PYTHAGOREAN TRIPLES, THE ACOUSTIC DECAPHONIC PIANO AND WHY 10 IS A UNIQUE CHOICE

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Abstract. In this note we discuss Pythagorean possibilities for the tunings of a piano and suggest an analytic metric - the tempered index - for comparing one tuning to another. In particular, we examine all Pythagorean tuning systems with less than 500 keys in the keyboard and with the generating harmonic frequency smaller that $f = 21$. We show that among all of these systems the smallest comma occurs for the 10-step scale system. This enables us to define Pythagorean t-step systems with a good comma for an arbitrary number t of steps in the scale. We also define a notion of a tempered index of such systems. We show that within these definitions the classical 12-step Pythagorean scale system generated by the harmonic $f = 3$, i.e by the interval $\frac{3}{2}$ of the 'perfect fifth', has much larger tempered index than the 10-step Pythagorean system generated by the harmonic $f = 13$ i.e. by the interval $\frac{13}{8}$. It turns out that the $t = 10$, $f = 13$, system has the smallest tempered index among all systems we considered, and that the classical $t = 12$, $f = 3$ system has the largest tempered index among all of them.

1. Motivation

In January 2023 I contacted the world renowned jazz pianist Leszek Możdżer, to ask if he would be willing to give a piano recital for the participants of a *mathematics* conference 'GRIEG meets Chopin' that I was helping to organize. To my surprise Leszek Możdżer's answer to my request was positive, but with one caveat. Specifically, would I help him with the mathematics needed to redesign his Östlind and Almquist concert piano from the usual 12-step equally tempered (TET) scale to the 10-step TET scale? Możd er further proposed that, at the concert during the mathematical conference, two pianos would be played: his redesigned 10-TET acoustic Östlind and Almquist piano and the usual 12-TET Steinway piano.

Without thinking much about 'why the *ten*-step scale?' I gladly accepted Możdżer's proposal and in short order prepared a table with data needed to retune the piano from 12-TET to 10-TET. But while the mathematics I used was straightforward the *actual* retuning of the 12-TET piano to the 10-TET scale encountered many technical isuues. These were eventually resolved by the combined efforts of two teams of Leszek Możdżer (with Roman Galiński, Jan Grzyśka, Ryszard Mariański, Mirosław Mastalerz, Sławomir Rosa) and mine (Aleksander Bogucki, Andrzej Włodarczyk). My team even filed a patent aplication with the major ideas of this retuning.

As a result the *World Premiere of the Acoustic Decaphonic Piano* by Leszek Możdżer took place on July 13, 2023 in the *Nowa Miodowa Concert Hall* in Warsaw, Poland. In a briliant program he premiered a number of his compositions writen for two pianos, both traditional 12-TET and the decaphonic 10-TET, some of which were played on both instruments simultanously. He also performed a number of world's piano masterpieces paraphrased for the 10TET acoustic piano. In the opinion of many of those in attendence, he proved that with his virtuosity and for his musical compositions/paraphrases, the 10TET piano is a wonderful instrument [1].

Leszek Możdżer's answer to my question 'why you want a 10-scale piano?' is beyond the scope of this note; shortly: it was quite unsatisfactory for me. So, since January 2023 I have been looking for a mathematical argument that would characterize the 10-step musical scale among all other scales. The present paper is an attempt for such a characterization.

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In this characterization the main idea consists in defining what a t-step *Pythagorean* musical scale is, and comparing such a scale, for each t, with its corresponding t-step equally tempered scale. An implementation of this idea required the following notions: a generating frequency, a good comma, Pythagorean t-step scale with a good comma, and an tempered index of a musical tuning system. All of these notions are defined in the next sections and, eventually, they are used to answer the question 'why 10 is a unique choice?'.

