mathbb{R})$ and η_2 is the Lie–Poisson structure corresponding to the modified commutator $[,]_A$. We need to prove that for a generic point $x \in \mathfrak{g}^*$ we have $\operatorname{rank}(t_1\eta_1|_x + t_2\eta_2|_x) = const$ for $(t_1,t_2) \in \mathbb{C}^2 \setminus \{0\}$.

Let $e_1,\dots,e_n$ be a basis of $\mathfrak{g}$ and let the corresponding structure constants are defined by $[e_i,e_j] = c_{ij}^k e_k, [e_i,e_j] = C_{ij}^k e_k$. The condition above can be rewritten as $\operatorname{rank}(t_1 c_{ij}^k x_k + t_2 C_{ij}^k x_k) = const, (t_1,t_2) \in \mathbb{C}^2 \setminus \{0\}$. To prove it let us pass to the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(n,\mathbb{C})$ (skew-symmetric matrices with complex entries). The same considerations as above show that the map $L^t : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}, X \mapsto \sqrt{t_1 I + t_2 A} X \sqrt{t_1 I + t_2 A}$, where $(t_1,t_2) \in \mathbb{C}^2$ is nonproportional to $(a_1,-1),\dots,(a_n,-1)$, is an isomorphism of the corresponding Lie algebras. In other words, $t_1 c_{ij}^k x_k + t_2 C_{ij}^k x_k = L_{ii'}^t L_{jj'}^t (L^t)_{kk'}^{-1} c_{i'j'}^k x_k$, here the matrix $L_{ii'}^t$ is defined as $L^t e_i = L_{ii'}^t e_{i'}$ and similarly $(L^t)_{kk'}^{-1}$. Thus we conclude that the rank of $t_1 c_{ij}^k x_k + t_2 C_{ij}^k x_k$ is constant as far as t belongs to $T := \mathbb{C}^2 \setminus (\operatorname{Span}_{\mathbb{C}}\{(a_1,-1)\} \cup \dots \cup \operatorname{Span}_{\mathbb{C}}\{(a_n,-1)\})$ and x belongs to $V := \mathfrak{g}_{\mathbb{C}} \setminus (\bigcup_{t\in T} (L^t)^{-1} S_{\mathbb{C}})$. Recall that $S := \operatorname{Sing}\eta_{\mathfrak{g}}$ is the set $\{x \in \mathfrak{g} \mid \operatorname{rank}(c_{ij}^k x_k) < \max_x \operatorname{rank}(c_{ij}^k x_k)\}$ and $S_{\mathbb{C}}$ is its complexification.

Finally we use the fact that $\operatorname{ind}\mathfrak{e}(n-1,\mathbb{C}) = \operatorname{ind}\mathfrak{so}(n,\mathbb{C})$ (which can be proved in the same way as in real case) to conclude that Θ is Kronecker at any point $x \in U := \mathfrak{g} \cap V \setminus (V_1 \cup \dots \cup V_n)$. Here $V_i := \operatorname{Sing}\eta_{\mathfrak{g}_i}, \mathfrak{g}_i := (\mathfrak{g}, [,]^{(a_i,-1)}), i = 1,\dots,n$. The set U is dense because $\mathfrak{g} \cap V = \mathfrak{g} \setminus (\bigcup_{t\in T'} (L^t)^{-1} S)$, where $T' := \mathbb{R}^2 \setminus (\operatorname{Span}_{\mathbb{R}}\{(a_1,-1)\} \cup \dots \cup \operatorname{Span}_{\mathbb{R}}\{(a_n,-1)\})$, and $\operatorname{codim}(\bigcup_{t\in T'} (L^t)^{-1} S) \geqslant 2$ due to the condition $\operatorname{codim}_{\mathbb{R}} S \geqslant 3$.

The corresponding complete family $\mathcal{C}^{\Theta}(\mathfrak{g}^*)$ of functions in involution is generated by the functions $f((L^t)^{-1}x), t \in \mathbb{R}^2$, where f is a Casimir function of $\eta_{\mathfrak{g}}$.

One can show that the hamiltonian $\mathrm{Tr}((L^{-1}x)x), Lx = Dx + xD$, of the Euler–Manakov top is contained in the family $\mathcal{C}^{\Theta}(\mathfrak{g}^*)$ (with $A := D^2$), but this is a little bit technical question and we will skip it.

17 Introduction to the KdV equation and infinite-dimensional argument translation method

References: [Ke03, Mag78]

The Gelfand–Fuchs cocycle: Let $\mathfrak{g} := \Gamma(TS^1)$ be the Lie algebra of vector fields on a circle. Elements of $\mathfrak{g}$ can be viewed as $v(x)\partial_x$, where v is a function on S^1 and x is a coordinate. The bracket will be expressed as $[v(x)\partial_x, w(x)\partial_x] = (-vw_x + wv_x)(x)\partial_x$.