2. The 12-scale Pythagorean and equally tempered systems

The well known *Pythagorean tuning system* assigns the following multipliers to each step of its *12-step scale*:

1	$\frac{256}{243}$	$\frac{9}{8}$	$\frac{32}{27}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{729}{512}$	$\frac{3}{2}$	$\frac{128}{81}$	$\frac{27}{16}$	$\frac{16}{9}$	$\frac{243}{128}$	2

Cells in the above table represent 12 keys of a piano keyboard spanning an *octave*. The frequencies of pure tones played by a given key expressed in Herzes are given by the frequency in Herzes played by the first key, multiplied by the multiplier from the cell correponding to this key. For example if, the key corresponding to the cell with multiplier $\frac{27}{16}$ plays a pure tone of 432Hz, then the first key plays a pure tone of frequency $f_0 = 256\text{Hz}$. The frequencies of the keys in other octaves of the piano are determined in the same way, but now the frequencies of the keys in the octave with a number $n = -3, -2, -1, 0, 1, 2, 3$ are multiplied by the frequency $2^n \times f_0 = 2^n \times 256\text{Hz}$.

The multipliers h_ℓ in the above table are obtained by taking successive powers of

$$h_1 = \tfrac{3}{2},$$

and possibly dividing or multiplying them by 2, to keep the obtained number h_ℓ in the range of the interval $[1, 2]$. In this way all the multipliers are of the form

$$h_\ell = \left(\tfrac{3}{2}\right)^\ell 2^{m_\ell},$$

where ℓ is an integer, and m_ℓ is a unique integer such that $|h_\ell - 1| < 1$.

The number $h_1 = \frac{3}{2}$ corresponds to the musical interval of a *perfect fifth*. It is chosen to put as many perfect fifths intervals on the piano keyboard as possible, because for the Western World Ear perfect fifths intervals sound *nice* and *harmonious*. The reason for the number $\frac{3}{2}$ is twofold:

- First, the Western World Ear perceives the frequencies with frequency ratio $2:1$ as the same, and
- second, the number *three* is the next number in the *harmonic series* $1, 2, 3, 4, 5, \dots$ after the number 1 and its musically equivalent number 2.

Since the number 3 is beyond the octave $[1, 2]$ one takes its musical equivalent $h_1 = 3:2$ to represent the sound of the harmonic 3 in the octave $[1, 2]$.

A typical piano has a span of *seven* octaves. The reason for this is that

$$(\tfrac{3}{2})^{12} \simeq 129.746$$

and that this number differs from

$$2^7 = 128,$$

by no more than 1.4%. The *aproximate equality*

$$(\tfrac{3}{2}^{12} \simeq 2^7,$$

or better, the aproximate equality

$$(\tfrac{3}{2})^{\frac{12}{7}} \simeq 2,$$

means that starting with the first key on the piano keyboard, after hiting *twelve* consecutive keys distanced from each other by a perfect fifth $h_1 = \frac{3}{2}$, one arrives at the key with (almost) the same pitch class as of the starting key. This last key of the passage, has the aproximate multiplier 2^7, so it appears after passing *seven* octaves on the keyboard.

The difference

$$c_{(3,12,7)} = \left| \left(\tfrac{3}{2}\right)^{\frac{12}{7}} - 2 \right| \simeq 0.00387547$$

is related to the *Pytahgorean comma*, and for the purpose of this paper will be called just a *comma*. We observe that

$$0.003 < c_{(3,12,7)} < 0.004,$$

and define a positive real number ϵ to be the upper bond in this inequality,

$$\epsilon = 0.004.$$

Note that the comma $c_{(3,12,7)}$ is entirely defined by the three numbers 3, 12 and 7. It therefore can be easilly generalized for another triple of natural numbers $(2k+1, t, s)$. We have the following definition.

Definition 2.1. Let $n_k = 2k+1$, with $k = 1, 2, 3, \ldots$, be an odd natural number. Let m_0 be the unique natural number such that $1 < \frac{n_k}{2^{m_0}} < 2$. If

$$\Big| \left(\tfrac{n_k}{2^{m_0}}\right)^{\frac{t}{s}} - 2 \Big| < \epsilon = 0.004,$$

with t and s being some natural numbers, then the triple of numbers (n_k, t, s) is called a *triple with a good comma*. The *good comma* for this triple is defined to be

$$c_{(n_k,t,s)} = \Big| \left(\tfrac{n_k}{2^{m_0}}\right)^{\frac{t}{s}} - 2 \Big|.$$

Triples with a good comma enable us to define a generalization of the 12-step Pythagorean scale, which as far as comma is concerned, are not worse than the original.

Such systems are defined as follows.

Definition 2.2. Let (n_k, t, s) be a triple with a good comma. Define a *generating frequency* to be a number $h_1 = \frac{n_k}{2^{m_0}}$, where the natural number m_0 is such that $1 < h_1 < 2$. Consider a t-step scale tuning system with t keys in each octave, and such that its base octave have the frequency multipliers:

$$h_\ell = (h_1)^\ell \, 2^{m_\ell},$$

with m_ℓ an integer such that $|h_\ell - 1| < 1$. Here the integer ℓ runs as

$$\ell = -\tfrac{t-1}{2}, -\tfrac{t-1}{2} + 1, \ldots, -1, 0, 1, \ldots, \tfrac{t-1}{2} \quad \text{when } t \text{ is } \textit{odd},$$

and as

$$\ell = -\tfrac{t}{2} + 1, -\tfrac{t}{2} + 2, \ldots, -1, 0, 1, \ldots, \tfrac{t}{2} \quad \text{when } t \text{ is } \textit{even}.$$

Such tuning system is called a *Pythagorean* t*-step scale*. Its (good) *comma* is equal to $c_{(n_k,t,s)}$.

Although one can consider Pythagorean t-step scales for any triple with a good comma, but it is reasonable to consider bounds on *not so high harmonics* (not too big $n_k = 2k+1$), and *not too many keys on the keyboard* (not too large number ts).

For this reason in the following theorem we made the restrictions:

$$k \le 10 \qquad \& \qquad ts < 500.$$

We have the following theorem.

Theorem 2.3. *If* $k \le 10$ *and* $ts < 500$ *then the* only *triples with a good comma are given in the table below:*

triple with a good comma (n_k, t, s)	comma $c_{(n_k,t,s)}$ $\times 10^3$
$(13, 10, 7)$	0.871016
$(9, 53, 9)$	0.928274
$(21, 28, 11)$	1.90363
$(5, 28, 9)$	2.15556
$(9, 47, 8)$	2.34232
$(5, 31, 10)$	2.80238
$(3, 29, 17)$	2.94065
$(15, 21, 19)$	3.26428
$(11, 24, 11)$	3.32468
$(15, 11, 10)$	3.3525
$(21, 23, 9)$	3.5923
$(7, 21, 17)$	3.71073
$(3, 12, 7)$	3.87547

.

Remark 2.4. The suprisng thing is that the triple $(3, 12, 7)$ of the classical 12-step Pythagorean system has the largest comma $c_{(3,12,7)} = 3.87547 \times 10^{-3}$ in the table, and as such, occupies the table's last row. Actually, what is even more surprising, is that the smallest comma occurs for the triple $(13, 10, 7)$, which corresponds to the Pythagorean **ten**-step system. As can be seen from the triple $(13, 10, 7)$, its corresponding **decimal** (or better to say **decaphonic**) system is generated by the **thirteen**'s harmonic, with the generating interval $h_1 = \frac{13}{8}$ playing the role of the 'fifth' in this system. The number of octaves needed to traverse the full circle of these 'fifth's is equal to **seven**, which is given by $s = 7$ appearing in the corresponding triple. This last fact is also a surprise, as $s = 7$ appears also in the classical 12-step Pythagorean system.

Remark 2.5. It is further worth noting that the comma of **decaphonic** $(13, 10, 7)$ system is more than *four times smaller* than the comma of the usual Pythagorean 12-step system. When we take into account only the systems whose keyboards have the usual-pianos-total-number-of-about-100-keys, the comma $c_{(13,10,7)}$ leads the list of the smallest commas really signigficantly.

3. Comparing decaphonic system with the others

The decaphonic Pythagorean system corresponding to the triple $(13, 10, 7)$ has the following multipliers:

1	$\frac{2197}{2048}$	$\frac{32768}{28561}$	$\frac{16}{13}$	$\frac{169}{128}$	$\frac{371293}{262114}$	$\frac{256}{169}$	$\frac{13}{8}$	$\frac{28561}{16384}$	$\frac{4096}{2197}$	2

In this section we want to compare it with all the Pythagorean t-scales from the Theorem 2.3. We recall (see Definition 2.2) that any such scale has multipliers

$$h_\ell = \left(\tfrac{2k+1}{2^{m_0}}\right)^\ell 2^{m_\ell},$$

with numbers k, m_0, m_ℓ, and the range of the numbers ℓ determined by the corresponding triple with a good comma $(2k + 1, t, s)$. To each of such systems there is associated an *equally tempered system* with t-steps, characterized by thhe condition that the ratios of any two consecutive steps in it are the same. The equally tempered system associated with the Pythagorean $(2k + 1, t, s)$ system has multipliers

$$H_\ell = \left(2^a\right)^\ell 2^{M_\ell},$$

with

$$a = \tfrac{s}{t},$$

and with the same indices ℓ as in h_ℓ, and with the integer M_ℓ such that $|H_\ell - 1| < 1$.