Proposition. *The expression $c(v\partial_x, w\partial_x) := \int_{S^1} vw_{xxx}dx$ is a cocycle on $\mathfrak{g}$.*

Proof $c([v\partial_x, w\partial_x], u\partial_x) = \int_{S^1}(-vw_x+wv_x)u_{xxx}dx = \int_{S^1}(-vw_x+wv_x)du_{xx} =$ [integration by parts] $= -\int_{S^1} u_{xx}(-vw_x + wv_x)_x dx = -\int_{S^1} u_{xx}(-v_xw_x + w_xv_x - v_{xx}w + w_{xx}v)dx = \int_{S^1} u_{xx}(v_{xx}w - w_{xx}v)dx$. Summing the last expression over cyclic permutations of v, w, u gives zero. *Exercise:* Prove the skew-symmetry. □

The Virasoro Lie algebra: The central extension $\mathfrak{g}' := \mathfrak{g}\oplus\mathbb{R}$ of $\mathfrak{g}$ with respect to the Gelfand–Fuchs cocycle: $[(v(x)\partial_x, a), (w(x)\partial_x, b)]' := ((-vw_x + wv_x)(x)\partial_x, c(v\partial_x, w\partial_x))$.

The "$H^1_{\alpha\beta}$-energy" on $\mathfrak{g}'$: The quadratic form

$$\langle (v(x)\partial_x, a), (w(x)\partial_x, b)\rangle := \int_{S^1}(\alpha vw + \beta v_x w_x)dx + ab.$$

If $\alpha = 1, \beta = 0$ we get the L^2 scalar product, if $\alpha = 1, \beta = 1$, this is the Sobolev one.

The Virasoro group: This is a central extension $G' := G \times \mathbb{R}$ of the group $G := Diff(S^1)$ of diffeomorphisms of a circle by means of the Bott cocycle

$$B(\psi, \varphi) := \int_{S^1} \log((\psi\circ\varphi)_x) d\log(\varphi_x).$$

The group operation on G' is given by

$$(\psi(x), a) \circ (\varphi(x), b) := ((\psi\circ\varphi)(x), a + b + B(\psi, \varphi)).$$

Remark: If $\alpha \neq 0$ the $H^1_{\alpha\beta}$-energy can be extended to a right-invariant metric on G.

The dual space $\mathfrak{g}^*$: It can be naturally identified with the space of quadratic differentials $\{u(x)(dx)^2\}$ on the circle. The pairing is given by the formula:

$$\langle u(x)(dx)^2, v(x)\partial_x\rangle := \int_{S^1} u(x)v(x)dx.$$

The coadjoint orbits coincide with the orbits of the action of diffeomorphisms on quadratic differentials:

$$\mathrm{Ad}^*_\varphi : u(dx)^2 \mapsto u(\varphi)\cdot\varphi_x^2(dx)^2 = u(\varphi)(d\varphi)^2.$$

Remark: If $u(x) > 0$ for any $x \in S^1$, the square root $\sqrt{u(x)(dx)^2}$ transforms as a 1-form. In particular, $\Phi(u(x)(dx)^2) := \int_{S^1} \sqrt{u(x)}dx$ is a Casimir function: the value of Φ is stable under the diffeomorphism action. The corresponding orbit has codimension one: a diffeomorphism action sends the quadratic differential $u(x)(dx)^2$ to the constant quadratic differential $C(dx)^2$, where $C := (1/2\pi) \int_{S^1} \sqrt{u(x)}dx$.

If u changes sign, the integral $\int_a^b \sqrt{u(x)}dx$ between two consecutive zeroes a, b of u is invariant. Thus the codimension of the orbit is greater than 1 in this case.

The dual space $(\mathfrak{g}')^*$: It can be naturally identified with the space of pairs $\{(u(x)(dx)^2, a)\}$ with the natural pairing

$$\langle (u(x)(dx)^2, a), (v(x)\partial_x, b)\rangle := \int_{S^1} u(x)v(x)dx + ab.$$

Generic coadjoint orbits are of codimension 2 (they are contained in the hyperplanes $a = const$).

Digression on the Euler equations: *Recall:* Let $\mathfrak{g}$ be a Lie algebra with a positively defined scalar product b. Extend b to the right invariant contravariant metric $b_r : T^*G \times_G T^*G \to \mathbb{R}$, denote by $B : T^G \to \mathbb{R}$ the corresponding quadratic form. The hamiltonian equation on T^*G with the hamiltonian $H := B$ is right invariant, hence can be reduced to a hamiltonian equation on $\mathfrak{g}^*$.