There is a geometric way of describing Pythagorean and equally tempered systems. We briefly introduce it now.

First we look at the Pythagorean t-scale system, and to avoid the fuzz with the coefficient m_ℓ appearing in the multiplier h_ℓ, we represent its t steps by the points

$$z_\ell = e^{i\varphi_\ell}$$

on the *unit circle* in the *complex plane* $\mathbb{C}$. In this representation the *angle* corresponding to the value h_ℓ is[1]

$$\varphi_\ell \;=\; 2\pi \log_2 h_\ell \;=\; 2\pi\,(m_\ell + \ell\alpha),$$

where

$$\alpha = \log_2(2k+1).$$

Now, for the t-*scale equally tempered system* we associte to each of its steps with *multipliers* H_ℓ, the *points*

$$Z_\ell = e^{i\Phi_\ell} = e^{i2\pi\ell a}$$

on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$, with the corresponding *angles*

$$\Phi_\ell \;=\; 2\pi \log_2 H_\ell \;=\; 2\pi\,(M_\ell + \ell a).$$

For example, for $t = 12$ with $(n_k, t, s) = (3, 12, 7)$ and $\alpha = \log_2 3$, these definitions give:

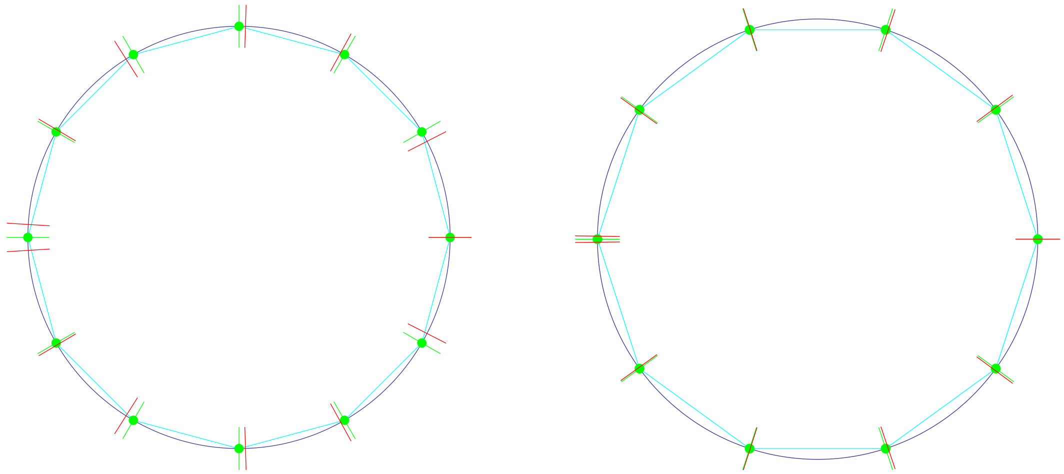
ℓ	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$\frac{\varphi_\ell}{2\pi}$	$10-6\alpha$	$8-5\alpha$	$7-4\alpha$	$5-3\alpha$	$4-2\alpha$	$2-1\alpha$	0α	$-1+1\alpha$	$-3+2\alpha$	$-4+3\alpha$	$-6+4\alpha$	$-7+5\alpha$	$-9+6\alpha$
z_ℓ	$e^{(-6)2\pi i\alpha}$	$e^{(-5)2\pi i\alpha}$	$e^{(-4)2\pi i\alpha}$	$e^{(-3)2\pi i\alpha}$	$e^{(-2)2\pi i\alpha}$	$e^{(-1)2\pi i\alpha}$	$e^{(0)2\pi i\alpha}$	$e^{(1)2\pi i\alpha}$	$e^{(2)2\pi i\alpha}$	$e^{(3)2\pi i\alpha}$	$e^{(4)2\pi i\alpha}$	$e^{(5)2\pi i\alpha}$	$e^{(6)2\pi i\alpha}$
$\frac{\Phi_\ell}{2\pi}$	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{5}{6}$	$\frac{5}{12}$	0	$\frac{7}{12}$	$\frac{1}{6}$	$\frac{3}{4}$	$\frac{1}{3}$	$\frac{11}{12}$	$\frac{1}{2}$
Z_ℓ	$e^{\pi i} = -1$	$e^{\frac{\pi i}{6}}$	$e^{\frac{4\pi i}{3}}$	$e^{\frac{\pi i}{2}}$	$e^{\frac{5\pi i}{3}}$	$e^{\frac{5\pi i}{6}}$	$e^{0\pi i} = 1$	$e^{\frac{7\pi i}{6}}$	$e^{\frac{\pi i}{3}}$	$e^{\frac{3\pi i}{2}}$	$e^{\frac{2\pi i}{3}}$	$e^{\frac{11\pi i}{6}}$	$e^{\pi i} = -1$

and for $t = 10$ with $(n_k, t, s) = (13, 10, 7)$ and $\alpha = \log_2 13$, they give:

ℓ	-5	-4	-3	-2	-1	0	1	2	3	4	5
$\frac{\varphi_\ell}{2\pi}$	$19-5\alpha$	$15-4\alpha$	$12-3\alpha$	$8-2\alpha$	$4-1\alpha$	0α	$-3+1\alpha$	$-7+2\alpha$	$-11+3\alpha$	$-14+4\alpha$	$-18+5\alpha$
z_ℓ	$e^{(-5)2\pi i\alpha}$	$e^{(-4)2\pi i\alpha}$	$e^{(-3)2\pi i\alpha}$	$e^{(-2)2\pi i\alpha}$	$e^{(-1)2\pi i\alpha}$	$e^{(0)2\pi i\alpha}$	$e^{(1)2\pi i\alpha}$	$e^{(2)2\pi i\alpha}$	$e^{(3)2\pi i\alpha}$	$e^{(4)2\pi i\alpha}$	$e^{(5)2\pi i\alpha}$
$\frac{\Phi_\ell}{2\pi}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{9}{10}$	$\frac{3}{5}$	$\frac{3}{10}$	0	$\frac{7}{10}$	$\frac{2}{5}$	$\frac{1}{10}$	$\frac{4}{5}$	$\frac{1}{2}$
Z_ℓ	$e^{\pi i} = -1$	$e^{\frac{2\pi i}{5}}$	$e^{\frac{9\pi i}{5}}$	$e^{\frac{6\pi i}{5}}$	$e^{\frac{3\pi i}{5}}$	$e^{0\pi i} = 1$	$e^{\frac{7\pi i}{5}}$	$e^{\frac{4\pi i}{5}}$	$e^{\frac{\pi i}{5}}$	$e^{\frac{8\pi i}{5}}$	$e^{\pi i} = -1$

Note that, due to the *magic* of the *Euler's formula*, $e^{2\pi i m_\ell} = 1 = e^{2\pi i M_\ell}$, the points z_ℓ of the Pythagorean scales and Z_ℓ of the equally tempered scales are totally determined by the integer ℓ alone; the information as to the exact values of m_ℓ and M_ℓ is not needed to determine the position of the points z_ℓ or Z_ℓ on the circle. In the figure below we plot the Pythagorean points z_ℓ with red dashes, and equally tempered points Z_ℓ with green points and dashes on the unit circle. The left figure is for the 12-step $(3, 12, 7)$ scales, and the right figure is for the 10-step $(13, 10, 7)$ scales. In these pictures the following is visible:

[1] This angle is in *radians*. If one needs this angle in *degrees* it is $\varphi_\ell = 360^0 \log_2(h_\ell)$. It is worth mention, that *piano tuners* measure this angle in *cents*, where they define it as $\varphi_\ell = 1200 \log_2(h_\ell)$.