The last is called the *Euler equation* and is given by a vector field $\eta_{\mathfrak{g}}(b(v, v))$. Let $A : g \to \mathfrak{g}^*$ be defined by $\langle v, A(w)\rangle = b(v, w)$. Call A *the inertia operator*. It turns out (*Exercise:* prove this) that this equation is of the form

$$\frac{dx}{dt} = -\mathrm{ad}^*_{A^{-1}x}x, x \in \mathfrak{g}^*.$$

The Euler equation related to the "$H^1_{\alpha\beta}$-energy":

THEOREM. *(Khesin–Misiołek) The Euler equation on $x := (v(x)(dx)^2, a)$ corresponding to the "$H^1_{\alpha\beta}$"-scalar product with $\alpha \neq 0$ has the form*

$$\alpha(v_t + 3vv_x) - \beta(v_{xxt} + 2v_xv_{xx} + vv_{xxx}) - bv_{xxx} = 0, a_t = 0.$$

Remark: By choosing $\alpha = 1, \beta = 0$ one obtains the *Korteweg–de Vries equation*. For $\alpha = \beta = 1$ one recovers the *Camassa–Holm equation*.

Proof Let us calculate the ad^* operator. We have

$$\langle \mathrm{ad}^*_{(v\partial_x, b)}(u(dx)^2, a), (w\partial_x, c)\rangle = \langle (u(dx)^2, a), [(v\partial_x, b), (w\partial_x, c)]'\rangle =$$
$$\int_{S^1} u(-vw_x + wv_x)dx + a\int_{S^1} vw_{xxx}dx = \int_{S^1} uwv_xdx - \int_{S^1} uvdw - a\int_{S^1} wv_{xxx}dx =$$
$$\int_{S^1} uwv_xdx + \int_{S^1} w(u_xv + uv_x)dx - a\int_{S^1} wv_{xxx}dx = \int_{S^1} w(2uv_x + u_xv - av_{xxx})dx.$$

Hence $\mathrm{ad}^*_{(v\partial_x, b)}(u(dx)^2, a) = ((2uv_x + u_xv - av_{xxx})(dx)^2, 0)$.

Now let us look at the inertia operator $A : \mathfrak{g}' \to (\mathfrak{g}')^*$ given by $\langle (v\partial_x, b), A((w\partial_x, a))\rangle = \int_{S^1}(\alpha vw + \beta v_xw_x)dx + ba = \int_{S^1} v\Lambda w dx + ba$, where $\Lambda : \alpha - \beta\partial_x^2$ is a second order differential operator. We have $A((w\partial_x, a)) = ((\Lambda w)(dx)^2, a)$. This operator is nondegenerate for $\alpha \neq 0$.

The corresponding Euler equation is

$$\frac{d}{dt}(u(dx)^2, a) = -\mathrm{ad}^*_{A^{-1}(u(dx)^2,a)}(u(dx)^2, a) = -\mathrm{ad}^*_{((\Lambda^{-1}u)(dx)^2,a)}(u(dx)^2, a),$$

or, using the formula for ad^*

$$\frac{d}{dt}(u(dx)^2, a) = -((2u\Lambda^{-1}u_x + u_x\Lambda^{-1}u - a\Lambda^{-1}u_{xxx})(dx)^2, 0).$$

Putting $v := \Lambda^{-1}u$ we get

$$\frac{d}{dt}(\Lambda v) = -2(\Lambda v)v_x - (\Lambda v_x)v + av_{xxx},\ a_t = 0.$$

Substituting $\Lambda = \alpha - \beta\partial_x^2$ we get the proof. □

Bihamiltonian property of the KdV and C–H equations:

THEOREM. *(Khesin–Misiołek) The Euler equation corresponding to the "$H^1_{\alpha\beta}$"-scalar product with $\alpha \neq 0$ is bihamiltonian: it is hamiltonian with respect to the Lie-Poisson structure $\eta_{\mathfrak{g}'}$ on $(\mathfrak{g}')^*$ (this is standard fact) and it is also hamiltonian with respect to the constant Poisson structure obtained by "freezing" of $\eta_{\mathfrak{g}'}$ at the point $((\alpha/2)(dx)^2, \beta)$.*

Remark: We leave this theorem without proof. The last but not least remark: one can apply the general Magri–Lenard scheme to obtain an infinite sequence of "first integrals" of the (α, β)-Euler equation (in fact Magri invented his scheme having in mind the KdV equation).

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