- The equally tempered system is represented by the vertices of a *regular* t*-gon*; there is a *regular dodecagon* with green vertices corresponding to the 12TET system on the left, and a *regular decagon* with green vertices corresponding to the 10TET system on the right. The *equal musical intervals* between the equally tempered scale points are visualised by the *equal angles* between the successive green bullets/green dashes.
- Except for $z_0 = Z_0 = 1$, the equally tempered green keys are not coincident with the Pythagorean red keys. The angular distances between the successive red dashes are not equal.
- The splitting between the vertices of the regular dekagon and the red dashes is much larger at the left figure corresponding to $t = 12$. It is particularly notable for the *tritone* key $Z_6 = Z_{-6} = -1$ (the green bullets/dashes most to the West in both pictures). Actually, for $\ell = \pm 6$ on the left and $\ell = \pm 5$ on the right, we have *two Pythagorean red tritones* on each of the figures. They are equally distanced from the corresponding *equally tempered green tritone*. It is a matter of taste which of these tritones, h_{-6} or h_6 on the left figure (h_{-5} or h_5 on the right figure), should be chosen to make the t-step Pythagorean scale complete. It is customary to chose h_6 as the 12^{th} element of the scale. Accordingly, we also have chosen h_5 as the 10^{th} element of the $(13, 10, 7)$ scale.

Let us close this section by the introduction of a quantity which measures how much a Pythagorean scale differs from its equally tempered system.

Definition 3.1. The *tempered index* of the Pythagorean system corresponding to a triple (n_k, t, s) is

$$\delta_{(n_k,t,s)} = \tfrac{1}{t-1} \sum_{\ell=-\frac{t}{2}+1}^{\frac{t}{2}} |\Phi_\ell - \varphi_\ell| \quad \text{when } t \text{ is even,}$$

and it is

$$\delta_{(n_k,t,s)} = \tfrac{1}{t-1} \sum_{\ell=-\frac{t-1}{2}}^{\frac{t-1}{2}} |\Phi_\ell - \varphi_\ell| \quad \text{when } t \text{ is odd.}$$

According to this definition the *tempered index* is the average of the absolute values of the nonzero differences between corresponding angles of the equally tempered and Pythagorean points of the scale. Here this average is expressed in *radians per step*.

Using the explicit values of Φ_ℓ and ϕ_ℓ from the table above we get:

$$\delta_{(3,12,7)} = \tfrac{2\pi}{11}(36 \log_2 3 - 57) \simeq 0.0335008 \, \tfrac{\text{radian}}{\text{step}},$$

and

$$\delta_{(13,10,7)} = \tfrac{2\pi}{9}(25 \log_2 13 - \tfrac{185}{2}) \simeq 0.00767453 \, \tfrac{\text{radian}}{\text{step}}.$$

One can also calculate this difference in *cents per step*[2] to obtain:

$$\delta_{(3,12,7)} \simeq 6.39818\,\frac{\text{cent}}{\text{step}},$$

and

$$\delta_{(13,10,7)} \simeq 1.46573\,\frac{\text{cent}}{\text{step}},$$

or in *angular degrees per step* to have:

$$\delta_{(3,12,7)} \simeq 1.91946\,\frac{\text{deg}}{\text{step}},$$

and

$$\delta_{(13,10,7)} \simeq 0.439718\,\frac{\text{deg}}{\text{step}}.$$

Clearly, the classical 12-step system has much larger tempered index than the 10-step one.

Remark 3.2. We end this paper with the remark that our characterization of $t = 10$ is *limited by our assumptions*. If, for example, we extended the list of generating harmonics from $f = 21$ to $f = 63$ and mantained the number of the keys in the keyboard $ts < 500$, we would find triples (n_k, t, s) that have better commas than $(13, 10, 7)$; associated with them there are Pytahgorean systems with smaller tempered index than the decimal system $(13, 10, 7)$. But these systems have either much larger keybord than the $(13, 10, 7)$ system, or are generated by very high harmonics such us $f = 57$ or $f = 59$. We therefore restricted our considerations to that which is included in this paper.

References

[1] https://polmic.pl/index.php?option=com_content&view=article&id=9396:wyjscie-z-muzycznego-matrixa-czy-dekafoniczny-system-rownomiernie-temperowany-jest-bardziej-doskonaly&catid=83&lang=pl&Itemid=196

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[2] The formula for the difference in cents/step is: $\frac{600}{\pi}\,\delta$, and the formula for the difference in degrees/step is: $\frac{180}{\pi}\,\delta$.